Neutrino Radiation in Gravitational Fields

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Abstract. A differential equation representing radiation solutions of the general relativistic Weyl equation is derived. Their optical properties and the group of motion of the corresponding energy-momentum tensor are studied. If there exists neutrino radiation the Riemann space must be algebraically special and the propagation of the neutrinos occurs only along one of the principal null directions. Gravitational- and neutrino *pp*-waves taken together, represent an exact solution of the Weyl-Einstein system of field equations.

§ 1. Introduction

The physics of neutrinos is determined by two interactions: the weak interaction dominating within elementary particle physics, and the long-range gravitational interaction. The latter is most important for some stages of stellar evolution (e.g. [1]), at the beginning of our Universe (e.g. [2]), and in connection with a cosmic neutrino background radiation (e.g. [3]).

In the framework of General Relativity, neutrinos are commonly treated phenomenologically as point particles moving with the speed of light and show in this simplified model, the same properties as photons. On the other hand, a rigorous quantum-mechanical treatment in curved spacetime must start from the generally covariant Weyl theory of the neutrino (§ 2). In the following, the characteristic properties of a special case of Weyl's theory in a 4-dimensional Riemann space shall be studied with *rigorous* methods in analogy to the electromagnetic case. Several deviations from the phenomenological description are found in this way.

We restrict ourselves to *neutrino radiation fields* defined in analogy to electrodynamics and we derive the differential equation for such solutions of Weyl's equation in § 3. By means of this equation, we determine the optical properties of the null-congruence given by the 4-vector j^α of the probability-density and the group of motion of the energy-momentum tensor. After this, it is possible to classify the neutrino radiation solutions by means of some supplementary conditions in § 4. We show in § 5 the algebraic conditions of the metric field, which must be satisfied in order that the corresponding solution can exist. Especially, there

results an interesting connection between the principal null directions of the Weyl curvature tensor and the j^α-field. Finally, an exact solution of the coupled Weyl-Einstein system is given in § 6. It represents a neutrino pp-wave accompannied by a gravitational pp-wave.

§ 2. Generally Covariant Weyl Theory

Confining ourselves to the gravitational interaction, the behaviour of neutrinos is described in the framework of a metrical theory of gravitation by the generally covariant Weyl equation (e.g. [4, 5])

$$\sigma^{\alpha A \dot{B}} \varphi_{A||\alpha} = 0 , \tag{2.1a}$$

where the generalized Pauli matrices $\sigma^{\alpha A \dot{B}}$ are implicitly defined by [1]

$$2\sigma_{(\alpha}{}^{\dot{A}B} \sigma_{\beta)\dot{A}C} := g_{\alpha\beta} \delta^B_C . \tag{2.2}$$

The covariant derivative of a 2-spinor is given by [2]

$$\varphi_{A||\mu} := \varphi_{A|\mu} - \Gamma_\mu{}^B{}_A \cdot \varphi_B , \tag{2.3}$$

with the spinor affinities

$$\Gamma_\mu{}^B{}_A = \tfrac{1}{2} \sigma_\varkappa{}^{B\dot{E}} (\sigma^\varkappa{}_{A\dot{E}|\mu} + \sigma^\lambda{}_{A\dot{E}} \Gamma^\varkappa{}_{\mu\lambda}) . \tag{2.4}$$

In the following, we use the 2-spinor calculus in Riemann space. For example, to every tensor $W_\varrho^{\alpha\beta\cdots}$ we associate an equivalent spinor by

$$W_{R\dot{X}\cdots}^{A\dot{F}B\dot{G}\cdots} := \sigma^\varrho{}_{R\dot{X}} \cdots \sigma_\alpha{}^{A\dot{F}} \sigma_\beta{}^{B\dot{G}} \cdots W_\varrho^{\alpha\beta\cdots} . \tag{2.5a}$$

This may be inverted, to give

$$W_\varrho^{\alpha\beta\cdots} = W_{R\dot{X}\cdots}^{A\dot{F}B\dot{G}\cdots} \sigma^\alpha{}_{A\dot{F}} \sigma^\beta{}_{B\dot{G}} \cdots \sigma_\varrho{}^{R\dot{X}} \cdots . \tag{2.5b}$$

Instead of (2.5) we write

$$W_{R\dot{X}}^{A\dot{F}B\dot{G}} \leftrightarrow W_\varrho^{\alpha\beta} , \tag{2.5c}$$

and call $W_{R\dot{X}}^{A\dot{F}B\dot{G}}$ the spinor equivalent of $W_\varrho^{\alpha\beta}$. Now the Weyl equation becomes

$$\varphi^A{}_{||A\dot{C}} = 0 . \tag{2.1b}$$

For more details about the 2-spinor calculus we refer the reader to Pirani's article [6] (see also Penrose [7]).

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[1] $M_{(\alpha\beta)} := \tfrac{1}{2}(M_{\alpha\beta} + M_{\beta\alpha})$, $M_{[\alpha\beta]} := \tfrac{1}{2}(M_{\alpha\beta} - M_{\beta\alpha})$ (same rule for spinor indices). Signature of $g_{\alpha\beta}$: $(--- +)$.

[2] $:= \cdot$ is the sign of definition; partial and covariant derivatives are denoted $_{|\alpha}$ and $_{||\alpha}$ respectively (same rule for spinor indices).

For each Weyl field φ_A the corresponding null 4-vector j^α of the probability density is

$$j_\alpha \leftrightarrow \varphi_A \varphi_{\dot{B}}\,, \tag{2.6a}$$

with

$$j^\alpha{}_{||\alpha} = 0\,, \qquad j^\alpha j_\alpha = 0\,; \tag{2.6b, c}$$

the energy-momentum tensor is (disregarding an irrelevant factor)

$$T_{\alpha\beta} \leftrightarrow i\left[\varphi_{(A||B)(\dot{C}}\varphi_{\dot{D})} - \varphi_{(\dot{C}||\dot{D})(A}\varphi_{B)}\right]\,, \tag{2.7a}$$

with

$$T^{\alpha\beta}{}_{||\beta} = 0\,, \qquad T^{[\alpha\beta]} = 0\,, \qquad T^\varepsilon{}_\varepsilon = 0\,, \qquad T_{\alpha\beta}j^\alpha j^\beta = 0\,. \tag{2.7b–e}$$

We treat the neutrino field φ_A in analogy to the Maxwell field, as an unquantized (classical) field in Riemann space[3]. The probability interpretation is implied by the structure of j^α and $T^{\alpha\beta}$, which are quantum mechanical expectation values.

§ 3. Neutrino Radiation Fields

Just as in electrodynamics, we define as *neutrino radiation field* any solution of Weyl's equation (2.1), whose energy flows for *all* observers pointwise in the same direction with the velocity of light. That is[4],

$$p^\alpha := T^{\alpha\beta}o_\beta \sim A^\alpha\,, \qquad A^\alpha A_\alpha = 0 \tag{3.1a, b}$$

for all o^α with $o^\alpha o_\alpha = 1$. This implies

$$T^{\alpha\beta} \sim A^\alpha A^\beta \leftrightarrow \alpha^A \alpha^B \alpha^{\dot{C}} \alpha^{\dot{D}}\,. \tag{3.2}$$

Because of (2.7a), α^A must be proportional to φ^A, and as a consequence A^ε must be proportional to j^ε. Physically this means that the 4-momentum p^α of the neutrino radiation is always collinear with the probability density j^α. Therefore, with (3.2) we have

$$T^{\alpha\beta} = b(x)j^\alpha j^\beta \leftrightarrow b(x)\varphi^A \varphi^B \varphi^{\dot{C}} \varphi^{\dot{D}}\,. \tag{3.3}$$

For each observer the real function $b(x)$ can be interpreted as the quotient of the energy density and the square of the probability density.

Equating (3.3) with (2.7a), we obtain the following implicit system of differential equations for φ_A:

$$i\left[\varphi_{(A||B)(\dot{C}}\varphi_{\dot{D})} - \varphi_{(\dot{C}||\dot{D})(A}\varphi_{B)}\right] = b(x)\varphi_A \varphi_B \varphi_{\dot{C}} \varphi_{\dot{D}}\,. \tag{3.4}$$

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[3] In regard to the topology of the Riemann space, we suppose the existence of a spinor structure [8]; this seems to be the case for all physically interesting metrics [9].

[4] $\sim$ is the sign of proportionality.

We solve (3.4) for $\varphi_{A||B\dot{C}}$ using (2.1) and obtain the general solution (see the Appendix)

$$\varphi_{A||\mu}\leftrightarrow\varphi_{A||B\dot{C}} = i\,b\,\varphi_A\varphi_B\varphi_{\dot{C}} + \varphi_{(A}K_{B)\dot{C}}\,, \qquad (3.5)$$

with the four additional *real* functions $K_\varepsilon(x)$:

$$K_\varepsilon \leftrightarrow K_{A\dot{B}}\,. \qquad (3.6)$$

Evidently every φ_A satisfying (3.5) is also a solution of Weyl's equation (2.1). Therefore, Eq. (3.5) is the *differential equation* for neutrino radiation fields.

In every special Riemann space the existence of such radiation solutions must be proven each time by means of the integrability conditions corresponding to (3.5) (see § 6). Apart from this, we consider in the next two paragraphs necessary properties of such solutions.

§ 4. Neutrino Ray Optics and Groups of Motion of the Energy-momentum Tensor

We analyse the kinematics of neutrino radiation by means of the 4-current j^α of the probability density. One obtains with (3.5)

$$j^\alpha_{\ ||\varepsilon}j^\varepsilon = c_1 j^\alpha\,, \qquad c_1 := K_\varepsilon j^\varepsilon\,, \qquad (4.1\,\mathrm{a, b})$$

and consequently j^α is a tangent vector to a null *geodesic* congruence with *non-affine* parametrization. The geometry, i.e. the differential optical mapping properties, of neutrino rays in curved space-time is described by the three optical parameters twist ω, shear (or distortion) $|\sigma|$ and expansion Θ (Sachs [10]). In a geodesic and *affinely* parametrized null congruence, they are determined by the corresponding tangent vector l'^α as follows [5]:

$$\Theta = \tfrac{1}{2}l'^\alpha_{\ ||\alpha} \qquad\qquad = -\mathrm{Re}\,\varrho\,, \qquad (4.2)$$

$$2\omega^2 = l'_{[\alpha||\beta]}l'^{\alpha||\beta} \qquad = 2(\mathrm{Im}\,\varrho)^2\,, \qquad (4.3)$$

$$2|\sigma|^2 = l'_{(\alpha||\beta)}l'^{\alpha||\beta} - 2\Theta^2 = 2\sigma\bar{\sigma}\,; \qquad (4.4)$$

where

$$\varrho := (\lambda'_{A||X\dot{Y}}\lambda'^A)\mu'^X\lambda'^{\dot{Y}}\,, \qquad (4.5)$$

$$\sigma := (\lambda'_{A||X\dot{Y}}\lambda'^A)\lambda'^X\mu'^{\dot{Y}}\,. \qquad (4.6)$$

establish the connection between the tensor and the spinor formulation with

$$l'_\alpha \leftrightarrow \lambda'_A \lambda'_{\dot{B}}\,, \qquad \lambda'_A\mu'^A = 1\,. \qquad (4.7\,\mathrm{a, b})$$

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[5] Construction with respect to an affine parameter is denoted by a dash. $\bar{\sigma}$ is the complex conjugate of σ.

In order to evaluate (4.2), (4.3) and (4.4) in the case of neutrino radiation fields, we change from j^α to a tangent vector j'^α (which corresponds to l'^α) by introducing an affine parameter r':

$$j'^\alpha := \frac{dx^\alpha}{dr'} = \alpha(r) j^\alpha, \quad j^\alpha := \frac{dx^\alpha}{dr}, \quad \alpha = e^{-\int c_1 dr}. \qquad \text{(4.8 a–c)}$$

The spinor equivalent of j'^α is given by:

$$\varphi'_A = a\varphi_A, \quad \text{Re}\, a = \sqrt{\alpha}, \quad a \neq 0. \qquad \text{(4.9 a–c)}$$

Introducing μ_A and μ'_A, which satisfy

$$\varphi_A \mu^A = 1, \quad \varphi'_A \mu'^A = 1, \qquad \text{(4.10 a, b)}$$

we establish two basis in spinor space with

$$\mu'_A = \frac{1}{a}\mu_A. \qquad \text{(4.11)}$$

It follows from Eq. (4.5) and (4.6), that

$$\varrho = a\bar{a}(\varphi_{A||X\dot{Y}}\varphi^A)\mu^X \varphi^{\dot{Y}}, \qquad \text{(4.12)}$$

$$\sigma = a^3 \bar{a}^{-1}(\varphi_{A||X\dot{Y}}\varphi^A)\varphi^X \mu^{\dot{Y}}. \qquad \text{(4.13)}$$

Substitution of (3.5) yields the optical scalars of neutrino radiation:

$$|\sigma| = 0, \qquad \text{(4.14)}$$

$$\omega = 0, \qquad \text{(4.15)}$$

$$\Theta = -\tfrac{1}{2} a\bar{a} K_\varepsilon j^\varepsilon. \qquad \text{(4.16)}$$

In every Riemann space all radiation solutions of the Weyl equation possess a shear-free and twist-free geodesic congruence[*].

In particular, the congruence is hypersurface-orthogonal because of (4.15):

$$j_\alpha = h(x) g(x)_{|\alpha}. \qquad \text{(4.17)}$$

(Existence of wave fronts $g = \text{const.}$) By comparison, the rays of a (source-free) electromagnetic radiation field are in general only geodesic and shear-free (Mariot [11], Robinson [12]).

Restricting the free functions $b(x)$ and $K_\varepsilon(x)$ of (3.5) by the supplementary condition

$$K_\varepsilon j^\varepsilon = 0, \qquad \text{(4.18)}$$

we obtain special radiation solutions of the Weyl equation, which are characterized by additional physical properties going beyond (4.1 a),

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* *Note added in proof:* With a tensor method the same result is obtained recently by Griffiths and Newing in J. Phys. A: Gen. Phys. **3**, 269 (1970).

Table *Classification of the radiation solutions φ_A of the Weyl equation (2.1) in order of increasing specialization according to supplementary conditions of their differential equation (3.5) and corresponding physical properties.* ⇔ *characterizes the one-to-one correspondence*

Type	φ_A is a solution of (3.5) with the following supplementary conditions		φ_A is a solution if (3.5) with the following properties
a	—	⇔	a) j^α-congruence: $\omega = |\sigma| = 0$ twist- and shear-free $j^\alpha{}_{\|\varepsilon}j^\varepsilon \sim j^\alpha$ geodesic hypersurface-orthogonal
b	$K_\varepsilon j^\varepsilon = 0$	⇔	*planefronted wave (p-wave)* a) j^α-congruence: $\Theta = 0$ expansion-free $j^\alpha{}_{\|\varepsilon}j^\varepsilon = 0$ affinely parametrized b) Group of motion of $T^{\alpha\beta}$: $\underset{j}{\pounds}\,T^{\alpha\beta} = \underset{j}{\pounds}\,T_{\alpha\beta} = 0$
c	$K_\varepsilon = dj_\varepsilon$ with $d_{|\varepsilon} \sim j_\varepsilon$	⇔	*planefronted wave with parallel rays (pp-wave)* a) j^α-congruence: There exists a constant tangent vector $\hat{j}^{1\alpha} = j^\alpha\, e^{-2\delta(x)}, \hat{j}_{\alpha\|\varepsilon} = 0,\ \bar{\delta} = \delta$ (i.e. j^α has the form $j^\alpha = \hat{j}^\alpha \cdot e^{2\delta(x)}$) where $\delta_{|\varepsilon} = dj_\varepsilon$
d	$K_\varepsilon = dj_\varepsilon$ with $d_{|\varepsilon} \sim j_\varepsilon$ and $b_{|\varepsilon} \sim j_\varepsilon$	⇔	a) Wave field φ_A: There exists a constant spinor $\hat{\varphi}_A$ proportional to φ_A: $\hat{\varphi}_A = \varphi_A \cdot e^{-\Phi(x)},\ \hat{\varphi}_{A\|\varepsilon} = 0,\ \Phi := \delta + i\beta$, $\delta = \bar{\delta},\ \beta = \bar{\beta}$ (i.e. φ_A has the form $\varphi_A = \hat{\varphi}_A \cdot e^{\Phi(x)}$) where $\Phi_{|\varepsilon} = (d + ib)j_\varepsilon$. b) Group of motion of $T^{\alpha\beta}$: (η^α: connecting vector of the j^α-congruence with $j_\alpha \eta^\alpha = 0$) $\underset{\eta}{\pounds}\,T^{\alpha\beta} = \underset{\eta}{\pounds}\,T_{\alpha\beta} = 0$

(4.14) and (4.15). According to (4.16) the expansion vanishes if and only if Eq. (4.18) is satisfied. Furthermore, as a consequence of (2.6b), (2.7b), (3.3) and (4.18), the Lie-derivative of the energy-momentum tensor vanishes in the direction of the density-current

$$\underset{j}{\pounds}\,T^{\alpha\beta} = \underset{j}{\pounds}\,T_{\alpha\beta} = 0 \,. \tag{4.19}$$

Because all optical parameters are equal to zero, these solutions are *planefronted waves (p-waves)*.

Further restrictions beyond (4.18) allow a *classification* of the various types of neutrino radiation solutions by means of supplementary conditions and uniquely corresponding properties. The result calculated in the 2-spinor formalism is listed in the *Table* in the order of increasing specialisation of the free functions. Especially for the type *d*, the form of the solution is known and there exists a constant null bivector field composed of φ_A (comp. § 5). Evidently the *plane wave* is of the type *d* with $\delta = 0$.

§ 5. Propagation Direction and Principal Null Directions of the Weyl Tensor

Applying (3.5), the Ricci identity for 2-spinors

$$\varphi_{A||[\mu\nu]} = \varphi_E R^E{}_{A\mu\nu} \tag{5.1}$$

yields necessary algebraic conditions for the metrical quantities of the Riemann space. They must be fullfilled in order that a radiation solution of the Weyl equation exists (integrability conditions). The spin-curvature is determined by the Riemann tensor

$$R^E{}_{A\mu\nu} = -\tfrac{1}{2} R^{\varkappa\lambda}{}_{\mu\nu} \sigma_{\varkappa A\dot{B}} \sigma_\lambda{}^{E\dot{B}} . \tag{5.2}$$

With the aid of the spinor equivalents of Weyl tensor and trace-free Ricci tensor [6]

$$C_{\alpha\beta\gamma\delta} \leftrightarrow \Psi_{ABCD} \varepsilon_{\dot{E}\dot{F}} \varepsilon_{\dot{G}\dot{H}} + \psi_{\dot{E}\dot{F}\dot{G}\dot{H}} \varepsilon_{AB} \varepsilon_{CD} , \tag{5.3a}$$

$$\Psi_{ABCD} = \Psi_{(ABCD)} , \tag{5.3b}$$

$$R_{\alpha\beta} - \tfrac{1}{4} R g_{\alpha\beta} \leftrightarrow -2 \Phi_{AB\dot{C}\dot{D}} , \tag{5.4a}$$

$$\Phi_{AB\dot{C}\dot{D}} = \Phi_{(AB)(\dot{C}\dot{D})} = \Phi_{ABCD} , \tag{5.4b}$$

we change to the equivalent spinor form of (5.1). We break this equation up into invariant parts according to the symmetry properties of its spinor indices:

$$\varphi_{D||}{}^H{}_{(\dot{X}||\dot{W})H} = \Phi_{DE\dot{W}\dot{X}} \varphi^E , \tag{5.5}$$

$$\varphi_{(C||}{}^{\dot{P}}{}_{B||A)\dot{P}} = \Psi_{ABCD} \varphi^D , \tag{5.6}$$

$$\varphi^B{}_{||\dot{P}(B||A)}{}^{\dot{P}} = \frac{R}{8} \varphi_A . \tag{5.7}$$

The vanishing divergence of the energy-momentum tensor yields, for radiation solutions

$$(b_{|X\dot{Y}} + b K_{X\dot{Y}}) \varphi^X \varphi^{\dot{Y}} = 0 , \tag{5.8}$$

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[6] $(A \dots E)$ denotes complete symmetry in all indices $A \dots E$.

whence we may write

$$(b_{|X\dot{Y}} + b\,K_{X\dot{Y}})\varphi^X = c_2\,\varphi_{\dot{Y}} \tag{5.9}$$

for some function $c_2(x)$. Now applying this equation and (3.5), the Ricci identities (5.5–5.7) for radiation solutions become

$$ic_2\,\varphi_{\dot{F}}\varphi_{\dot{Y}}\varphi_A - \tfrac{1}{4}K_{A(\dot{F}}K_{\dot{Y})E}\varphi^E$$
$$-\tfrac{1}{2}\varphi_A K_{E(\dot{Y}||\dot{F})}{}^E + \tfrac{1}{2}K_{A(\dot{Y}||\dot{F})E}\varphi^E = \Phi_{AD\dot{Y}\dot{F}}\varphi^D, \tag{5.10}$$

$$i\bar{c}_2\,\varphi_A\,\varphi_E\,\varphi_X + \varphi_{(A}K_X{}^{\dot{F}}{}_{||E)\dot{F}} = \Psi_{AEXD}\varphi^D, \tag{5.11}$$

$$2\varphi^A K_{\dot{Y}(E||A)}{}^{\dot{Y}} - 3\varphi_E K_{A\dot{Y}||}{}^{A\dot{Y}} - \tfrac{3}{2}\varphi_E K_{A\dot{Y}}K^{A\dot{Y}} = R\varphi_E. \tag{5.12}$$

(5.11) represents a relation between the radiation solution φ_A and the Petrov classification of the Weyl tensor [13] in its spinor form (Penrose [7]). First we have

$$\Psi_{ABCD}\varphi^A\varphi^B\varphi^C\varphi^D = 0. \tag{5.13}$$

Therefore:

For all radiation solutions of the Weyl equation the 4-current of the probability density is collinear with one of the principal null directions of the Weyl tensor.

With exception of conformally flat spaces, at every point there exist, at most, 4 different null directions determined uniquely by the Weyl tensor in which the neutrino radiation can propagate.

Furthermore, from (5.12) we have

$$2K_{\dot{Y}(E||A)}{}^{\dot{Y}} + 3\varepsilon_{EA}K_{\dot{Y}B||}{}^{B\dot{Y}} + \tfrac{3}{2}\varepsilon_{EA}K_{B\dot{Y}}K^{B\dot{Y}} + \varepsilon_{EA}R = 2\varphi_A\pi_E, \tag{5.14}$$

for some spinor π_E. Substituting the symmetrical part of (5.14),

$$K_{\dot{Y}(A||E)}{}^{\dot{Y}} = \varphi_{(A}\pi_{E)}, \tag{5.15}$$

into (5.11), we find

$$ic_2\,\varphi_A\,\varphi_E\,\varphi_X - \varphi_{(A}\varphi_E\pi_{X)} = \Psi_{AEXD}\varphi^D, \tag{5.16}$$

so that

$$\Psi_{AEXD}\varphi^D\varphi^E\varphi^X = 0. \tag{5.17}$$

Therefore the Weyl tensor cannot be of Petrov type I.

Radiation solutions of the Weyl equation can exist only in Riemann spaces with algebraically special Weyl tensors.

In the case of *special* radiation solutions (comp. table) it is possible to derive further results from the Ricci identities. For example assuming

a pp-wave (type c of the Table) as an *exact solution* of the coupled Weyl-Einstein system

$$R_{\alpha\beta} = -\lambda\, T_{\alpha\beta} \qquad\qquad (5.18)$$

$(R = 0)$ with (2.1) and (2.7), then upon substituting

$$K_{A\dot{B}} = d\varphi_A\,\varphi_{\dot{B}}\,, \qquad d_{|A\dot{B}} \sim \varphi_A\,\varphi_{\dot{B}} \qquad (5.19\,\text{a, b})$$

and

$$R_{\alpha\beta} \sim j_\alpha j_\beta \qquad\qquad (5.20)$$

into (5.10) and (5.14) results in

$$c_2 = 0\,, \qquad \pi_A = 0\,. \qquad\qquad (5.21\,\text{a, b})$$

Moreover, use of (5.9) gives

$$b_{|\varepsilon} \sim j_\varepsilon\,. \qquad\qquad (5.22)$$

Therefore, according to (5.16), the Weyl tensor of the corresponding metric $g_{\alpha\beta}$ is of Weyl type N or 0 and the wave equation φ_A satisfies the conditions of type d of the table.

§ 6. Exact Solution of Weyl-Einstein's Equations

Finally we want to deduce an exact solution of the coupled system of Weyl-Einstein's equation for neutrino pp-waves. Let $\tilde{\varphi}_A$ be the spinor part of the solution, then according to (5.22) it is of type d (Table):

$$\tilde{\varphi}_A = \hat{\varphi}_A e^{\Phi}\,, \qquad \hat{\varphi}_{A||\mu} = 0\,, \qquad \Phi := \delta + i\beta \qquad (6.1\text{--c})$$

with real functions δ and β. Hence a constant complex null bivector field exists

$$N_{\alpha\beta} \leftrightarrow \sqrt{2}\ \varepsilon_{AB}\hat{\varphi}_{\dot{C}}\hat{\varphi}_{\dot{D}}\,, \qquad N_{\alpha\beta||\varepsilon} = 0 \qquad (6.2\,\text{a, b})$$

which is connected by

$$N_{\alpha\beta}\bar{N}^{\gamma\beta} = -2\hat{j}_\alpha\hat{j}^\gamma \qquad\qquad (6.3)$$

with the constant vector field

$$\hat{j}^\alpha := j^\alpha e^{-2\delta}\,, \qquad \hat{j}_{\alpha||\varepsilon} = 0\,. \qquad (6.4\,\text{a, b})$$

According to the theorem of Ehlers and Kundt [14], a Riemann space admits a constant null bivector if and only if its line element is that of a gravitational pp-wave [15]:

$$ds^2 = -dx^2 - dy^2 + 2\,du\,dv + 2H(x, y, u)\,du^2\,, \qquad (6.5)$$

with

$$\hat{j}_\alpha = u_{|\alpha}\,. \qquad\qquad (6.6)$$

In accordance with (5.21) the related Weyl tensor is of type N or 0 with the unique principal null direction $\hat{j}_\alpha$. Furthermore the Ricci tensor is

$$R_{\alpha\beta} = -(H_{|xx} + H_{|yy})\hat{j}_\alpha \hat{j}_\beta , \qquad (6.7)$$

and using (2.7), Einstein's Equation (5.18) reduces to

$$H_{|xx} + H_{|yy} = \lambda\, b\, e^{4\delta} . \qquad (6.8)$$

Since the derivatives of d, b, δ and β are proportional to j_α (comp. table), these four founctions depend only on the null coordinate u. The general solution of the homogeneous part of the differential equation (6.8) is

$$H_0 = \mathrm{Re}\, f(z, u) , \qquad z := x + i y \qquad (6.9\,\mathrm{a, b})$$

for any analytic function f of z.

$$H_S(x, y, u) = \tfrac{1}{4}\, \lambda\, b(u)\, e^{4\delta(u)}(x^2 + y^2) \qquad (6.10)$$

represents one special solution of the inhomogeneous differential equation (6.8). The general solution is given by $H = H_0 + H_S$.

We are left to prove the existence of the assumed radiation solution (6.1) in a Riemann space with metric (6.5). Using (2.2), (5.1) and the differential equation (3.5), the general integrability condition

$$\varphi_{A|[\mu\nu]} = 0 \qquad (6.11)$$

takes, in the case of type d, the special form:

$$\left[\Psi_{ABFH}\varepsilon_{\dot{G}j} + \varepsilon_{FH}\Phi_{AB\dot{G}j} + \frac{R}{12}\,\varepsilon_{AB}\varepsilon_{FH}\varepsilon_{\dot{G}j} \right] \varphi^A = 0 , \qquad (6.12)$$

where $\Phi_{AB\dot{G}j}$ is the spinor equivalent of the Ricci tensor (6.7) with

$$\Phi_{AB\dot{G}j} \sim \tilde{\varphi}_A \tilde{\varphi}_B \tilde{\varphi}_{\dot{G}} \tilde{\varphi}_j . \qquad (6.13)$$

The spinor equivalent Ψ_{ABCD} corresponding to the Weyl tensor is given by

$$\Psi_{ABCD} \sim \tilde{\varphi}_A \tilde{\varphi}_B \tilde{\varphi}_C \tilde{\varphi}_D . \qquad (6.14)$$

For that reason the integrability conditions are satisfied if and only if $\varphi_A \sim \tilde{\varphi}_A$.

This completes the proof that the metric $g_{\alpha\beta}$ of the line element (6.5) and the radiation solution $\tilde{\varphi}_A$ given by (6.1) are exact solutions of the Weyl-Einstein system. A neutrino pp-wave therefore is accompanied by a gravitational pp-wave[7].

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[7] In connection with photons, the special properties of the line element (6.5) were studied by Peres [16] and Bonnor [17]. Their results and interpretations can be used immediately in the neutrino case. Recently further radiation solutions have been found with a tensor method by Griffiths and Newing [18].

Appendix

We solve the system (3.4)

$$i[\varphi_{(A||B)(\dot{C}}\varphi_{\dot{D})} - \varphi_{(\dot{C}||\dot{D})(A}\varphi_{B)}] = b(x)\varphi_A\varphi_B\varphi_{\dot{C}}\varphi_{\dot{D}} \qquad (A.1)$$

for $\varphi_{A||B\dot{C}}$, using the Weyl equation (2.1): first we perform at one chosen point a spin transformation such that [8]

$$\varphi_1 \overset{*}{=} \mu(x), \qquad \varphi_2 \overset{*}{=} 0 \qquad (A.2)$$

with some function $\mu(x)$. Now (A.1) reduces to the following independent equations

$$\frac{i}{2}[\mu\varphi_{1||1\,i} - \bar{\mu}\varphi_{1||1\,i}] \overset{*}{=} b(\mu\bar{\mu})^2, \qquad (A.3\,a)$$

$$\mu\varphi_{i||2\,i} - \bar{\mu}\varphi_{2||1\,i} - \mu\varphi_{1||2\,i} \overset{*}{=} 0, \qquad (A.3\,b)$$

$$\mu\varphi_{i||2\dot{2}} - \bar{\mu}\varphi_{1||2\dot{2}} \overset{*}{=} 0, \qquad (A.3\,c)$$

$$\varphi_{2||2\,i} \overset{*}{=} 0, \qquad (A.3\,d)$$

$$\varphi_{2||2\dot{2}} \overset{*}{=} 0. \qquad (A.3\,e)$$

Splitting the complex Eqs. (A.3 b), (A.3 d) and (A.3 e) into real and imaginary parts, (A.3) represents 8 real functions for the 6 complex independent components of $\varphi_{A||B\dot{C}}$ (satisfying Weyl's equation). The solution therefore contains 4 new real functions $\varkappa, \nu, \varrho, \tau$. We obtain:

$$\varphi_{1||1\,i} \overset{*}{=} \mu(\varkappa + ib\mu\bar{\mu}), \qquad (A.4\,a)$$

$$\varphi_{1||1\dot{2}} \overset{*}{=} 2\mu(\nu + i\tau), \qquad (A.4\,b)$$

$$\varphi_{1||2\dot{2}} \overset{*}{=} \mu\varrho, \qquad (A.4\,c)$$

$$\varphi_{2||1\,i} \overset{*}{=} \mu(\nu - i\tau), \qquad (A.4\,d)$$

$$\varphi_{2||2\,i} \overset{*}{=} 0, \qquad (A.4\,e)$$

$$\varphi_{2||2\dot{2}} \overset{*}{=} 0. \qquad (A.4\,f)$$

Inspection of the spinor indices shows that the free functions may be combined into a hermitean spinor

$$K_{1\,i} := K_{i\,1} :\overset{*}{=} \varkappa, \qquad (A.5\,a)$$

$$K_{1\dot{2}} :\overset{*}{=} 2(\nu + i\tau), \qquad (A.5\,b)$$

$$K_{2\dot{2}} = K_{\dot{2}2} :\overset{*}{=} 2\varrho. \qquad (A.5\,c)$$

Thus the solution (A.4) becomes finally

$$\varphi_{A||B\dot{C}} = ib\,\varphi_A\varphi_B\varphi_{\dot{C}} + \varphi_{(A}K_{B)\dot{C}}. \qquad (A.6)$$

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[8] $\overset{*}{=}$ denotes equality for this special choice.

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