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Neutrino Energy Loss in Neutron Star Matter

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The theory of the energy loss due to neutrino processes in neutron stars is formulated in such a form that one can include the effect of strong interaction systematically. As an application the rates of energy loss are calculated for the following cases of interest in the study of neutron stars; i) the neutron and the proton liquid are both normal, ii) the proton or iii) the neutron liquid is in the superfluid state. Our results largely confirm the earlier calculations by Bahcall and Wolf and by Wolf. The rate of energy loss in the state of nuclei coexisting with free neutrons is also discussed.

§ 1. Introduction

It is expected that a neutron star cools rapidly down to a point where the surface temperature is about 10^6 °K predominantly through neutrino processes. The rate of energy loss due to neutrino and anti-neutrino emissions has been calculated by several authors.[1)~8)] We refer to the work of Tsuruta and Cameron[4)] for a general discussion of the cooling of neutron stars. The interior of a massive neutron star consists mainly of neutrons, protons and electrons, the density of protons being of the order of several percent of the total density.[5),6)] At relatively low density, protons exist in the form of neutron-rich nuclei, which coexist with the neutron liquid, and above a certain density which is estimated to be close to the nuclear density all the three particles are in the state of uniform degenerate fermi liquids.[7)] As emphasized by Bahcall and Wolf,[1)] nucleon-nucleon interactions is necessary for the neutrino emissions which take place in the presence of the degenerate nucleon and electron liquids. Thus, the possible processes of neutrino emissions in neutron stars are, for example,

$$n+n\rightarrow n+p+e^-+\bar{\nu}_e$$

and its inverse

$$n+p+e^-\rightarrow n+n+\nu_e\,.$$

Since the strong interaction between nucleons plays an important role, we attempt to formulate the theory of the neutrino processes in such a form that the effect of the interaction can be taken into account systematically. This is desirable when one has to calculate the rate of energy loss for the cases where at relatively high density the inter-nucleon distance is not very large compared with the radius

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of the hard core of the interaction or where nucleons are in ordered states such as the superfluid or the solid state. The possibility of the neutron and/or the proton liquid being in the superfluid state in neutron stars has recently been investigated by a number of authors,[8)~13)] and among possible effects due to the superfluidity[14),15)] one can mention the drastic reduction of the neutrino energy loss caused by the appearance of the energy gap in the density of states of nucleons in the superfluid state, as pointed out by Wolf.[2),3)]

In § 2 we present the general formula for calculating the rate of energy loss due to the neutrino processes and in the succeeding sections we apply it to the following cases:

1) The neutron and proton liquids are both in the normal state,
2) the proton liquid is in the superfluid state (the BCS-type state), while the neutron in the normal and
3) the neutron liquid is in the superfluid state, while the proton in the normal.

The superfluid states of the neutron and the proton liquid are both assumed to be described by the BCS theory. In the calculation we keep only the lowest order terms in the effective interaction between nucleons involved in the neutrino processes. Our results confirm the earlier calculation by Bahcall and Wolf except for minor corrections. We also calculate the rate of energy loss for the case where neutron-rich nuclei coexist with free neutrons, using the Thomas-Fermi model.

§ 2. Formulation

The total hamiltonian of our system, consisting of neutrons, protons and electrons and of neutrino field, may be written in the form

$$H = H_0 + H_\nu + H_i , \tag{1}$$

where H_0 is the hamiltonian of n, p and e including the strong interaction, H_ν that of neutrino field and

$$H_i = W + \text{h.c.}$$

with

$$W = \int d\bar{x} K \psi_p{}^+ \psi_n \psi_e{}^+ \psi_\nu \tag{2}$$

is the weak interaction term. In the last expression K denotes symbolically the coefficient depending on spinor indices of the field operators. We suppose that our system is uniform and that the neutron, proton and electron liquids are in the state of thermal equilibrium at temperature T. We may do so since we are interested in the situation where the neutrino opacity is negligible and the rate of cooling is sufficiently small. The rate of energy loss α due to the neutrino process is given by $d\langle H_0\rangle/dt$, where $\langle H_0\rangle$ is the statistical average of the total energy of the n, p and e-liquids at temperature $T(=k_B\beta^{-1})$.*) It is easy to show that to the lowest order in the weak interaction

$$\alpha=-\int_{-\infty}^{t}dt'\langle[[H_0, W^+(t)], W(t')]\rangle-\int_{-\infty}^{t}dt'\langle[[H_0, W(t)], W^+(t')]\rangle, \tag{3}$$

where the interaction representation with respect to H_t is used and the vacuum expectation value is to be taken for the neutrino field. The first term in the expression (3) corresponds to the process in which an anti-neutrino is emitted, i.e., 1) $n\rightarrow p+e+\bar{\nu}$ and the second to the inverse process 2) $p+e\rightarrow n+\nu$. When the n, p and e liquids are virtually in equilibrium, the respective chemical potentials μ_n, μ_p and μ_e satisfy

$$\mu_n=\mu_p+\mu_e\,. \tag{4}$$

Since this condition is maintained as the cooling proceeds, the rate of energy loss due to the process 2) must be equal to that due to the process 1), so that we have only to calculate the rate α_1 due to the process 1) given by the first term of (3).

Now let us introduce the following correlation function:

$$K(t-t')=\langle T[H_\nu, W^+(t)], W(t')\rangle, \tag{5}$$

where T is the usual time ordering operator and for convenience we suppose that the neutrino field is also at the temperature T. One can show that the rate α_1 is given by the real part of the fourier inverse of this function:

$$\alpha_1=2\mathrm{Re}\lim_{\substack{\beta_\nu\to\infty\\ \omega\to 0}}K(\omega), \tag{6}$$

where $\beta_\nu\rightarrow\infty$ means that we let all factors $\beta\xi^{(\nu)}$ appearing in the distribution functions infinite, $\xi^{(\nu)}$ being energy of a neutrino. Substituting (2) into (5) we have, using the notation $1=(\bar{x}, t)$,

$$\begin{aligned}K(\omega)=&\int_{-\infty}^{\infty}d(t-t')e^{i\omega(t-t')}d\bar{x}d\bar{x}'K^+K\\&\times\langle T\psi_n^+(1)\psi_p(1)\psi_p^+(1')\psi_n(1')\rangle\langle T\psi_e(1)\psi_e^+(1')\rangle\\&\times\langle T[H_\nu, \psi_\nu^+(1)], \psi_\nu(1')\rangle,\end{aligned} \tag{7}$$

since electrons and neutrinos are considered to be free. In what follows we make a simplifying assumption that the strong interaction between nucleons is spin-independent and replace K by the constant of the weak interaction:[16]

$$|K|^2=g^2(C_V^2+3C_A^2)=1.10\times10^{-97}\ \mathrm{erg}^2\ \mathrm{cm}^6. \tag{8}$$

Correspondingly we suppose the proton spin to be the same as the neutron spin in (7). Hence we have

*) Hereafter we take the unit $k_B=\hbar=1$.

$$K(\omega)=|K|^2\sum_{\bar{Q}-\bar{q}}\int\int\frac{dQ_0dq_0}{(2\pi)^2}\xi^{(\nu)}_{\bar{Q}-\bar{q}}G^{(e)}(q)G^{(\nu)}(Q-q)\Pi(\omega-Q) \tag{9}$$

in terms of the Green's function of electrons and of antineutrinos and of the neutron-proton propagator

$$\Pi(Q)=\int_{-\infty}^{\infty}dte^{iQ_0t}d\bar{x}e^{-i\bar{Q}\bar{x}}\langle T\psi_n^+(1)\psi_n(0)\psi_p(1)\psi_p^+(0)\rangle. \tag{10}$$

We have used the abbreviations such as $q=(\bar{\boldsymbol{q}},q_0)$ and $\omega=(0,\omega)$. Expression (9) is represented by the diagram shown in Fig. 1. Using the technique of thermal Green's function it is straightforward to obtain the following expression:

$$K(\omega)=-i|K|^2\int\int\frac{d\bar{\boldsymbol{Q}}d\bar{\boldsymbol{q}}}{(2\pi)^6}\xi^{(\nu)}_{\bar{Q}-\bar{q}}\frac{dQ_0}{2\pi}\frac{1-n(\xi_{\bar{q}}^{(e)})}{Q_0-\xi_{\bar{q}}^{(e)}-\xi^{(\nu)}_{\bar{Q}-\bar{q}}}\Pi(\omega-Q), \tag{11}$$

where

$$n(x)=[\exp(\beta x)+1]^{-1},\qquad \xi_{\bar{k}}^{(\nu)}=|\boldsymbol{k}|c$$

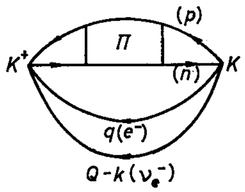

Fig. 1.

and $\xi_k^{(e)}$ the energy of electrons measured from the fermi energy. Thus the problem is reduced to finding Π in the presence of the strong interaction.

As noted by Bahcall and Wolf, the fact that the fermi momentum of protons is smaller roughly by a factor of $1/(20)^{1/3}$ than that of neutrons restricts the possible processes of neutrino emission greatly when k_BT is much smaller than any of the chemical potentials. Because of the Pauli principle and the energy-momentum conservation, in order for a neutron in the neighbourhood of the fermi surface to decay it is necessary that a part of its momentum is transferred to the surrounding medium through the strong interaction. Clearly the most important process is the creation of a particle-hole pair (or a quasi-particle pair in the superconducting state) of neutrons and protons by the proton produced in the decay. Bearing this in mind we proceed to calculate Π.

In terms of the complete Green's functions of neutrons and protons we can write Π in the form

$$\Pi(Q)=\int dkG_n(k)G_p(k-Q)\{1+\Lambda(k,Q)\}, \tag{12}$$

where Λ is the vertex correction given by the sum of various diagrams such as shown in Fig. 2. For simplicity let us first proceed neglecting Λ; in other words

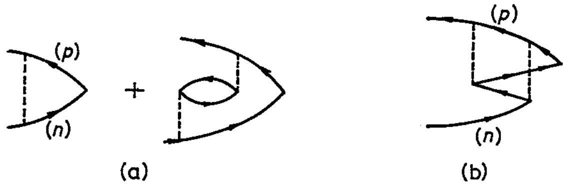

Fig. 2.

we neglect the interaction between the proton and the neutron-hole created in the decay. Using the spectral representation for the Green's functions,

$$G_{n(p)}(\bar{k}, k_0) = \int_{-\infty}^{\infty} \frac{\rho_{n(p)}(\bar{k}, x)}{x-k_0} dx \tag{13}$$

in the expression (11), converting to thermal Green's functions, performing the sum over the frequencies and then making the appropriate analytic continuation with respect to the frequency ω, we obtain

$$K(\omega) = -i \int \frac{d\bar{k}d\bar{Q}d\bar{q}}{(2\pi)^9} |K|^2 \xi_{\bar{Q}-\bar{q}}^{(\nu)}$$
$$\times [1-n(\xi_q^{(e)})] \int dxdy \rho_n(\bar{k}, x) \rho_p(\bar{k}-\bar{Q}, y) n(x)$$
$$\times [1-n(y)] \frac{1}{\omega - \xi_q^{(e)} - \xi_{\bar{Q}-\bar{q}}^{(\nu)} - y + x + i\delta}. \tag{14}$$

The spectral density is related to the imaginary part of the retarded Green's function by

$$\rho(\bar{p}, y) = \pi^{-1} \operatorname{Im} G(\bar{p}, y)$$
$$= \pi^{-1} \frac{\operatorname{Im} \Sigma(\bar{p}, y)}{\{y - \xi_p - \operatorname{Re} \Sigma(\bar{p}, y)\}^2 + \{\operatorname{Im} \Sigma(\bar{p}, y)\}^2}, \tag{15}$$

where $\Sigma(\bar{p}, y)$ is the self-energy part. Since the fermi momentum of neutrons is considerably larger than that of protons, $\rho_n(\bar{k}, x)$ for $|\bar{k}-\bar{Q}| \sim k_f^{(p)}$ and $x \sim 0$ is negligibly small whereas $\rho_p(\bar{k}-\bar{Q}, y)$ for $y \sim 0$ may be large because the state with $\bar{k}-\bar{Q}$ is far above the proton fermi surface when $|\bar{k}| \sim k_f^{(n)}$. In the next section we shall calculate $K(\omega)$ for the simplest case that both neutrons and protons are in the normal state.

§ 3. Normal nucleon liquids

Since the magnitude of momentum which has to be transferred to the surrounding medium at the decay is of the order of the proton fermi momentum, the dominant role is played by the hard-core part of the strong interaction. At the density comparable to the nuclear density the so-called gas parameter is of the order of 0.4, so that the expansion in powers of this parameter is valid for the present problem. Therefore, we assume in the following calculation that the interaction between nucleons is given by the s-wave part of the hard-core interaction, i.e., we put

$$v_{np}(\bar{q}) = v_{pp}(\bar{q}) = v_{nn}(\bar{q}) = 4\pi a/m\,,$$

where a is the radius of the hard core and m the mass of a nucleon.

Since only those neutrons in the region of width $k_BT(\ll\mu_n)$ around its fermi surface can decay, we approximate the neutron spectral density ρ_n in the normal state by

$$\rho_n(\boldsymbol{k}, x) = -\delta(x-\xi_k^{(n)}), \tag{16}$$

where

$$\xi_k^{(n)} = k^2/2m_n^* - \mu_n\,, \tag{17}$$

where m_n^* is the effective mass of a neutron close to the fermi surface. We shall use the similar expression for the proton energy. In Eq. (14) the proton momentum $\bar{\boldsymbol{k}}-\bar{\boldsymbol{Q}}$ is considerably larger than $k_f^{(p)}$ and y must be of the order of k_BT. Therefore the proton spectral density is approximately given by

$$\rho(\bar{\boldsymbol{k}}-\bar{\boldsymbol{Q}}, y) = \frac{1}{\pi}\mathrm{Im}\,\Sigma(\bar{\boldsymbol{k}}-\bar{\boldsymbol{Q}}, y)/\xi^{(p)2}_{\bar{k}-\bar{Q}}. \tag{18}$$

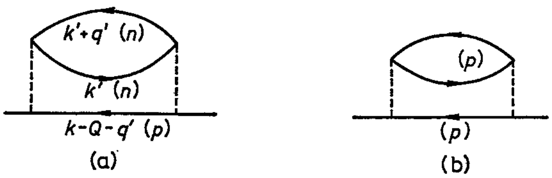

Fig. 3.

The lowest-order diagrams which contribute to the imaginary part of Σ are shown in Fig. 3. For simplicity we first suppose that the proton density is negligible ($k_f^{(p)}\ll k_f^{(n)}$) and consider the contribution from the diagram 3(a). The imaginary part of Σ due to this diagram may easily be calculated with the help of thermal Green's function. The result is

$$\begin{aligned}\mathrm{Im}\,\Sigma^{(3-a)}(\bar{\boldsymbol{k}}-\bar{\boldsymbol{Q}}, y) = &-\frac{\pi}{2(2\pi)^6}\left(\frac{4\pi a}{m}\right)^2\int d\bar{\boldsymbol{k}}'d\bar{\boldsymbol{q}}' \\ &\times\left(\tanh\frac{\xi^{(n)}_{\bar{k}'+\bar{q}'}}{2T}-\tanh\frac{\xi^{(n)}_{\bar{k}'}}{2T}\right)\left(\coth\frac{\xi^{(n)}_{\bar{k}'+\bar{q}'}-\xi^{(n)}_{\bar{k}'}}{2T}+\tanh\frac{\xi^{(p)}_{\bar{k}-\bar{Q}-\bar{q}'}}{2T}\right) \\ &\times\delta(y-\xi^{(p)}_{\bar{k}-\bar{Q}-\bar{q}'}-\omega_{k'q'}),\end{aligned} \tag{19}$$

where

$$\omega_{k'q'} = \xi^{(n)}_{\bar{k}'+\bar{q}'} - \xi^{(n)}_{\bar{k}'}.$$

Substituting this into (14), we get

$$\mathrm{Re}\, K_1^{(3-\mathrm{a})} = -\frac{(4\pi a/m)^2}{(2\pi)^{14}} \int d\bar{k} d\bar{Q} d\bar{q} |K|^2 \xi_{\bar{Q}-\bar{q}}^{(\nu)} n(\xi_{\bar{k}}^{(n)})$$
$$\times [1-n(\xi_{\bar{q}}^{(e)})] \int d\bar{k}' d\bar{q}' [1-n(\xi_{\bar{k}-\bar{Q}-q'}^{(p)})] n(\xi_{\bar{k}'}^{(n)}) [1-n(\xi_{\bar{k}'+\bar{q}'}^{(n)})]$$
$$\times (\xi_{\bar{k}}^{(n)} - \xi_{\bar{q}}^{(e)} - \xi_{\bar{Q}-\bar{q}}^{(\nu)} - \xi_{\bar{k}-\bar{Q}}^{(p)})^{-2} \delta(\xi_{\bar{k}}^{(n)} - \xi_{\bar{q}}^{(e)} - \xi_{\bar{Q}-\bar{q}}^{(\nu)} - \xi_{\bar{k}-\bar{Q}-q'}^{(p)} - \omega_{k'q}^{(n)}). \tag{20}$$

We turn now to the discussion of the vertex correction Λ. The diagrams of the type shown in Fig. 2(a) do not contribute to the real part of K because of the energy-momentum conservation, although they lead to the correction of the coupling constant of the weak interaction, which we may neglect if the gas parameter is small. In the second order in the interaction, however, there is a diagram 2(b) contributing to Re K, which we can call the exchange type as compared with the process given by 3(a). When one takes into account the fact that the energies $\xi_{\bar{k}-\bar{Q}}^{(p)}$ and $\xi_{\bar{k}'-\bar{Q}}^{(p)}$ are much larger than any other energy appearing in the process, one finds the contribution to Re K from this diagram to be just $-1/2$ of (20).

When evaluating the integrals in (20) we again take advantage of the fact that the fermi momentum of protons is much smaller than that of neutrons, and replace the factor $(\xi_{\bar{k}}^{(n)} - \xi_{\bar{q}}^{(e)} - \xi_{\bar{Q}-\bar{q}}^{(\nu)} - \xi_{\bar{k}-\bar{Q}}^{(p)})$ by the average and take out of the integral. We transform the variables of integrations from k and Q to $k-Q-q$ and $Q-k$, and then change the integrals over the momenta to the integrals over the energies, remembering that the most important region is near the respective fermi surfaces. The final result for the energy loss α_1 due to the processes under consideration (Figs. 3(a) and 2(b)) is

$$\alpha_1 = -\frac{2^{13}\pi^5 g^2 (C_V^2 + 3C_A^2) a^2}{(2\pi)^{12} c^4} \frac{(m_n^*)^3 (m_p^*)^3}{m^2} \frac{(k_f^{(p)})^3}{(k_f^{(n)})^4} \times (k_B T)^8 I_{\mathrm{normal}}, \tag{21}$$

where

$$I_{\mathrm{normal}} = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_3 \int_0^{\infty} dx_4 \frac{x_1 e^{x_1} e^{x_2} e^{x_3}}{(e^{x_1}-1)(e^{x_2}+1)(e^{x_3}+1)} \times \frac{x_4^3}{e^{x_1+x_2+x_3+x_4}+1}. \tag{22}$$

This integral is the same as that given by Bahcall and Wolf and can be calculated by the method of contour integration:

$$I_{\mathrm{normal}} = \frac{11513}{120960} \pi^8 \cong 903. \tag{23}$$

The total rate is given by $\alpha = 2\alpha_1$. Our result is in agreement with that obtained by Bahcall and Wolf, although the agreement seems to be fortuitous in view of the facts that they used a different method to evaluate the effect of the strong interaction and that they did not consider the exchange process.

Let us now discuss the contribution from the proton-proton interaction represented by the diagram 3(b) and the corresponding exchange diagram. From the energy-momentum conservation this contribution vanishes when $k_f^{(p)} < \frac{1}{4}k_f^{(n)}$. As $k_f^{(p)}$ increases from $\frac{1}{4}k_f^{(n)}$ to $\frac{1}{2}k_f^{(n)}$, however, it becomes comparable to the contribution from the proton-neutron interaction discussed above.

§ 4. Proton superfluidity

In this section we shall consider the case where the proton liquid is in the BCS-type superconducting state, while the neutron liquid is in the normal state. Since the concentration of protons in neutron stars is of the order of a few percent,[1),2)] the ${}^1S_0$ attractive interaction is expected between protons even at high densities at which the interaction between neutrons is repulsive.[6)]

The calculation of the present case is quite similar to the one in the preceding section. One must note that the proton state with momentum $\bar{\boldsymbol{k}}-\bar{\boldsymbol{Q}}$ is far above the proton fermi surface so that $\xi_{k-Q}^{(p)}$ is much larger than the energy gap 2Δ of the proton superfluid. Consequently we can use the same equation for the spectral density given by (19). In the diagrams 3(a) and 2(b) we have to substitute the Green's function of the superfluid state only for the proton with momentum $\bar{\boldsymbol{k}}-\bar{\boldsymbol{Q}}-\bar{\boldsymbol{q}}'$. Since the calculation is quite familiar in the theory of superconductivity of metals, we only write the final result for the present case corresponding to the result (20) of the preceding section:

$$\begin{aligned}\mathrm{Re}\,K_1^{(3(a))} = &-\frac{1}{(2\pi)^{14}}\int d\bar{\boldsymbol{k}}d\bar{\boldsymbol{Q}}d\bar{\boldsymbol{q}}|K|^2\xi_{Q-q}^{(\nu)}n(\xi_k^{(n)})\\ &\times\{1-n(\xi_q^{e})\}\int d\bar{\boldsymbol{k}}'d\bar{\boldsymbol{q}}'\left(\frac{4\pi a}{m}\right)^2[u_{k-Q-q'}^2\{1-n(E_{k-Q-q'})\}\\ &\times\delta(\xi_k^{(n)}-\xi_q^{(e)}-\xi_{Q-q}^{(\nu)}-E_{k-Q-q'}-\omega_{k'q'})+v_{k-Q-q'}^2 n(E_{k-Q-q'})\\ &\times\delta(\xi_k^{(n)}-\xi_q^{(e)}-\xi_{Q-q}^{(\nu)}+E_{k-Q-q'}-\omega_{k'q'})]\\ &\times n(\xi_{k'}^{(n)})\{1-n(\xi_{k'+q'}^{(n)})\}\frac{1}{(\xi_k^{(n)}-\xi_q^{(e)}-\xi_{Q-q}^{(\nu)}-\xi_{k-Q}^{(p)})^2},\end{aligned}\tag{24}$$

where

$$\frac{u_p^2}{v_p^2}=\frac{1}{2}\left(1\pm\frac{\xi_p^{(p)}}{E_p}\right),\tag{25}$$

$$E_{\bar{p}}=\sqrt{\xi_{\bar{p}}^{(p)2}+\Delta^2}\tag{26}$$

and 2Δ is the energy gap of the proton superfluid. After making similar approximations as before we can evaluate the integrals analytically when $k_BT \ll \Delta$. The result for the rate α_1 corresponding to (21) of the normal case is given by replacing I_{normal} in (21) by

$$I_{p-\text{super}} = \frac{\sqrt{2\pi}}{1680}(\beta\Delta)^{15/2}e^{-\beta\Delta}. \tag{27}$$

Wolf gave the reduction factor Y of the rate of energy loss due to the proton superfluidity as a function of $\beta\Delta$.[12] It seems from his figure of $Y(\beta\Delta)$ that our result agrees with his within a factor of three for $\beta\Delta \cong 20$. We remark that in the present case the proton-proton interaction is not important because of the proton energy gap.

§ 5. Neutron superfluidity

According to the analysis of Baym, Bethe and Pethick,[7] when the density of nucleon matter is below the nuclear density, protons exist in the form of neutron-rich nuclei, which form a lattice by the Coulomb interaction. If this is the case, the uniform nucleon matter exists only at relatively high densities at which the $^3P_2$ part instead of the $^1S_0$ of the interaction between neutrons is expected to be most attractive. Therefore, it is more likely that the neutron superfluid is of the type of $^3P_2$ pairing, instead of the usual $^1S_0$ pairing.[8)~10)] The ground state of the $^3P_2$ superfluid has, according to recent studies, the energy gap which is only weakly anisotropic. Therefore, we would expect that the effect of the $^3P_2$ superfluidity on the neutrino energy loss is not significantly different from that of the $^1S_0$ superfluidity. For this reason we assume in the present calculation that the neutron fluid is in the $^1S_0$ superfluid state and use the BCS theory as in the preceding section. For simplicity we further assume that the proton liquid is in the normal state.

The spectral density of neutrons is now given by

$$\rho(\bar{k}, x) = -\{u_k^2\delta(x - E_k) + v_k^2\delta(x + E_k)\}, \tag{28}$$

where the notation is the same as for (25) and (26), all quantities now referring to neutrons. In the present case the proton self-energy corresponding to (19) is given by

$$\begin{aligned}
\operatorname{Im}\Sigma(\bar{p}, y) = &\frac{\pi}{2(2\pi)^6}\left(\frac{4\pi a}{m}\right)^2 \int d\bar{k}' d\bar{q}' \Bigg[\left(\tanh\frac{E_{k'}}{2T} - \tanh\frac{E_{k'+q'}}{2T}\right) \\
&\times \Bigg\{\left(u_{k'+q'}^2 u_{k'}^2 - \frac{\Delta^2}{4E_{k'}E_{k'+q'}}\right)\left(\coth\frac{E_{k'+q'} - E_{k'}}{2T} + \tanh\frac{\xi_{p-q'}^{(p)}}{2T}\right) \\
&\times \delta(y - \xi_{p-q'}^{(p)} - E_{k'+q'} + E_{k'}) \\
&- \left(v_{k'}^2 v_{k'+q'}^2 - \frac{\Delta^2}{4E_{k'}E_{k'+q'}}\right)\left(-\coth\frac{E_{k'+q'} - E_{k'}}{2T} + \tanh\frac{\xi_{p-q'}^{(p)}}{2T}\right) \\
&\times \delta(y - \xi_{p-q'}^{(p)} + E_{k'+q'} - E_{k'}) + \left(\tanh\frac{E_{k'}}{2T} + \tanh\frac{E_{k'+q'}}{2T}\right)
\end{aligned}$$

$$\times\left\{\left(u_{k'}^2v_{k'+q'}^2+\frac{\Delta^2}{4E_{k'}E_{k'+q'}}\right)\left(-\coth\frac{E_{k'+q'}+E_{k'}}{2T}+\tanh\frac{\xi_{p-q'}^{(p)}}{2T}\right)\right.$$
$$\times\delta(y-\xi_{p-q'}^{(p)}+E_{k'+q'}+E_{k'})$$
$$-\left(u_{k'+q'}^2v_{k'}^2+\frac{\Delta^2}{4E_{k'}E_{k'+q'}}\right)\left(\coth\frac{E_{k'+q'}+E_{k'}}{2T}+\tanh\frac{\xi_{p-q'}^{(p)}}{2T}\right)$$
$$\left.\times\delta(y-\xi_{p-q'}^{(p)}-E_{k'+q'}-E_{k'})\right\}\Bigg]. \tag{29}$$

Substituting (28) and (29) into (14) and (15) we can calculate Im K_1. In the calculation we replace the coherence factors $u_{k'+q'}^2$ and $v_{k'+q'}^2$ by 1/2 and further neglect $\xi_{p-q'}^{(p)}$, $\xi_q^{(e)}$ and $\xi_{Q-q}^{(\nu)}$ when they appear with E_k and $E_{k'}$ in the coherence factors. This approximation is valid when $k_BT\ll\Delta$. Then we can calculate Im K_1 analytically. Since the resulting expression is rather lengthy, we only mention the final result for α_1 which is given by the formula (21) with I_{normal} replaced by

$$I_{n-\text{super}}=\frac{\pi}{15}(\beta\Delta)^7e^{-2\beta\Delta}. \tag{30}$$

The result obtained by Wolf[8] is

$$I_{n-\text{super}}\cong 0.123(\beta\Delta)^7e^{-2\beta\Delta}, \tag{31}$$

which agrees with ours within a factor of 2.

It should be remarked here that the contribution from the proton-proton interaction given by the diagrams 2(b) with a proton bubble and 3(b) is particularly important in the present case if $k_f^{(p)}>\frac{1}{4}k_f^{(n)}$. When we have this contribution, the factor $\exp(-2\beta\Delta)$ in (30) is replaced by $\exp(-\beta\Delta)$ among other changes.

§ 6. Neutrino loss in nuclei

So far we have considered the uniform nucleon liquids. At lower densities, however, neutron-rich nuclei and a degenerate neutron liquid coexist and protons are present only in the nuclei. Such a state of nucleon matter has recently been studied in detail by Baym, Bethe and Pethick (hereafter referred to as BBP).[7] In this section we shall briefly discuss the rate of neutrino energy loss in this mixed phase of neutron-rich nuclei and a neutron liquid, which seems to prevail in a sizable volume of a typical neutron star.

Let us use the approximate picture of the Thomas-Fermi model. If a neutron outside the nuclei decays, the produced proton will have an energy much higher than the chemical potential of protons μ_p. Hence, the decay of a neutron outside the nuclei is highly forbidden. Consequently, we have only to consider the neutrino processes inside the neutron-rich nuclei. According to BBP, the mass number of the nuclei A is larger than 200 when the total mass density ρ is higher than $10^{13}g\,\text{cm}^{-3}$. Therefore, we may, as the first approximation, treat the neutron-rich nuclei as a nucleon matter of a finite size R.

We remind ourselves that when $\Delta k_f \equiv k_f^n - (k_f^p + k_f^e) < 0$, the ordinary neutrino process $n \rightarrow p + e^- + \bar{\nu}_e$ and its inverse can occur without involving the strong interaction. We note that in the present case $k_f^e < k_f^p$ because electrons are not confined in nuclei. According to BBP, Δk_f varies from 0.20 to 0.31 fm^{-1} as the density increases from 4.66×10^{11} to $1.30 \times 10^{14} g\ \text{cm}^{-3}$, so that the condition $\Delta k_f < 0$ does not seem to hold. However, in the mixed state nucleons are under the influence of the self-consistent non-uniform field responsible for the formation of nuclei. Consequently, the momenta of neutrons and protons now have a width of the order of $2\pi/R$ where R is the size of the nuclei. For the same density range as above the value of $2\pi/R$ varies from 1.05 to 0.52 fm^{-1} and is always larger than Δk_f. Therefore, we would expect that the direct processes are not prohibited by the momentum conservation and the Pauli principle. If this is the case, they may be more important than the processes discussed in § 3, since the ratio of the rate due to the direct processes to the one given by (21) is proportional to $(k_B T/\mu_n)^{-2}$. For a quantitative evaluation one can regard these processes in the solid region as umklapp processes and make a similar calculation as in the preceding sections.

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