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Neutral Stochastic Differential Delay Equations with Markovian Switching

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ABSTRACT

Neutral stochastic differential delay equations (NSDDEs) have recently been studied intensively (see Kolmanovskii, V.B. and Nosov, V.R., *Stability and Periodic Modes of Control Systems with Aftereffect*; Nauka: Moscow, 1981 and Mao X., *Stochastic Differential Equations and Their Applications*; Horwood Pub.: Chichester, 1997). Given that many systems are often subject to component failures or repairs, changing subsystem interconnections and abrupt environmental disturbances etc., the structure and parameters of underlying NSDDEs may change abruptly.

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One way to model such abrupt changes is to use the continuous-time Markov chains. As a result, the underlying NSDDEs become NSDDEs with Markovian switching which are hybrid systems. So far little is known about the NSDDEs with Markovian switching and the aim of this paper is to close this gap. In this paper we will not only establish a fundamental theory for such systems but also discuss some important properties of the solutions e.g. boundedness and stability.

Key Words: Brownian motion; Generalized Itô's formula; Markov chain; Hybrid system.

1. INTRODUCTION

Many dynamical systems not only depend on present and past states but also involve derivatives with delays. Neutral differential delay equations (NDDEs) are often used to describe such systems. For example, Brayton[1] used a partial differential equation (PDE) to describe the problem of loseless transmission and then transferred the PDE into the following NDDE

$$\frac{d}{dt}[x(t) - Kx(t-\tau)] = f(x(t), x(t-\tau))$$

Another similar equation encountered by Rubanik[2] in his study of vibrating masses attached to an elastic bar is

$$\ddot{x}(t) + {\omega_1}^2 x(t) = \varepsilon f_1(x(t), \dot{x}(t), y(t), \dot{y}(t)) + \gamma_1 \ddot{y}(t-\tau)$$
$$\ddot{y}(t) + {\omega_2}^2 x(t) = \varepsilon f_2(x(t), \dot{x}(t), y(t), \dot{y}(t)) + \gamma_2 \ddot{x}(t-\tau)$$

In general, an NDDE has the form

$$\frac{d}{dt}[x(t) - D(x(t-\tau))] = f(x(t), x(t-\tau), t) \tag{1.1}$$

For the theory of NDDEs please see Hale and Lunel[3] and the references therein. Taking the environmental disturbances into account, Kolmanovskii and Nosov[4] and Mao[5] discussed the neutral stochastic differential delay equations (NSDDEs)

$$d[x(t) - D(x(t-\tau))] = f(x(t), x(t-\tau), t)\,dt + g(x(t), x(t-\tau), t)\,dB(t) \tag{1.2}$$

On the other hand, many practical systems may experience abrupt changes in their structure and parameters caused by phenomena such as component failures or repairs, changing subsystem interconnections, and abrupt environmental disturbances. The hybrid systems driven by continuous-time Markov chains have recently been developed to cope with such situation. The hybrid systems combine a part of the state that takes values continuously and another part of the state that takes discrete values. Such hybrid systems have been considered for the modelling of electric power systems by Willsky & Levy[6] as well as for the control of a solar thermal central receiver by Sworder & Rogers.[7] Athans[8] suggested that the hybrid systems would become a basic framework in posing and solving control-related issues in Battle Management Command, Control and Communications (BM/C^3) systems. An important class of hybrid systems is the jump linear systems **Q1**

$$\dot{x}(t) = A(r(t))x(t) \tag{1.3}$$

where a part of the state $x(t)$ takes values in $\mathbb{R}^n$ while another part of the state $r(t)$ is a Markov chain taking values in $S = \{1, 2, \ldots, N\}$. One of the important issues in the study of hybrid systems is the automatic control, with consequent emphasis being placed on the analysis of stability. For more detailed account on hybrid systems please see Basak et al.,[9] Ji and Chizeck,[10] Mao et al.,[11,12] Mariton,[13] Shaikhet,[14] among the others.

Motivated by the hybrid systems, let us return to the NSDDE (1.2). If this system experiences abrupt changes in their structure and parameters and we use the continuous-time Markov chains to model these abrupt changes, we then need to deal with NSDDE with Markovian switching

$$\begin{aligned} d[x(t) - D(x(t-\tau), r(t))] = {} & f(x(t), x(t-\tau), t, r(t))\, dt \\ & + g(x(t), x(t-\tau), t, r(t))\, dB(t) \end{aligned} \tag{1.4}$$

So far little is known about such systems and the aim of this paper is to close this gap. We will establish a fundamental theory for the NSDDEs with Markovian switching e.g., the definition of the solutions and conditions for the existence and uniqueness of the solutions. We will also discuss some important properties of the solutions e.g., asymptotic boundedness and stability.

2. NSDDES WITH MARKOVIAN SWITCHING

Throughout this paper, unless otherwise specified, we use the following notations. Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$. If **A** is a vector or matrix, its transpose is denoted by $\mathbf{A}^T$. If $\mathbf{A}$ is a matrix, its trace norm is denoted by $|\mathbf{A}| = \sqrt{\mathrm{trace}(\mathbf{A}^T\mathbf{A})}$. If $\mathbf{A}$ is a symmetric matrix, denote by $\lambda_{\max}(\mathbf{A})$ and $\lambda_{\min}(\mathbf{A})$ its largest and smallest eigenvalue, respectively. Let $\mathbb{R}_+ = [0, \infty)$ and $\tau > 0$. Let $C([-\tau, 0]; \mathbb{R}^n)$ denote the family of continuous functions φ from $[-\tau, 0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). For $p > 0$, denote by $L_{\mathcal{F}_0}{}^p([-\tau, 0]; \mathbb{R}^n)$ the family of all $\mathcal{F}_0$-measurable and $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables ξ such that $E\|\xi\|^p < \infty$. Denote by $C_{\mathcal{F}_0}{}^b([-\tau, 0]; \mathbb{R}^n)$ the family of all $\mathcal{F}_0$-measurable, bounded and $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables. If $x(t)$ is a continuous $\mathbb{R}^n$-valued stochastic process on $t \in [-\tau, \infty)$, we let $x_t = \{x(t+\theta): -\tau \leq \theta \leq 0\}$ for $t \geq 0$ which is regarded as a $C([-\tau, 0]; \mathbb{R}^n)$-valued stochastic process. Let $w(t) = (w_1(t), \ldots, w_m(t))^T$, $t \geq 0$, be an m-dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\Delta) = j \mid r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while

$$\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$$

We assume that the Markov chain $r(\cdot)$ is $\mathcal{F}_t$-adapted but independent of the Brownian motion $w(\cdot)$. It is well-known (see Skorohod[15]) that almost every sample path of $r(t)$ is a right-continuous step function with a finite number of simple jumps in any finite subinterval of $\mathbb{R}_+$. In other words, there is a sequence of stopping times $0 = \tau_0 < \tau_1 < \cdots < \tau_k \to \infty$ almost surely such that

$$r(t) = \sum_{k=0}^{\infty} r(\tau_k) 1_{[\tau_k, \tau_{k+1})}(t)$$

where 1_A denotes the indicator function of set A.

In this paper we consider the n-dimensional NSDDE with Markovian switching

$$\begin{aligned} d[x(t) - D(x(t-\tau), r(t))] &= f(x(t), x(t-\tau), t, r(t))\, dt \\ &\quad + g(x(t), x(t-\tau), t, r(t))\, dB(t) \end{aligned} \tag{2.1}$$

on $t \geq 0$ with initial data $x_0 = \xi \in L_{\mathcal{F}_0}{}^2([-\tau, 0]; \mathbb{R}^n)$ and $r(0) = r_0$, where r_0 is an S-valued $\mathcal{F}_0$-measurable random variable and

$$\begin{aligned} &D\colon \mathbb{R}^n \times S \to \mathbb{R}^n, \quad f\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^n, \\ &g\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^n \end{aligned}$$

are all Borel-measurable functions. By the definition of Itô's stochastic differential, Eq. (2.1) means that for every $T > 0$,

$$\begin{aligned} x(T) - D(x(T-\tau), r(T)) &= \xi(0) - D(x(-\tau), r_0) \\ &\quad + \int_0^T f(x(t), x(t-\tau), t, r(t))\, dt \\ &\quad + \int_0^T g(x(t), x(t-\tau), t, r(t))\, dB(t) \end{aligned} \tag{2.2}$$

holds with probability one. Let us first give the definition of the solution.

Definition 2.1

An $\mathbb{R}^n$-valued stochastic process $\{x(t)\}_{t \geq -\tau}$ is called a solution of Eq. (2.1)if it has the following properties:

1. $\{x(t)\}$ is continuous and $\mathcal{F}_t$-adapted (as usual we set $\mathcal{F}_t = \mathcal{F}_0$ when $t \in [-\tau, 0]$);
2. for every $T > 0$

$$\int_0^T |f(x(t), x(t-\tau), t, r(t))|\, dt < \infty \quad a.s.$$

and

$$\int_0^T |g(x(t), x(t-\tau), t, r(t))|^2\, dt < \infty \quad a.s.$$

3. $x_0 = \xi$ and Eq. (2.2) holds with probability 1 for every $T \geq 0$.

A solution $\{x(t)\}$ is said to be unique if any other solution $\{\bar{x}(t)\}$ is indistinguishable from $\{x(t)\}$.

To establish the existence-and-uniqueness theorem we need to impose the following assumptions.

Assumption 2.2

Assume that there exists a positive constant K such that

$$|f(x, y, t, i)|^2 \vee |g(x, y, t, i)|^2 \vee |D(y, i)|^2 \leq K(1 + |x|^2 + |y|^2) \tag{2.3}$$

for all $(x, y, t, i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S$. Moreover, for every $h > 0$, there is a positive constant K_h such that

$$|f(x, y, t, i) - f(\bar{x}, y, t, i)|^2 \vee |g(x, y, t, i) - g(\bar{x}, y, t, i)|^2 \leq K_h|x - y|^2 \tag{2.4}$$

for all $(y, t, i) \in \mathbb{R}^n \times \mathbb{R}_+ \times S$ and those $x, \bar{x} \in \mathbb{R}^n$ with $|x| \vee |\bar{x}| \leq h$. We refer to (2.3) as the linear growth condition and (2.4) the local Lipschitz condition in x for $f(x, y, t, i)$.

We can now state our theorem on the existence and uniqueness of the solution.

Theorem 2.3

Under Assumption 2.2 Eq. (2.1) has a unique solution $x(t)$ on $t \geq -\tau$. Moreover, the solution has the property

$$\begin{aligned} E\|x_{k\tau}\|^2 &= E\left(\sup_{(k-1)\tau \leq t \leq k\tau} |x(t)|^2\right) \\ &\leq \frac{C^{k+1} - 1}{C - 1} + C^k E\|\xi\|^2 \quad \forall k = 1, 2, \ldots \end{aligned} \tag{2.5}$$

where

$$C = C(K, \tau) = [10K + 11 \vee 5K\tau(\tau + 5)]e^{10K\tau(\tau+5)}$$

Proof.

Given the initial data ξ on $[-\tau, 0]$, we first show that Eq. (2.1) has a unique solution $x(t)$ on $t \in [0, \tau]$ and it has the property

$$E\|x_\tau\|^2 \leq C(1 + E\|\xi\|^2) \tag{2.6}$$

where $C = C(K, \tau)$ has been defined in the statement of the theorem. In fact, when $t \in [0, \tau]$, Eq. (2.1) can be written as

$$\begin{aligned} x(t) = \xi(0) + D(\xi(t-\tau), r(t)) - D(\xi(-\tau), r_0) \\ + \int_0^t f(x(s), \xi(s-\tau), s, r(s))\, ds \\ + \int_0^t g(x(s), \xi(s-\tau), s, r(s))\, dB(s) \end{aligned} \tag{2.7}$$

This is a stochastic differential equation with Markovian switching and it is known (see Mao[11]) that this equation has a unique solution $x(t)$ on $t \in [0, \tau]$ under Assumption 2.2. It now follows from (2.7) that

$$\begin{aligned} \frac{1}{5}|x(t)|^2 \le |\xi(0)|^2 + |D(\xi(t-\tau), r(t))|^2 + |D(\xi(-\tau), r_0)|^2 \\ + \left| \int_0^t f(x(s), \xi(s-\tau), s, r(s))\, ds \right|^2 \\ + \left| \int_0^t g(x(s), \xi(s-\tau), s, r(s))\, dB(s) \right|^2 \end{aligned}$$

Taking $x = 0$ in (2.3) we observe that

$$|D(y, i)| \le K(1 + |y|^2) \quad \forall (y, i) \in \mathbb{R}^n \times S \tag{2.8}$$

So

$$|\xi(0)|^2 + |D(\xi(t-\tau), r(t))|^2 + |D(\xi(-\tau), r_0)|^2 \le 2K + (2K+1)\|\xi\|^2$$

Moreover, by the Hölder inequality and (2.3),

$$\begin{aligned} \left| \int_0^t f(x(s), \xi(s-\tau), s, r(s))\, ds \right|^2 &\le t \int_0^t |f(x(s), \xi(s-\tau), s, r(s))|^2\, ds \\ &\le \tau K \int_0^t (1 + |x(s)|^2 + |x(s-\tau)|^2)\, ds \end{aligned}$$

We therefore see that for any $t_1 \in [0, \tau]$,

$$E\left(\sup_{0\le t\le t_1}|x(t)|^2\right) \le 10K + 5(2K+1)E\|\xi\|^2 + 5\tau KE\int_0^{t_1}(1+|x(s)|^2+|x(s-\tau)|^2)\,ds + 5E\left(\sup_{0\le t\le t_1}\left|\int_0^t g(x(s),\,\xi(s-\tau),\,s,\,r(s))\,dB(s)\right|^2\right)$$

But, by the Doob martingale inequality and (2.3),

$$E\left(\sup_{0\le t\le t_1}\left|\int_0^t g(x(s),\,\xi(s-\tau),\,s,\,r(s))\,dB(s)\right|^2\right) \le 4E\int_0^{t_1}|g(x(s),\,\xi(s-\tau),\,s,\,r(s))|^2\,ds \le 4KE\int_0^{t_1}(1+|x(s)|^2+|x(s-\tau)|^2)\,ds$$

Thus we have

$$E\left(\sup_{0\le t\le t_1}|x(t)|^2\right) \le 10K + 5(2K+1)E\|\xi\|^2 + 5K(\tau+5)E\int_0^{t_1}(1+|x(s)|^2+|x(s-\tau)|^2)\,ds \le 10K + 5(2K+1)E\|\xi\|^2 + 5K\tau(\tau+5) + 10K(\tau+5)\int_0^{t_1}E\left(\sup_{-\tau\le u\le s}|x(u)|^2\right)ds$$

Consequently

$$E\left(\sup_{-\tau\le t\le t_1}|x(t)|^2\right) \le E\left(\|\xi\|^2 + \sup_{0\le t\le t_1}|x(t)|^2\right) \le 10K + (10K+11)E\|\xi\|^2 + 5K\tau(\tau+5) + 10K(\tau+5)\int_0^{t_1}E\left(\sup_{-\tau\le u\le s}|x(u)|^2\right)ds$$

The well-known Gronwall inequality implies

$$E\left(\sup_{-\tau\le t\le\tau}|x(t)|^2\right) \le [10K + 11 \vee 5K\tau(\tau+5)](1 + E\|\xi\|^2)e^{10K\tau(\tau+5)}$$

and (2.6) follows.

Once we obtain the unique solution on $[0, \tau]$ we can regard them as the initial data and consider Eq. (2.1) for $t \in [\tau, 2\tau]$. In this case, Eq. (2.1) can be written as

$$\begin{aligned} x(t) = \xi(\tau) &+ D(x(t-\tau), r(t)) - D(x(0), r(\tau)) \\ &+ \int_\tau^t f(x(s), x(s-\tau), s, r(s))\,ds \\ &+ \int_\tau^t g(x(s), x(s-\tau), s, r(s))\,dB(s) \end{aligned}$$

This is a stochastic differential equation with Markovian switching and it has a unique solution $x(t)$ on $t \in [\tau, 2\tau]$ under Assumption 2.2. Moreover, we can show in the same way as (2.6) was proved that

$$E\|x_{2\tau}\|^2 \le C(1 + E\|x_\tau\|^2)$$

Repeating this procedure on intervals $[2\tau, 3\tau]$, $[3\tau, 4\tau]$ and so on we obtain the unique solution $x(t)$ on $t \ge -\tau$. Moreover, we have, for any $k = 1, 2, \ldots$

$$\begin{aligned} E\|x_{k\tau}\|^2 &\le C(1 + E\|x_{(k-1)\tau}\|^2) \\ &\le C + C^2(1 + E\|x_{(k-2)\tau}\|^2) \\ &\;\vdots \\ &\le C + C^2 + \cdots + C^{k-1} + C^k(1 + E\|x_0\|^2) \\ &= \frac{C^{k+1}-1}{C-1} + C^k E\|\xi\|^2 \end{aligned}$$

which is the required (2.5). The proof is complete.

Theorem 2.3 shows that if the initial data are in L^2 then the solution will be in L^2. The following theorem shows that if the initial data are in L^p $(p \ge 2)$ then the solution will be in L^p.

Theorem 2.4

Under Assumption 2.2, if the initial data $\xi \in L^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ for some $p \geq 2$, then the unique solution $x(t)$ of Eq. (2.1) has the property that

$$E\|x_{k\tau}\|^p \leq \frac{\bar{C}^{k+1} - 1}{\bar{C} - 1} + \bar{C}^k E\|\xi\|^p \quad \forall k = 1, 2, \dots \tag{2.9}$$

where $\bar{C}$ is a positive constant dependent of only K, τ and p.

Proof.

When $t \in [0, \tau]$, it follows from (2.7) that

$$\begin{aligned}\frac{1}{5^{p-1}}|x(t)|^2 \leq |\xi(0)|^p + |D(\xi(t-\tau), r(t))|^p + |D(\xi(-\tau), r_0)|^p \\ + \left|\int_0^t f(x(s), \xi(s-\tau), s, r(s))\,ds\right|^p \\ + \left|\int_0^t g(x(s), \xi(s-\tau), s, r(s))\,dB(s)\right|^p\end{aligned}$$

By (2.8), the Hölder inequality and (2.3) we can show that

$$\begin{aligned}&|\xi(0)|^p + |D(\xi(t-\tau), r(t))|^p + |D(\xi(-\tau), r_0)|^p \\ &\quad \leq (2K)^{p/2} + [1 + (2K)^{p/2}]\|\xi\|^p\end{aligned}$$

and

$$\begin{aligned}&\left|\int_0^t f(x(s), \xi(s-\tau), s, r(s))\,ds\right|^p \\ &\quad \leq \tau^{p-1}(2K)^{p/2}\int_0^t \left(1 + \sup_{-\tau\leq u\leq s}|x(u)|^p\right) ds\end{aligned}$$

We therefore see that for any $t_1 \in [0, \tau]$,

$$\begin{aligned}E\left(\sup_{0\leq t\leq t_1}|x(t)|^p\right) &\leq c_1(1 + E\|\xi\|^p) \\ &\quad + c_1\int_0^{t_1} E\left(\sup_{-\tau\leq u\leq s}|x(u)|^p\right) ds \\ &\quad + 5^{p-1}E\left(\sup_{0\leq t\leq t_1}\left|\int_0^t g(x(s), \xi(s-\tau), s, r(s))\,dB(s)\right|^p\right)\end{aligned}$$

where c_1 and the following c_2, c_3 etc. are all positive constants dependent of only K, τ and p. But, by the Burkholder-Davis-Gunday inequality (see Mao[5]), the Hölder inequality and (2.3), we have

$$E\left(\sup_{0\le t\le t_1}\left|\int_0^t g(x(s),\, \xi(s-\tau),\, s,\, r(s))\, dB(s)\right|^p\right)$$
$$\le \left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{p/2} E\int_0^{t_1} |g(x(s),\, \xi(s-\tau),\, s,\, r(s))|^p\, ds$$
$$\le c_2 + c_2\int_0^{t_1} E\left(\sup_{-\tau\le u\le s}|x(u)|^p\right) ds$$

We hence have

$$E\left(\sup_{0\le t\le t_1}|x(t)|^2\right) \le c_3(1+E\|\xi\|^p) + c_3\int_0^{t_1} E\left(\sup_{-\tau\le u\le s}|x(u)|^p\right) ds$$

Consequently

$$E\left(\sup_{-\tau\le t\le t_1}|x(t)|^p\right) \le E\left(\|\xi\|^p + \sup_{0\le t\le t_1}|x(t)|^p\right)$$
$$\le (c_3+1)(1+E\|\xi\|^p) + c_3\int_0^{t_1} E\left(\sup_{-\tau\le u\le s}|x(u)|^p\right) ds$$

The well-known Gronwall inequality implies

$$E\left(\sup_{-\tau\le t\le \tau}|x(t)|^2\right) \le (c_3+1)e^{c_3\tau}(1+E\|\xi\|^p)$$

In particular,

$$E\|x_\tau\|^p \le \bar{C}(1+E\|\xi\|^p)$$

where $\bar{C} = (c_3+1)e^{c_3\tau}$ which is a positive constant dependent of only K, τ and p.

In general, we can show in the same way that for any $k = 1, 2, \ldots$

$$E\|x_{k\tau}\|^p \le \bar{C}(1+E\|x_{(k-1)\tau}\|^p)$$

and, by induction, the required (2.9) follows. The proof is complete.

3. THE GENERALIZED ITÔ FORMULA

To investigate the properties of the solutions in more detail we need to introduce the generalized Itô formula. Given any solution $x(t)$, if we set

$$X(t) = x(t) - D(x(t-\tau), r(t))$$
$$F(t) = f(x(t), x(t-\tau), t, r(t))$$
$$G(t) = G(x(t), x(t-\tau), t, r(t))$$

then Eq. (2.1) becomes

$$dX(t) = F(t)\,dt + G(t)\,dB(t)$$

In other words, $X(t)$ is an Itô process. Denote by $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R})$ the family of real-valued functions $V(x, t, i)$ which are continuously twice differentiable in x and once in t. If $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R})$, the generalized Itô formula (see Mao[16] and Skorohod[15] states that for any bounded stopping times $0 \le \rho_1 \le \rho_2 < \infty$ a.s.

$$\begin{aligned} &EV(X(\rho_2), \rho_2, r(\rho_2)) - EV(X(\rho_1), \rho_1, r(\rho_1)) \\ &\quad = E\int_{\rho_1}^{\rho_2} \Bigg(V_t(X(s), s, r(s)) + V_x(X(s), s, r(s))F(s) \\ &\quad + \frac{1}{2}\text{trace}[G^T(s)V_{xx}(X(s), s, r(s))G(s)] \\ &\quad + \sum_{j=1}^{N} \gamma_{r(s),j} V(X(s), s, j) \Bigg) ds \end{aligned}$$

holds provided that $V(X(t), t, r(t))$, $V_t(X(t), t, r(t))$ etc. are bounded on $t \in [\rho_1, \rho_2]$ with probability 1, where

$$V_t(x, t, i) = \frac{\partial V(x, t, i)}{\partial t}$$
$$V_x(x, t, i) = \left(\frac{\partial V(x, t, i)}{\partial x_1}, \ldots, \frac{\partial V(x, t, i)}{\partial x_n} \right)$$
$$V_{xx}(x, t, i) = \left(\frac{\partial^2 V(x, t, i)}{\partial x_i \partial x_j} \right)_{n \times n}$$

Substituting $X(t)$, $F(t)$ and $G(t)$ into the formula above we obtain the following very useful lemma.

Lemma 3.1

Let $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R})$ and $x(t)$ be a solution of Eq. (2.1). Then for any stopping times $0 \le \rho_1 \le \rho_2 < \infty$ a.s.

$$\begin{aligned} &EV(x(\rho_2) - D(x(\rho_2 - \tau), r(\rho_2)), \rho_2, r(\rho_2)) \\ &\quad = EV(x(\rho_1) - D(x(\rho_1 - \tau), r(\rho_1)), \rho_1, r(\rho_1)) \\ &\qquad + E\int_{\rho_1}^{\rho_2} LV(x(s), x(s - \tau), s, r(s))\, ds \end{aligned} \tag{3.1}$$

holds provided that $V(x(t) - D(x(t), r(t)), t, r(t))$ and $LV(x(t), x(t - \tau), t, r(t))$ are bounded on $t \in [\rho_1, \rho_2]$ with probability 1, where the operator $LV: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}$ is defined by

$$\begin{aligned} LV(x, y, t, i) = {} & V_t(x - D(y, i), t, i) + V_x(x - D(y, i), t, i) f(x, y, t, i) \\ & + \frac{1}{2}\operatorname{trace}[g^T(x, y, t, i) V_{xx}(x - D(y, i), t, i) g(x, y, t, i)] \\ & + \sum_{j=1}^{N} \gamma_{ij} V(x - D(y, i), t, j) \end{aligned}$$

We shall also refer to (3.1) as the generalized Itô formula.

This formula will play a key role in the remaining of this paper.

4. ASYMPTOTIC BOUNDEDNESS

In what follows we will impose Assumption 2.2 as a standing hypothesis without mentioning it explicitly. Moreover, we will let the initial data $\xi \in C_{\mathcal{F}_0}{}^b([-\tau, 0]; \mathbb{R}^n)$, and denote by $x(t; \xi)$ the solution of Eq. (2.1). It is easy to observe from Theorem 2.4 that for any $p > 0$,

$$E\left(\sup_{-\tau \le t \le T} |x(t; \xi)|^p \right) < \infty, \quad \forall T > 0$$

That is, any pth moment of the solution is finite. But this does not mean that the pth moment will not tend to infinity as $t \to \infty$. In practice it is useful to know whether the pth moment of the solution will be bounded in long term. In the literature this is known as the property of asymptotic boundedness. To be precise, let us give the definition.

Definition 4.1

Equation (2.1) is said to be asymptotically bounded in pth moment if there is a constant $H > 0$ such that

$$\limsup_{t\to\infty} E|x(t;\xi)|^p \le H$$

for all $\xi \in C_{\mathcal{F}_0}{}^b([-\tau, 0]; \mathbb{R}^n)$. When $p = 2$, it said to be asymptotically bounded in mean square.

The following theorem gives a criterion on the asymptotic boundedness in terms of a Lyapunov function.

Theorem 4.2

Assume that there is a function $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R})$ and positive constants $p, \lambda_1, \lambda_2, \alpha_1, \alpha_2, \beta$ such that $p \ge 1$, $\lambda_1 > \lambda_2$,

$$\alpha_1|x|^p \le V(x, t, i) \le \alpha_2|x|^p, \quad \forall (x, t, i) \in \mathbb{R}^n \times \mathbb{R}_+ \times S \tag{4.1}$$

and

$$LV(x, y, t, i) \le -\lambda_1|x|^p + \lambda_2|x|^p + \beta, \quad \forall (x, t, i) \in \mathbb{R}^n \times \mathbb{R}_+ \times S \tag{4.2}$$

Assume also that there is a constant $\kappa \in (0, 1)$ such that

$$|D(y, i)| \le \kappa|y|, \quad \forall (y, i) \in \mathbb{R}^n \times S \tag{4.3}$$

Then Eq. (2.1) is asymptotically bounded in pth moment.

The proof of this theorem consists of the following four lemmas.

Lemma 4.3

Let $p \ge 1$ and (4.3) hold. Then

$$|x - D(y, i)|^p \le (1+\kappa)^{p-1}(|x|^p + \kappa|y|^p), \quad \forall (x, y, i) \in \mathbb{R}^n \times \mathbb{R}^n \times S$$

Proof.

The required inequality follows from (4.3) directly when $p = 1$ so we only need to prove the lemma for $p > 1$. By the Hölder inequality we derive

$$|x - D(y, i)|^p = \left| x - \kappa^{(p-1)/p}\frac{D(y, i)}{\kappa^{(p-1)/p}} \right|^p$$
$$\le (1+\kappa)^{p-1}\left(|x|^p + \frac{|D(y, i)|^p}{\kappa^{p-1}} \right) \le (1+\kappa)^{p-1}(|x|^p + \kappa|y|^p)$$

as required.

Lemma 4.4

Let the assumptions of Theorem 4.2 hold. Let $\bar{\gamma} > 0$ be the unique root to the equation

$$\bar{\gamma}\alpha_2(1+\kappa)^{p-1} + e^{\bar{\gamma}\tau}(\lambda_2 + \bar{\gamma}\alpha_2\kappa(1+\kappa)^{p-1}) = \lambda_1 \tag{4.4}$$

If $\gamma \in (0, \bar{\gamma}]$, then for $\xi \in C_{\mathcal{F}_0}{}^b([-\tau, 0]; \mathbb{R}^n)$,

$$e^{\gamma t} E|x(t;\xi) - D(x(t-\tau;\xi), r(t))|^p \le C_\gamma E\|\xi\|^p + \frac{\beta}{\gamma\alpha_1} e^{\gamma t}, \quad \forall t \ge 0 \tag{4.5}$$

where

$$C_\gamma = \frac{1}{\alpha_1}[\alpha_2(1+\kappa)^p + \tau e^{\gamma\tau}(\lambda_2 + \gamma\alpha_2\kappa(1+\kappa)^{p-1})]$$

Proof.

Fix any $\xi \in C_{\mathcal{F}_0}{}^b([-\tau, 0]; \mathbb{R}^n)$ and write $x(t;\xi) = x(t)$. Using the generalized Itô formula, the assumptions and Lemma 4.3, we derive

$$\begin{aligned}
& e^{\gamma t} EV(x(t) - D(x(t-\tau), r(t)), t, r(t)) \\
&\quad = EV(\xi(0) - D(\xi(-\tau), r_0), 0, r_0) \\
&\qquad + E\int_0^t e^{\gamma s}[\gamma V(x(s) - D(x(s-\tau), s), s, r(s)) \\
&\qquad + LV(x(s), x(s-\tau), s, r(s))]\, ds \\
&\quad \le \alpha_2 E|\xi(0) - D(\xi(-\tau), r_0)|^p \\
&\qquad + E\int_0^t e^{\gamma s}[\gamma\alpha_2 E|\xi(s) - D(\xi(s-\tau), r(s))|^p \\
&\qquad - \lambda_1|x(s)|^p + \lambda_2|x(s-\tau)|^2 + \beta]\, ds \\
&\quad \le \alpha_2(1+\kappa)^p E\|\xi\|^p + \frac{\beta}{\gamma} e^{\gamma t} \\
&\qquad - (\lambda_1 - \gamma\alpha_2(1+\kappa)^{p-1})E\int_0^t e^{\gamma s} E|\xi(s)|^2\, ds \\
&\qquad + (\lambda_2 + \gamma\alpha_2\kappa(1+\kappa)^{p-1})E\int_0^t e^{\gamma s}|x(s-\tau)|^2\, ds
\end{aligned} \tag{4.6}$$

But

$$E\int_0^t e^{\gamma s}|x(s-\tau)|^2\,ds = e^{\gamma\tau}E\int_{-\tau}^{t-\tau} e^{\gamma s}|x(s)|^2\,ds$$
$$\le \tau e^{\gamma\tau}E\|\xi\|^p + e^{\gamma\tau}E\int_0^t e^{\gamma s}|x(s)|^2\,ds$$

Substituting this into (4.6) and noting from (4.4) that

$$\lambda_1 - \gamma\alpha_2(1+\kappa)^{p-1} \ge e^{\gamma\tau}(\lambda_2 + \gamma\alpha_2\kappa(1+\kappa)^{p-1})$$

we obtain

$$e^{\gamma t}EV(x(t) - D(x(t-\tau), r(t)), t, r(t))$$
$$\le [\alpha_2(1+\kappa)^p + \tau e^{\gamma\tau}(\lambda_2 + \gamma\alpha_2\kappa(1+\kappa)^{p-1})]E\|\xi\|^p + \frac{\beta}{\gamma}e^{\gamma t}$$

This, together with (4.1), yields the required assertion (4.5).

Lemma 4.5
Let $p \ge 1$ and (4.3) hold. Then,

$$|x|^p \le \kappa|y|^p + \frac{|x - D(y,i)|^p}{(1-\kappa)^{p-1}}, \quad \forall (x, y, i) \in \mathbb{R}^n \times \mathbb{R}^n \times S$$

Proof.
When $p = 1$, we have

$$|x| \le |D(y,i)| + |x - D(y,i)| \le \kappa|y| + |x - D(y,i)|$$

so the required inequality holds. When $p > 1$, by the Hölder inequality we derive

$$|x|^p = |D(y,i) + x - D(y,i)|^p$$
$$= \left| D(y,i) + \left[\frac{(1-\kappa)}{\kappa}\right]^{(p-1)/p} \frac{x - D(y,i)}{[(1-\kappa)/\kappa]^{(p-1)/p}} \right|^p$$
$$\le \frac{1}{\kappa^{p-1}}\left(|D(y,i)|^p + \frac{|x - D(y,i)|^p}{[(1-\kappa)/\kappa]^{p-1}} \right) \le \kappa|y|^p + \frac{|x - D(y,i)|^p}{(1-\kappa)^{p-1}}$$

as required.

Lemma 4.6

Let the assumptions of Theorem 4.2 hold. Let $\bar{\gamma} > 0$ be the unique root to Eq. (4.4) and set

$$\gamma = \bar{\gamma} \wedge \frac{1}{2\tau} \log\left(\frac{1}{\kappa}\right) \tag{4.7}$$

Then, for $\xi \in C_{\mathcal{F}_0}{}^b([-\tau, 0]; \mathbb{R}^n)$,

$$\begin{aligned} e^{\gamma t} E|x(t;\xi)|^p \le & \frac{1}{1-\sqrt{\kappa}}\left(\sqrt{\kappa} + \frac{C_\gamma}{1-\sqrt{\kappa}}\right) E\|\xi\|^p \\ & + \frac{\beta e^{\gamma t}}{\gamma \alpha_1 (1-\sqrt{\kappa})(1-\kappa)^{p-1}}, \quad \forall t \ge 0, \end{aligned} \tag{4.8}$$

where C_γ is the same as defined in Lemma 4.4.

Proof.

Fix any $\xi \in C_{\mathcal{F}_0}{}^b([-\tau, 0]; \mathbb{R}^n)$ and write $x(t;\xi) = x(t)$. It is easy to see from (4.7) that

$$\gamma \in (0, \bar{\gamma}] \quad \text{and} \quad e^{\gamma\tau}\kappa \le \sqrt{\kappa}$$

Now, by Lemma 4.5,

$$|x(t)|^p \le \kappa |x(t-\tau)|^p + \frac{1}{(1-\kappa)^{p-1}} |x(t) - D(x(t-\tau), r(t))|^p$$

We then compute, by Lemma 4.4, that

$$\begin{aligned} e^{\gamma t} E|x(t)|^p \le\ & \kappa e^{\gamma t} E|x(t-\tau)|^p + \frac{1}{(1-\kappa)^{p-1}} e^{\gamma t} \\ & \times E|x(t) - D(x(t-\tau), r(t))|^p \\ \le\ & \sqrt{\kappa} e^{\gamma(t-\tau)} E|x(t-\tau)|^p + \frac{1}{(1-\kappa)^{p-1}} \left[C_\gamma E\|\xi\|^p + \frac{\beta}{\gamma\alpha_1} e^{\gamma t} \right] \end{aligned}$$

Hence, for any $T \geq 0$,

$$\sup_{0\leq t\leq T}[e^{\gamma t}E|x(t)|^p] \leq \sqrt{\kappa}\sup_{0\leq t\leq T}[e^{\gamma(t-\tau)}E|x(t-\tau)|^p] + \frac{1}{(1-\kappa)^{p-1}}\left[C_\gamma E\|\xi\|^p + \frac{\beta}{\gamma\alpha_1}e^{\gamma T}\right]$$
$$\leq \sqrt{\kappa}\left(E\|\xi\|^p + \sup_{0\leq t\leq T}[e^{\gamma(t)}E|x(t)|^p]\right) + \frac{1}{(1-\kappa)^{p-1}}\left[C_\gamma E\|\xi\|^p + \frac{\beta}{\gamma\alpha_1}e^{\gamma T}\right]$$

This implies

$$\sup_{0\leq t\leq T}[e^{\gamma t}E|x(t)|^p] \leq \frac{\sqrt{\kappa}}{1-\sqrt{\kappa}}E\|\xi\|^p + \frac{1}{(1-\sqrt{\kappa})(1-\kappa)^{p-1}} \times \left[C_\gamma E\|\xi\|^p + \frac{\beta}{\gamma\alpha_1}e^{\gamma T}\right]$$

In particular,

$$e^{\gamma T}E|x(T)|^p \leq \frac{\sqrt{\kappa}}{1-\sqrt{\kappa}}E\|\xi\|^p + \frac{1}{(1-\sqrt{\kappa})(1-\kappa)^{p-1}} \times \left[C_\gamma E\|\xi\|^p + \frac{\beta}{\gamma\alpha_1}e^{\gamma T}\right]$$

This is the required assertion since $T > 0$ is arbitrary.

Now the assertion of Theorem 4.2 follows from Lemma 4.6 directly, because (4.8) implies

$$\limsup_{t\to\infty} E|x(t;\,\xi)|^p \leq \frac{\beta}{\gamma\alpha_1(1-\sqrt{\kappa})(1-\kappa)^{p-1}}$$

for any $\xi \in C_{\mathcal{F}_0}{}^b([-\tau, 0]; \mathbb{R}^n)$. The proof of Theorem 4.2 is therefore complete.

In many practical situations the Lyapunov functions take the quadratic form, namely

$$V(x, i, t) = x^{\mathrm{T}}Q_i x$$

where Q_is are symmetric positive-definite matrices. Clearly, in this case,

$$\min_{1\le i\le N} \lambda_{\min}(Q_i)|x|^2 \le V(x, i, t) \le \max_{1\le i\le N} \lambda_{\max}(Q_i)|x|^2$$

It is also easy to verify

$$\begin{aligned} LV(x, y, t, i) = {} & 2(x - D(y, i))^{\mathrm{T}} Q_i f(x, y, t, i) \\ & + \mathrm{trace}[g^{\mathrm{T}}(x, y, t, i) Q_i g(x, y, t, i)] \\ & + \sum_{j=1}^{N} \gamma_{ij}(x - D(y, i))^{\mathrm{T}} Q_j (x - D(y, i)) \end{aligned}$$

Consequently, the following useful corollary follows from Theorem 4.2 directly.

Corollary 4.7

Assume that there are symmetric positive-definite matrices Q_i $(1 \le i \le N)$ and positive constants $\lambda_1 > \lambda_2 > 0$ and $\beta > 0$ such that

$$\begin{aligned} & 2(x - D(y, i))^{\mathrm{T}} Q_i f(x, y, t, i) + \mathrm{trace}[g^{\mathrm{T}}(x, y, t, i) Q_i g(x, y, t, i)] \\ & \quad + \sum_{j=1}^{N} \gamma_{ij}(x - D(y, i))^{\mathrm{T}} Q_j (x - D(y, i)) \\ & \le -\lambda_1 |x|^2 + \lambda_2 |x|^2 + \beta, \quad \forall (x, t, i) \in \mathbb{R}^n \times \mathbb{R}_+ \times S \end{aligned} \tag{4.9}$$

Assume also that (4.3) holds. Then Eq. (2.1) is asymptotically bounded in mean square.

5. EXPONENTIAL STABILITY

After the discussion of asymptotic boundedness we shall show that the techniques developed in the previous section can be adopted to deal with exponential stability. Let us first give the definition.

Definition 5.1

Equation (2.1) is said to be exponentially stable in pth moment if

$$\limsup_{t\to\infty} \frac{1}{t} \log(E|x(t; \xi)|^p) < 0$$

for all $\xi \in C_{\mathcal{F}_0}{}^b([-\tau, 0]; \mathbb{R}^n)$. When $p = 2$, it said to be exponentially stable in mean square. Moreover, the equation is said to be almost surely exponentially stable if

$$\limsup_{t\to\infty} \frac{1}{t} \log(|x(t; \xi)|) < 0 \quad \text{a.s.}$$

for all $\xi \in C_{\mathcal{F}_0}{}^b([-\tau, 0]; \mathbb{R}^n)$.

For the general theory on stochastic exponential stability we refer the reader to Arnold,[16] Has'minskii,[17] Ladde & Lakshmikantham[18] and Mao,[19,20] to name a few. The following theorem gives a criterion on the exponential stability in mean square in terms of a Lyapunov function.

Theorem 5.2

Assume that there is a function $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R})$ and positive constants $p, \lambda_1, \lambda_2, \alpha_1, \alpha_2$ such that $p \geq 1$, $\lambda_1 > \lambda_2$,

$$\alpha_1 |x|^p \leq V(x, t, i) \leq \alpha_2 |x|^p, \quad \forall (x, t, i) \in \mathbb{R}^n \times \mathbb{R}_+ \times S \tag{5.1}$$

and

$$LV(x, y, t, i) \leq -\lambda_1 |x|^p + \lambda_2 |x|^p, \quad \forall (x, t, i) \in \mathbb{R}^n \times \mathbb{R}_+ \times S \tag{5.2}$$

Assume also that there is a constant $\kappa \in (0, 1)$ such that

$$|D(y, i)| \leq \kappa |y|, \quad \forall (y, i) \in \mathbb{R}^n \times S \tag{5.3}$$

Let $\bar{\gamma} > 0$ be the unique root to Eq. (4.4) and set

$$\gamma = \bar{\gamma} \vee \frac{1}{2\tau} \log\left(\frac{1}{\kappa}\right) \tag{5.4}$$

Then, for any $\xi \in C_{\mathcal{F}_0}{}^b([-\tau, 0]; \mathbb{R}^n)$,

$$\limsup_{t\to\infty} \frac{1}{t} \log(E|x(t; \xi)|^p) \leq -\gamma \tag{5.5}$$

In other words, Eq. (2.1) is exponentially stable in pth moment.

Proof.

If we compare the assumptions between Theorems 4.2 and 5.2 we observe that the only difference is the paremeter β. More precisely, if we set $\beta = 0$ then the assumptions of Theorem 4.2 become those of Theorem 5.2. We also note that the proofs of Lemmas 4.4 and 4.6 work for $\beta = 0$ as well. Hence Lemma 4.6 shows that for any $\xi \in C_{\mathcal{F}_0}{}^b([-\tau, 0]; \mathbb{R}^n)$,

$$e^{\gamma t} E|x(t;\xi)|^p \le \frac{1}{1-\sqrt{\kappa}}\left(\sqrt{\kappa} + \frac{C_\gamma}{1-\sqrt{\kappa}}\right) E\|\xi\|^p, \quad \forall t \ge 0 \tag{5.6}$$

This implies the required assertion (5.5) immediately.

With some additional conditions we can now show the almost sure exponential stability.

Theorem 5.3

In addition to the assumptions of Theorem 5.2, assume that $p \ge 2$ and there is a positive constant K such that

$$\begin{aligned} &|f(x,y,t,i)| \vee |g(x,y,t,i)| \le K(|x| + |y|) \\ &\quad \forall (x,y,t,i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S \end{aligned} \tag{5.7}$$

Then, for any $\xi \in C_{\mathcal{F}_0}{}^b([-\tau, 0]; \mathbb{R}^n)$,

$$\limsup_{t\to\infty} \frac{1}{t} \log(|x(t;\xi)|) \le -\frac{\gamma}{p} \quad a.s. \tag{5.8}$$

where $\gamma > 0$ is the same as defined in Theorem 5.2. In other words, Eq. (2.1) is almost surely exponentially stable.

Proof.

Fix any $\xi \in C_{\mathcal{F}_0}{}^b([-\tau, 0]; \mathbb{R}^n)$ and write $x(t;\xi) = x(t)$. For $k = 1, 2, \ldots,$ by the Hölder inequality, the Burkholder-Davis-Gundy inequality and condition (5.7), it is not difficult to show that

$$\begin{aligned} &E\left(\sup_{0\le\theta\le\tau} |x(k\tau+\theta) - D(x((k-1)\tau+\theta), r(k\tau+\theta))|^p\right) \\ &\quad \le C_1 E|x(k\tau) - D(x((k-1)\tau), r(k\tau))|^p \\ &\qquad + C_1 \int_{k\tau}^{(k+1)\tau} (E|x(s)|^p + E|x(s-\tau)|^p)\, ds \end{aligned} \tag{5.9}$$

where C_1 is a positive constant dependent of only K, τ and p. Using (5.6) and Lemma 4.3, we obtain

$$E\left(\sup_{0\le\theta\le\tau}|x(k\tau+\theta)-D(x((k-1)\tau+\theta),r(k\tau+\theta))|^p\right)\le C_2e^{-\gamma k\tau}$$

where C_2 is a positive constant independent of k. Thus, for any $\varepsilon\in(0,\gamma)$,

$$P\left\{\sup_{0\le\theta\le\tau}|x(k\tau+\theta)-D(x((k-1)\tau+\theta),r(k\tau+\theta))|^p>e^{-(\gamma-\varepsilon)k\tau}\right\}$$
$$\le C_2e^{-\gamma k\tau}$$

for all $k\ge1$. The well-known Borel-Cantelli lemma shows that for almost all $\omega\in\Omega$,

$$\sup_{0\le\theta\le\tau}|x(k\tau+\theta)-D(x((k-1)\tau+\theta),r(k\tau+\theta))|^p\le e^{-(\gamma-\varepsilon)k\tau}$$

holds for all but finitely many k. Hence for almost all $\omega\in\Omega$ there exists an integer $k_0=k_0(\omega)$ such that

$$\sup_{0\le\theta\le\tau}|x(k\tau+\theta)-D(x((k-1)\tau+\theta),r(k\tau+\theta))|^p\le e^{-(\gamma-\varepsilon)k\tau}$$
$$\text{whenever } k\ge k_0$$

This yields that for almost all $\omega\in\Omega$,

$$|x(t)-D(x(t-\tau),r(t))|\le e^{-p^{-1}(\gamma-\varepsilon)(t-\tau)}\quad\text{whenever } t\ge k_0\tau$$

Noting that $|x(t)-D(x(t-\tau),r(t))|$ is finite on $t\in[0,k_0\tau]$, we observe that there is a finite random variable $\zeta=\zeta(\omega)$ such that, with probability 1,

$$|x(t)-D(x(t-\tau),r(t))|\le\zeta e^{-p^{-1}(\gamma-\varepsilon)t}\quad\text{for all } t\ge0$$

Hence, with probability 1,

$$e^{p^{-1}(\gamma-\varepsilon)t}|x(t)|\le\zeta+\kappa e^{p^{-1}(\gamma-\varepsilon)t}|x(t-\tau)|,\quad\forall t\ge0$$

which implies

$$\begin{aligned}\sup_{0\le s\le t}\left[e^{p^{-1}(\gamma-\varepsilon)s}|x(s)|\right] &\le \zeta + \sup_{0\le s\le t}\left[\kappa e^{p^{-1}(\gamma-\varepsilon)s}|x(s-\tau)|\right]\\ &\le \zeta + \kappa e^{p^{-1}(\gamma-\varepsilon)\tau}\left(\|\xi\| + \sup_{\tau\le s\le t}\left[\kappa e^{p^{-1}(\gamma-\varepsilon)(s-\tau)}|x(s-\tau)|\right]\right)\\ &\le \zeta + \kappa e^{p^{-1}(\gamma-\varepsilon)\tau}\left(\|\xi\| + \sup_{0\le s\le t}\left[\kappa e^{p^{-1}(\gamma-\varepsilon)s}|x(s)|\right]\right), \quad \forall t\ge 0\end{aligned}$$

Since $\kappa e^{p^{-1}(\gamma-\varepsilon)\tau} < 1$ by (5.4), it follows that

$$\sup_{0\le s\le t}\left[e^{p^{-1}(\gamma-\varepsilon)s}|x(s)|\right] \le \frac{\zeta + \kappa e^{p^{-1}(\gamma-\varepsilon)\tau}\|\xi\|}{1-\kappa e^{p^{-1}(\gamma-\varepsilon)\tau}}, \quad \forall t\ge 0$$

with probability 1. This yields immediately that

$$\limsup_{t\to\infty}\frac{1}{t}\log(|x(t)|) \le -\frac{\gamma-\varepsilon}{p} \quad a.s.$$

Letting $\varepsilon \to 0$ we obtain the required assertion (5.8). The proof is complete.

We should point out that the condition $p \ge 2$ in Theorem 5.3 can be replaced by the weaker one $p \ge 1$ but the proof will become rather technical. Due to the page limit here we shall report this case elsewhere.

The following useful corollary follows directly from Theorems 5.2 and 5.3 if the quadratic function is used as the Lyapunov function.

Corallary 5.4

Assume that there are symmetric positive-definite matrices $Q_i (1 \le i \le N)$ and positive constants $\lambda_1 > \lambda_2 > 0$ such that

$$\begin{aligned}&2(x - D(y,i))^T Q_i f(x,y,t,i) + \text{trace}[g^T(x,y,t,i)Q_i g(x,y,t,i)]\\ &\qquad + \sum_{j=1}^{N}\gamma_{ij}(x-D(y,i))^T Q_j (x-D(y,i))\\ &\quad \le -\lambda_1|x|^2 + \lambda_2|x|^2, \quad \forall (x,t,i)\in\mathbb{R}^n\times\mathbb{R}_+\times S \qquad (5.10)\end{aligned}$$

Assume also that (5.3) holds. Then Eq. (2.1) is exponentially stable in mean square. If, moreover, (5.7) holds, then Eq. (2.1) is almost surely exponentially stable as well.

6. EXAMPLES

Let us now discuss a number of examples. In these examples we let the space $S = \{1, 2\}$ in order to make the calculation become simple while the theory is illustrated clearly.

Example 6.1

Let $B(t)$ be a scalar Brownian motion. Let $r(t)$ be a right-continuous Markov chain taking values in $S = \{1, 2\}$ with generator

$$\Gamma = (\gamma_{ij})_{2\times 2} = \begin{pmatrix} -10 & 10 \\ \gamma & -\gamma \end{pmatrix}$$

where $\gamma > 0$. Assume that $B(t)$ and $r(t)$ are independent. Consider the one-dimensional linear NSDDE with Markovian switching

$$d[x(t) - 0.1x(t-\tau)] = [a(r(t))x(t) + b(r(t))x(t-\tau) + c_1]\,dt + c_2 dB(t) \tag{6.1}$$

where c_1 and c_2 are constants,

$$a(1) = 0.5, \quad a(2) = -3, \quad b(1) = 0.5, \quad b(2) = 1$$

To find out whether Eq. (6.1) is asymptotic bounded in mean square, we use the Lyapunov function

$$V(x, t, i) = q_i x^2$$

with $q_1 = 1$ and $q_2 = 0.5$. It is easy to see that the operator $LV\colon \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \times S \to \mathbb{R}$ has the form

$$LV(x, y, t, 1) = 2(x - 0.1y)(0.5x + 0.5y + c_1) + {c_2}^2 - 5(x - 0.1y)^2$$

and

$$LV(x, y, t, 2) = (x - 0.1y)(-3x + y + c_1) + 0.5{c_2}^2 + 0.5\gamma(x - 0.1y)^2$$

It is straightforward to show

$$LV(x, y, t, 1) \le -3.05x^2 + 0.8y^2 + 2c_1(x - 0.1y) + {c_2}^2$$

and

$$LV(x, y, t, 2) \leq -(2.35 - 0.55\gamma)x^2 + (0.55 + 0.055\gamma)y^2 + c_1(x - 0.1y) + {c_2}^2$$

We require

$$(2.35 - 0.55\gamma) > 0.8 \vee (0.55 + 0.055\gamma)$$

namely $0 < \gamma < 2.81$. In this case,

$$LV(x, y, t, i) \leq -(2.35 - 0.55\gamma)x^2 + 0.8y^2 + 2c_1(|x| + 0.1|y|) + {c_2}^2$$

Now, choose $\varepsilon \in (0, (1.55 - 0.55\gamma)/2)$ and note

$$2c_1(|x| + 0.1|y|) \leq \varepsilon x^2 + \varepsilon y^2 + \frac{1.01{c_1}^2}{\varepsilon}$$

Consequently

$$LV(x, y, t, i) \leq -(2.35 - 0.55\gamma - \varepsilon)x^2 + (0.8 + \varepsilon)y^2 + \frac{1.01{c_1}^2}{\varepsilon} + {c_2}^2$$

By Corollary 4.7 we can therefore conclude that Eq. (6.1) is asymptotically bounded in mean square as long as $0 < \gamma < 2.81$.

Example 6.2
If $c_1 = c_2 = 0$, then Eq. (6.1) reduces to

$$\frac{d}{dt}[x(t) - 0.1x(t - \tau)] = a(r(t))x(t) + b(r(t))x(t - \tau) \tag{6.2}$$

which is a linear NDDE with Markovian switching.
Moreover, the above calculations show

$$LV(x, y, t, i) \leq -(2.35 - 0.55\gamma)x^2 + 0.8y^2$$

By Corollary 5.4 we can therefore conclude that if $0 < \gamma < 2.81$, then Eq. (6.2) is exponentially stable in mean square and it is also almost surely exponentially stable.

It is interesting to regard system (6.2) as the result of two NDDEs

$$\frac{d}{dt}[x(t) - 0.1x(t-\tau)] = x(t) + 2x(t-\tau) \tag{6.3}$$

and

$$\frac{d}{dt}[x(t) - 0.1x(t-\tau)] = -3x(t) + x(t-\tau) \tag{6.4}$$

switching from one to the other according to the law of the Markov chain. It is known (see Hale and Lunel[3]) that Eq. (6.3) is not exponentially stable although Eq. (6.4) is. However, due to the Markovian switching the overall system (6.2) is exponentially stable. This clearly shows the important role of Markovian switching.

Example 6.3

Let us finally discuss a nonlinear NSDDE with Markovian switching

$$d[x(t) - 0.1x(t-\tau)] = f(x(t), t, r(t))\,dt + g(x(t-\tau), t, r(t))\,dB(t) \tag{6.5}$$

Here $r(t)$ is a right-continuous Markov chain taking values in $S = \{1, 2\}$ with generator

$$\Gamma = (\gamma_{ij})_{2\times 2} = \begin{pmatrix} -10 & 10 \\ 1 & -1 \end{pmatrix}$$

Assume that f: $\mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^n$ and g: $\mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^n$ satisfy

$$\begin{aligned} x^{\mathrm{T}} f(x, t, i) &\le \begin{cases} 0.1|x|^2 & \text{if } i = 1, \\ -5|x|^2 & \text{if } i = 2; \end{cases} \\ |f(x, t, i)| &\le \begin{cases} 0.1|x| & \text{if } i = 1, \\ 5|x| & \text{if } i = 2; \end{cases} \\ |g(y, t, i)| &\le 0.5|y| \end{aligned} \tag{6.6}$$

Let $p \in [3, 4]$ and define the Lyapunov function

$$V(x, t, i) = q_i |x|^2$$

with $q_1 = 1$ and $q_2 = 0.4$. It is not difficult to show the operator

$$\begin{aligned} LV(x, y, t, i) \le\ & q_i p|x - 0.1y|^{p-2}(x - 0.1y)^{\mathrm{T}} f(x, t, i) \\ & + \frac{1}{2} q_i p(p-1)|x - 0.1y|^{p-2}|g(y, t, i)|^2 \\ & + \sum_{j=1}^{2} \gamma_{ij} q_j |x - 0.1y|^p \end{aligned}$$

By (6.6) we then estimate

$$\begin{aligned} LV(x, y, t, 1) &\le 4|x - 0.1y|^{p-2}[0.1|x|^2 + 0.01|x||y|] \\ &\quad + 1.5|x - 0.1y|^{p-2}|y|^2 - 6|x - 0.1y|^p \\ &\le |x - 0.1y|^{p-2}[0.42|x|^2 + 1.52|y|^2] - 6|x - 0.1y|^p \end{aligned}$$

But, by Lemmas 4.3 and 4.5,

$$|x - 0.1y|^{p-2} \le 1.1^{p-3}(|x|^{p-2} + 0.1|y|^{p-2}) \le 1.1|x|^{p-2} + 0.11|y|^{p-2}$$

and

$$-|x - 0.1y|^p \le -0.9^{p-1}|x|^p + 0.1 \times 0.9^{p-1}|y|^p \le -0.7|x|^p + 0.1|y|^p$$

Moreover, by the elementary inequality $a^{\alpha} b^{1-\alpha} \le \alpha a + (1-\alpha) b$ for $a, b \ge 0$ and $\alpha \in (0, 1)$, we have

$$|x|^2|y|^{p-2} \le \frac{2}{p}|x|^p + \frac{p-2}{p}|y|^p$$

It is therefore straightforward to show

$$LV(x, y, t, 1) \le -\left(2.4818 + \frac{3.2516}{p}\right)|x|^p + \left(0.8134 + \frac{3.2516}{p}\right)|y|^p$$

Similarly, we derive

$$\begin{aligned}
LV(x, y, t, 2) &\le 0.4p|x - 0.1y|^{p-2}[-5|x|^2 + 0.5|x||y|] \\
&\quad + 0.6|x - 0.1y|^{p-2}|y|^2 + 0.6|x - 0.1y|^p \\
&\le 6|x|^2(-0.9^{p-3}|x|^{p-2} + 0.1 \times 0.9^{p-3}|y|^{p-2}) \\
&\quad + (1.1|x|^{p-2} + 0.11|y|^{p-2})(0.4|x|^2 + |y|^2) \\
&\quad + 0.6 \times 1.1^{p-1}(|x|^p + 0.1|y|^p) \\
&\le -3.029|x|^p + 0.644|x|^2|y|^{p-2} + 1.1|x|^{p-2}|y|^2 + 0.2031|y|^p \\
&\le -\left(1.929 + \frac{0.912}{p}\right)|x|^p + \left(0.8671 + \frac{0.912}{p}\right)|y|^p
\end{aligned}$$

Combining the above two inequalities we obtain

$$LV(x, y, t, i) \le -\left(1.929 + \frac{0.912}{p}\right)|x|^p + \left(0.8134 + \frac{3.2516}{p}\right)|y|^p$$

Since

$$1.929 + \frac{0.912}{p} > 0.8134 + \frac{3.2516}{p} \quad \text{when } p \in [3, 4]$$

we can, by Corollary 5.4, conclude that Eq. (6.2) is exponentially stable in pth moment if $p \in [3, 4]$, and it is also almost surely exponentially stable.

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