Neutron stars for undergraduates

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The calculation of the structure of white dwarf and neutron stars is a suitable topic for an undergraduate thesis or an advanced special topics or independent study course. The subject is rich in many different areas of physics, ranging from thermodynamics to quantum statistics to nuclear physics to special and general relativity. The computations for solving the coupled structure differential equations (both Newtonian and general relativistic) can be done using a symbolic computational package. In doing so, students will develop computational skills and learn how to deal with units. Along the way they also will learn some of the physics of equations of state and of degenerate stars.
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I. INTRODUCTION

In 1967 Jocelyn Bell, a graduate student, along with her thesis advisor, Anthony Hewish, discovered the first pulsar, an object in outer space that emits very regular pulses of radio energy.[1] After recognizing that these pulse trains were so unvarying that they could not support an origin from little green men, it soon became generally accepted that the pulsar was due to radio emission from a rapidly rotating neutron star[2] endowed with a very strong magnetic field. At present more than 1000 pulsars have been catalogued.[3] Pulsars are by themselves quite interesting,[4] but perhaps more so is the structure of the underlying neutron star. This paper discusses a student project on their structure.

While still at MIT, one of us (Reddy) had the pleasure of acting as mentor for a bright British high school student, Aiden J. Parker. She spent the summer of 2002 at MIT as a participant in a special research program. With minimal guidance she was able to write a Fortran program for solving the Tolman–Oppenheimer–Volkov equations[5] to calculate the masses and radii of neutron stars.

In discussing this impressive work after Reddy's arrival at LANL, the question arose of whether it would have been possible (and easier) to have done the computation using Mathematica (or another symbolic and numerical manipulation package). This question was taken as a challenge by Silbar, who also figured it would be a good opportunity to learn how these kinds of stellar structure calculations are done. (Silbar's only previous experience in this field of physics consisted of having read, with some care, the chapter on stellar equilibrium and collapse in Weinberg's treatise on gravitation and cosmology.[6])

In the process of meeting the challenge, it became clear that this subject would be an excellent topic for a junior or senior physics major's project or thesis. There is much more physics in the problem than just simply integrating a pair of coupled nonlinear differential equations. In addition to the physics (and some astronomy), students must think about the magnitudes of the quantities they are calculating in order to check and understand the answers they obtain. Another side benefit is that students learn about the stability of numerical solutions and how to deal with singularities. In the process they also learn about the inner mechanics of the software package they use.

The paper proceeds as follows. The student should begin with a derivation of the (Newtonian) coupled structure equations (Sec. II A), and be given the general relativistic corrections (Sec. II B). Before trying to solve these equations, they need to know the relation between the energy density and pressure of the matter that constitutes the stellar interior, that is, the equation of state. The first equation of state to use can be derived from the noninteracting Fermi gas, which brings in quantum statistics and special relativity (Sec. III B).

As a warm-up problem students can integrate the Newtonian equations and learn about white dwarf stars. They can then include the general relativistic corrections and proceed in the same way to work out the structure of pure neutron stars and reproduce the results of Oppenheimer and Volkov.[5] It is interesting at this point to determine the importance of the general relativistic corrections, that is, how different a neutron star is from what would be given by classical Newtonian mechanics.

Of course, realistic neutron stars also contain some protons and electrons. As a first approximation we can treat this multicomponent system as a noninteracting Fermi gas. In the process we learn about chemical potentials. To improve on this treatment, we must include nuclear interactions in addition to the degeneracy pressure from the Pauli exclusion principle. The nucleon–nucleon interaction is not well known to undergraduates, but there is a simple model (which we learned from Prakash[7]) for the nuclear matter equation of state. It has parameters that are fit to quantities such as the binding energy per nucleon in symmetric nuclear matter, the nuclear symmetry energy, and the (not so well known) nuclear compressibility.[8] If we use these nuclear interactions in addition to the Fermi gas energy in the equation of state, we find (pure) neutron star masses and radii that are quite different from those using the Fermi gas equation of state.

In the following we will indicate possible "gotcha's" that students might encounter and possible side-trips that might be taken. Of course, the project we outline here should be augmented by the faculty mentor[9] with suggestions for byways that might lead to publishable results, if that is desired.

Balian and Blaizot have given a similar discussion of this subject matter.[10] However, they used this material (and related materials) as the basis for a full-year course. In contrast, our emphasis is more toward nudging the student into a research frame of mind involving numerical calculations. Much of the material we discuss here is covered in the textbook by Shapiro and Teukolsky.[11] However, as the reader will notice, the emphasis here is on students learning through

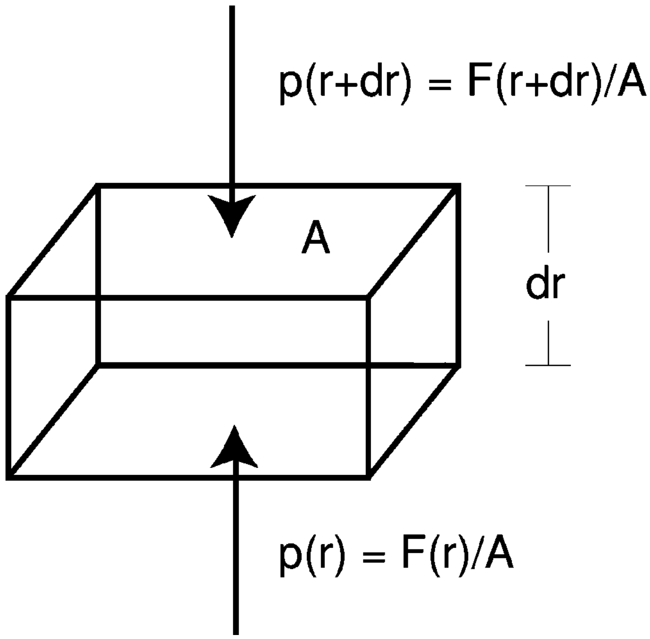

Fig. 1. Diagram for the derivation of Eq. (1).

computation. One of our intentions is to establish a framework for students to interact with their own computer program, and in the process learn about the physical scales involved in the structure of compact degenerate stars.

II. THE TOLMAN–OPPENHEIMER–VOLKOV EQUATION

A. Newtonian formulation

A good first exercise for the student is to derive the following structure equations for stars from classical mechanics,

$$\frac{dp}{dr} = -\frac{G\rho(r)\mathcal{M}(r)}{r^2} = -\frac{G\epsilon(r)\mathcal{M}(r)}{c^2 r^2}, \tag{1}$$

$$\frac{d\mathcal{M}(r)}{dr} = 4\pi r^2 \rho(r) = \frac{4\pi r^2 \epsilon(r)}{c^2}, \tag{2}$$

$$\mathcal{M}(r) = 4\pi \int_0^r r'^2\, dr'\, \rho(r') = 4\pi \int_0^r r'^2\, dr'\, \epsilon(r')/c^2. \tag{3}$$

Here $G = 6.673 \times 10^{-8}$ dyne cm^2/g^2 is Newton's gravitational constant, $\rho(r)$ is the mass density at the distance r in g/cm^3, and ϵ is the corresponding energy density in ergs/cm^3.[12] The quantity $\mathcal{M}(r)$ is the total mass inside the sphere of radius r. A sufficient hint for the derivation is shown in Fig. 1.

Challenge question: Eqs. (1)–(3) hold for any value of r, not just the large r situation depicted in Fig. 1. Derive these results in spherical coordinates where the box becomes a cutoff wedge.

Note that in the second halves of Eqs. (1)–(3), we have departed slightly from Newtonian physics and have expressed the energy density $\epsilon(r)$ in terms of the mass density $\rho(r)$ according to the famous Einstein equation from special relativity,

$$\epsilon(r) = \rho(r) c^2. \tag{4}$$

This definition allows Eq. (1) to be used when we take into account contributions of the interaction energy between the particles making up the star.

To solve Eqs. (1)–(3) for $p(r)$ and $\mathcal{M}(r)$, we can integrate from the origin, $r=0$, to the point $r=R$ where the pressure goes to zero. This point defines R as the radius of the star. We will need an initial value of the pressure at $r=0$, call it p_0; R and the total mass of the star, $\mathcal{M}(R) \equiv M$, will depend on the value of p_0. To be able to perform the integration, we need to know the energy density $\epsilon(r)$ in terms of the pressure $p(r)$. This relation is the equation of state for the matter making up the star. Thus, a lot of the effort in this project will be directed to developing an appropriate equation of state.

B. General relativistic corrections

The Newtonian formulation presented in Sec. II A works well in regimes where the mass of the star is not so large that it significantly "warps" space–time. That is, integrating Eqs. (1) and (2) will work well in cases for which general relativistic effects are not important, such as for the compact stars known as white dwarfs. (General relativistic effects become important when the ratio GM/c^2R becomes non-negligible, as is the case for typical neutron stars).

It is probably not to be expected that an undergraduate physics major will be able to derive the general relativistic corrections to Eqs. (1)–(3). For that, we can look at various derivations of the Tolman–Oppenheimer–Volkov (TOV) equation.[6,11] It is sufficient to simply state the corrections to Eq. (1) in terms of three additional (dimensionless) factors,

$$\frac{dp}{dr} = -\frac{G\epsilon(r)\mathcal{M}(r)}{c^2 r^2}\left[1 + \frac{p(r)}{\epsilon(r)}\right]\left[1 + \frac{4\pi r^3 p(r)}{\mathcal{M}(r)c^2}\right] \times \left[1 - \frac{2G\mathcal{M}(r)}{c^2 r}\right]^{-1}. \tag{5}$$

The first two factors in the square brackets in Eq. (5) represent special relativity corrections of order v^2/c^2. These factors enter because, in the nonrelativistic limit, the pressure p varies as $k_F^2/2m = mv^2/2$ [see Eq. (13)], while ϵ and $\mathcal{M}c^2$ vary as mc^2. These factors reduce to 1 in the nonrelativistic limit. (By now the student should realize that p and ϵ have the same dimensions.) The last set of brackets in Eq. (5) is a general relativistic correction. Equation (2) for $\mathcal{M}(r)$ remains unchanged.

Note that the correction factors are all positive definite. It is as if Newtonian gravity becomes stronger for any value of r. That is, relativity strengthens the relentless pull of gravity.

Equations (5) and (2), involve a balance between gravitational forces and the internal pressure. The pressure is a function of the equation of state, and for certain conditions it may not be sufficient to withstand the gravitational attraction. Thus the structure equations imply there is a maximum mass that a star can have. The resultant coupled nonlinear equations for $p(r)$ and $\mathcal{M}(r)$ can be integrated from $r=0$ to the point R where $p(R)=0$ to determine the star mass $M = \mathcal{M}(R)$ for a given value of p_0.

III. WHITE DWARF STARS

A. A few facts

For the cold, compact stallar objects known as white dwarf stars, it suffices to solve the Newtonian structure equations, Eqs. (1)–(3).[13] White dwarf stars[14] were first observed in 1844 by Friedrich Bessel (the same person who invented the special functions bearing that name). He noticed that the bright star Sirius wobbled back and forth and then deduced that the visible star was being orbited by some unseen object, that is, it is a binary system. The object itself was resolved optically some 20 years later and thus earned the name of "white dwarf." Since then, numerous other white (and the smaller brown) dwarf stars have been observed or detected.

A white dwarf star is a low- or medium-mass star near the end of its lifetime, having burned up, through nuclear processes, most of its hydrogen and helium forming carbon, silicon, or (perhaps) iron. They typically have a mass less than 1.4 times that of our Sun, $M_\odot = 1.989\times 10^{30}$ kg.[15] They are also much smaller than our Sun, with radii of the order of 10^4 km (to be compared with $R_\odot = 6.96\times 10^5$ km). These values can be determined from the period of the wobble for the dwarf-normal star binary in the usual Keplerian way. As a result (and as also is the case for neutron stars), the natural dimensions for discussing white dwarfs are for masses to be in units of solar mass, $M_\odot$, and distances to be in kilometers. By using these numbers, students should be able to make a quick estimate of the (average) densities of our Sun and of a white dwarf to obtain a feel for the numbers that will be encountered.

Because $GM/c^2R \approx 10^{-4}$ for a typical white dwarf, we can concentrate on solving the non-relativistic structure equations of Sec. II A. Question: why is it a good approximation to drop the special relativistic corrections for these dwarfs?

The reason a dwarf star is small is because, having burned up all the nuclear fuel it can, there is no longer enough thermal pressure to prevent its gravity from crushing it down. As the density increases, the electrons in the atoms are pushed closer together, which then tend to fall into the lowest energy levels available to them. (The star begins to become colder.) Eventually the Pauli principle takes over, and the electron degeneracy pressure (to be discussed in Sec. III B) provides the means for stabilizing the star against its gravitational attraction.[11,15]

B. Fermi gas model for electrons

For free electrons the number of states dn available between the momentum k and $k+dk$ per unit volume is[16]

$$dn = \frac{d^3k}{(2\pi\hbar)^3} = \frac{4\pi k^2\, dk}{(2\pi\hbar)^3}. \tag{6}$$

By integrating Eq. (6), we obtain the electron number density,

$$n = \frac{8\pi}{(2\pi\hbar)^3}\int_0^{k_F} k^2\, dk = \frac{k_F^3}{3\pi^2\hbar^3}. \tag{7}$$

The additional factor of two is included because there are two spin states for each electron energy level. Here $k_F c$, the Fermi energy, is the maximum energy electrons can have in the star under consideration. It is a parameter that varies according to the star's total mass, but which students are free to set in the calculations.

Each electron is neutralized by a proton, which in turn is accompanied in its atomic nucleus by a neutron (or perhaps a few more, as in the case of a nucleus like ^{56}Fe). Thus, if we neglect the electron mass m_e with respect to the nucleon mass m_N, the mass density of the star is given by

$$\rho = n m_N A/Z, \tag{8}$$

where A/Z is the number of nucleons per electron. For ^{12}C, $A/Z=2$, while for ^{56}Fe, $A/Z=2.15$. Note that, because n is a function of k_F, so is ρ. Conversely, given a value of ρ,

$$k_F = \hbar\left(\frac{3\pi^2\rho}{m_N}\,\frac{Z}{A}\right)^{1/3}. \tag{9}$$

The energy density of this star also is dominated by the nucleon masses, that is, $\epsilon \approx \rho c^2$.

The contribution to the energy density from the electrons (including their rest masses) is

$$\begin{aligned}\epsilon_{\rm elec}(k_F) &= \frac{8\pi}{(2\pi\hbar)^3}\int_0^{k_F}(k^2c^2+m_e^2c^4)^{1/2}k^2\, dk \\ &= \frac{m_e^4c^5}{\pi^2\hbar^3}\int_0^{k_F/m_ec}(u^2+1)^{1/2}u^2\, du \\ &= \frac{m_e^4c^5}{8\pi^2\hbar^3}[(2x^3+x)(1+x^2)^{1/2}-\sinh^{-1}(x)],\end{aligned} \tag{10}$$

where $x = k_F/m_ec$. The total energy density is then

$$\epsilon = n m_N A/Z + \epsilon_{\rm elec}(k_F). \tag{11}$$

One should check that the first term is much larger than the second.

To obtain the desired equation of state, we also need an expression for the pressure. The following presents a problem for the student. From the first law of thermodynamics, $dU = dQ - p\, dV$, and the temperature T fixed at $T=0$ (where $dQ=0$ because $dT=0$), we have

$$p = \left[-\frac{\partial U}{\partial V}\right]_{T=0} = n^2\frac{d(\epsilon/n)}{dn} = n\frac{d\epsilon}{dn} - \epsilon = n\mu - \epsilon, \tag{12}$$

where the energy density is given by Eq. (11). The quantity $\mu = d\epsilon/dn$ is known as the chemical potential of the electrons. The chemical potential will be especially useful in Sec. V, where we consider an equilibrium mix of neutrons, protons, and electrons.

If we utilize Eq. (10), Eq. (12) yields the pressure (another problem)

$$\begin{aligned}p(k_F) &= \frac{8\pi}{3c(2\pi\hbar)^3}\int_0^{k_F}(k^2c^2+m_e^2c^4)^{-1/2}k^4\, dk \\ &= \frac{m_e^4c^5}{3\pi^2\hbar^3}\int_0^{k_F/m_ec}(u^2+1)^{-1/2}u^4\, du \\ &= \frac{m_e^4c^5}{24\pi^2\hbar^3}[(2x^3-3x)(1+x^2)^{1/2}+3\sinh^{-1}(x)].\end{aligned} \tag{13}$$

[Hint: use the $n^2 d(\epsilon/n)/dn$ form and integrate by parts.]

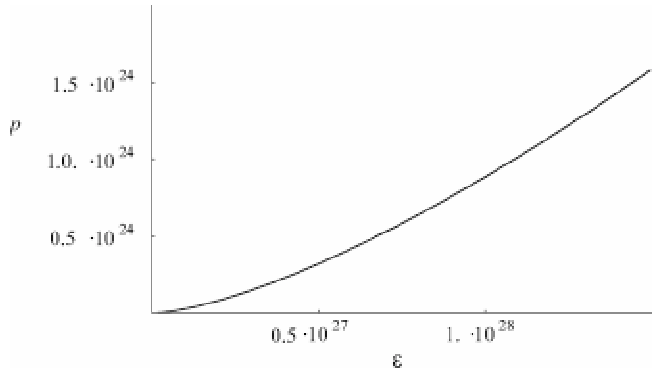

Fig. 2. Relation between the pressure p and the energy density ϵ in the free electron Fermi gas model. The units are ergs/cm^3. Note that the pressure is much smaller than the energy density, because the latter is dominated by the massive nucleons.

By using Mathematica,[17] students can show that the constant in front of the integral in the second line of Eq. (10) is 1.42×10^{24} ergs/cm^3. (Another problem: verify that the units of this constant are as claimed.[18]) Mathematica also can perform the integrals analytically. [We already gave the results in Eqs. (10) and (13).] They are a bit messy, however, as they both involve an inverse hyperbolic sine function, and thus are not terribly enlightening. It is useful, however, for the student to make a plot of ϵ versus p (such as shown in Fig. 2) for various values of the parameter $0\leqslant k_F\leqslant 2m_e$. This curve has a shape much like $\epsilon^{4/3}$ (the student should compare the curve to this function), and there is a good reason for that, as we will see.

Consider the (relativistic) case when $k_F\gg m_e$. Then Eq. (13) simplifies to

$$\begin{aligned} p(k_F) &= \frac{m_e^4c^5}{3\pi^2\hbar^3}\int_0^{k_F/m_ec} u^3\,du \\ &= \frac{m_e^4c^5}{12\pi^2\hbar^3}(k_F/m_ec)^4 \\ &= \frac{\hbar c}{12\pi^2}\left(\frac{3\pi^2 Z\rho}{m_N A}\right)^{4/3} \approx K_{\text{rel}}\epsilon^{4/3}, \end{aligned} \tag{14}$$

where

$$K_{\text{rel}} = \frac{\hbar c}{12\pi^2}\left(\frac{3\pi^2 Z}{A m_N c^2}\right)^{4/3}. \tag{15}$$

A star having a simple equation of state such as $p=K\epsilon^\gamma$ is called a polytrope, and we see that the relativistic electron Fermi gas gives a polytropic equation of state with $\gamma=4/3$. As will be seen in Sec. III C, a polytropic equation of state allows us to solve the structure equations (numerically) in a relatively straightforward way.[19]

There is another polytropic equation of state for the noninteracting electron Fermi gas model corresponding to the nonrelativistic limit, where $k_F\ll m_e$. In a way similar to the derivation of Eq. (14), we find

$$p = K_{\text{nonrel}}\epsilon^{5/3}, \tag{16}$$

where

$$K_{\text{nonrel}} = \frac{\hbar^2}{15\pi^2 m_e}\left(\frac{3\pi^2 Z}{A m_N c^2}\right)^{5/3}. \tag{17}$$

Question: what are the units of K_{rel} and K_{nonrel}? Confirm that, in the appropriate limits, Eqs. (10) and (13) reduce to Eqs. (14) and (17).

C. The structure equations for a polytrope

As mentioned, we want to express our results in units of km and $M_\odot$. Thus it is useful to define $\overline{\mathcal{M}}(r)=\mathcal{M}(r)/M_\odot$. The first Newtonian structure equation, Eq. (1), then becomes

$$\frac{dp(r)}{dr} = -R_0\frac{\epsilon(r)\overline{\mathcal{M}}(r)}{r^2}, \tag{18}$$

where the constant $R_0=GM_\odot/c^2=1.47$ km. (For those who know, R_0 is one-half the Schwartzschild radius of our sun.) In Eq. (18) p and ϵ carry dimensions of ergs/cm^3. Therefore, we define the dimensionless energy density, $\bar{\epsilon}$, and pressure, $\bar{p}$, by

$$p=\epsilon_0\bar{p}, \tag{19}$$

$$\epsilon=\epsilon_0\bar{\epsilon}, \tag{20}$$

where ϵ_0 has dimensions of energy density. Its choice is arbitrary, and a suitable strategy is to make that choice based on the dimensionful numbers that define the problem at hand. We will employ this strategy to choose it below. For a polytrope, we can write

$$\bar{p}=\bar{K}\bar{\epsilon}^\gamma, \tag{21}$$

where $\bar{K}=K\epsilon_0^{\gamma-1}$ is dimensionless.

It is easier to solve Eq. (18) for $\bar{p}$, so we express $\bar{\epsilon}$ in terms of it,

$$\bar{\epsilon}=(\bar{p}/\bar{K})^{1/\gamma}. \tag{22}$$

Equation (18) can now be recast in the form

$$\frac{d\bar{p}(r)}{dr} = -\frac{\alpha\bar{p}(r)^{1/\gamma}\overline{\mathcal{M}}(r)}{r^2}, \tag{23}$$

where the constant α is

$$\alpha=R_0/\bar{K}^{1/\gamma}=R_0/(K\epsilon_0^{\gamma-1})^{1/\gamma}. \tag{24}$$

Equation (23) has dimensions of 1/km, with α in km (because R_0 is).

We can choose any convenient value for α because ϵ_0 is still free. For a given value of α, ϵ_0 is then fixed at

$$\epsilon_0=\left[\frac{1}{K}\left(\frac{R_0}{\alpha}\right)^\gamma\right]^{1-\gamma}. \tag{25}$$

We also need to cast the other coupled equation, Eq. (2), in terms of the dimensionless quantities $\bar{p}$ and $\overline{\mathcal{M}}$,

$$\frac{d\overline{\mathcal{M}}(r)}{dr} = \beta r^2\bar{p}(r)^{1/\gamma}, \tag{26}$$

where[20]

$$\beta=\frac{4\pi\epsilon_0}{M_\odot c^2\bar{K}^{1/\gamma}} = \frac{4\pi\epsilon_0}{M_\odot c^2(K\epsilon_0^{\gamma-1})^{1/\gamma}}. \tag{27}$$

Equation (26) also carries dimensions of 1/km, the constant β having dimensions 1/km^3. Note that, in integrating from $r=0$, the initial value of $\bar{\mathcal{M}}(0)=0$.

D. Integrating the polytrope numerically

Our task is to integrate the coupled first-order differential equations, Eqs. (23) and (26), from the origin, $r=0$, to the point R where $\bar{p}(R)=0$.[21] To do so we need two initial values, $\bar{p}(0)$ (which must be positive), and $\bar{\mathcal{M}}(0)$ (which we know must be zero). The star's radius, R, and its mass $M=\bar{\mathcal{M}}(R)$ in units of $M_\odot$ will vary, depending on the choice for $\bar{p}(0)$.

For purposes of numerical stability in solving Eqs. (23) and (26), we want the constants α and β to be not much different from each other (and not much different from unity). We will see that this can be arranged for both of the two polytropic equation of states that we discussed for white dwarfs.

Our coupled differential equations are quite nonlinear. Because of the $\bar{p}^{1/\gamma}$ factors, the exact solution will be complex when $\bar{p}(r)<0$, that is, when $r>R$. For example, Mathematica and symbolic programs like it have built-in first-order differential equation solvers. The solver might be as simple as a fixed, equal-step Runge–Kutta routine. These packages also allow for program control constructs such as do loop and while statements.

E. The relativistic case $k_F \gg m_e$

The case $k_F \gg m_e$ is the regime for white dwarfs with the largest mass. A larger mass needs a greater central pressure to support it. However, large central pressures mean that the squeezed electrons become relativistic.

Recall that the polytrope exponent $\gamma=4/3$ for this case and the equation of state is given by $P=K_{\rm rel}\epsilon^\gamma$ with $K_{\rm rel}$ given by Eq. (15). After some trial and error, we choose (the student may want to try another value)

$$\alpha=R_0=1.473 \text{ km} \quad (k_F \gg m_e), \tag{28}$$

which in turn from Eq. (25) fixes,

$$\epsilon_0=7.463\times 10^{39} \text{ ergs/cm}^3=4.17 M_\odot c^2/\text{km}^3 \quad (k_F \gg m_e). \tag{29}$$

When $k_F \gg m_e$, Eqs. (15) and (27) give

$$\beta=52.46/\text{km}^3, \tag{30}$$

which is about 30 times larger than α, but manageable from the standpoint of performing the numerical integration.

In our first attempt to integrate the coupled differential equations for this case, we choose $\bar{p}(0)=1.0$. This choice gives a white dwarf of radius $R\approx 2$ km, which is miniscule compared with the expected radius of $\approx 10^4$ km. Why? What went wrong?

Students who make this kind of mistake will eventually realize that our choice of scale, $\epsilon_0=4.17 M_\odot c^2/\text{km}^3$, represents a huge energy density. We can simply estimate the average energy density of a star with a 10^4 km radius and a mass $M_\odot$ by the ratio of its rest mass energy to its volume,

$$\langle\epsilon\rangle \approx \frac{M_\odot c^2}{R^3}=10^{-12} M_\odot c^2 \text{ km}^{-3}, \tag{31}$$

Table I. Radius R (in km) and mass M (in $M_\odot$) for white dwarfs with a relativistic electron Fermi gas equation of state.

$\bar{p}(0)$	R	M
10^{-14}	4840	1.2431
10^{-15}	8600	1.2432
10^{-16}	15 080	1.2430

which is much, much smaller than our choice of ϵ_0 here. In addition, the pressure p is about 2000 times smaller than the energy density ϵ (see Fig. 2). Thus, choosing a starting value of $\bar{p}(0)\sim 10^{-15}$ would be more physical. Doing so does give much more reasonable results. Table I shows our program's results for R and M and how they depend on $\bar{p}(0)$.

The surprise is that, within the expected numerical error, all these cases have the same mass. Increasing the central pressure does not allow the star to be more massive, just more compact. This result is correct: the white dwarf mass is independent of the choice of the central pressure. However, it is not easy to understand this result from the numerical integration.

The discussion in terms of Lane–Emden functions shows why, although the mathematics here might be a bit steep for many undergraduates. For this reason, we give the analytic results without proof.[19] For the polytropic equation of state $p=K\epsilon^\gamma$, the mass is

$$M=4\pi\epsilon^{2(\gamma-4/3)/3}\left(\frac{K\gamma}{4\pi G(\gamma-1)}\right)^{3/2}\zeta_1^2|\theta(\zeta_1)|, \tag{32}$$

and the radius is given by

$$R=\sqrt{\frac{K\gamma}{4\pi G(\gamma-1)}}\,\zeta_1\epsilon^{(\gamma-2)/2}. \tag{33}$$

In Eqs. (32) and (33), ζ_1 and $\theta(\zeta_1)$ are numerical constants that depend on the polytropic index γ. From Eq. (32), we see that for $\gamma=4/3$, the mass is independent of the central energy density, and hence also the central pressure p_0. Also, note that from Eq. (33), the radius decreases with increasing central pressure as $R\propto p_0^{(\gamma-2)/2\gamma}=p_0^{-1/4}$. Students should notice this point and use it to check their numerical results.

Figure 3 shows the dependence of $\bar{p}(r)$ and $\bar{\mathcal{M}}(r)$ on r for $\bar{p}(0)=10^{-16}$. It is interesting that $\bar{p}(r)$ becomes small and essentially flat around 8000 km before going through zero at $R=15\,080$ km. Such a star has a very tall "atmosphere."

F. The nonrelativistic case, $k_F \ll m_e$

As the central pressure $\bar{p}(0)$ becomes smaller, the electron gas eventually is no longer relativistic. Also as $\bar{p}(0)$ becomes smaller, the electron gas can support less mass, which moves the gas in the direction of the less massive white dwarfs. It turns out that these dwarfs are larger (in radius) than the relativistic ones just discussed.

In the extreme case when $k_F \ll m_e$, we can integrate the structure equations for the polytropic equation of state where $\gamma=5/3$. The program for this case is much the same as in the 4/3 case, but the numbers involved are quite different as are the results.

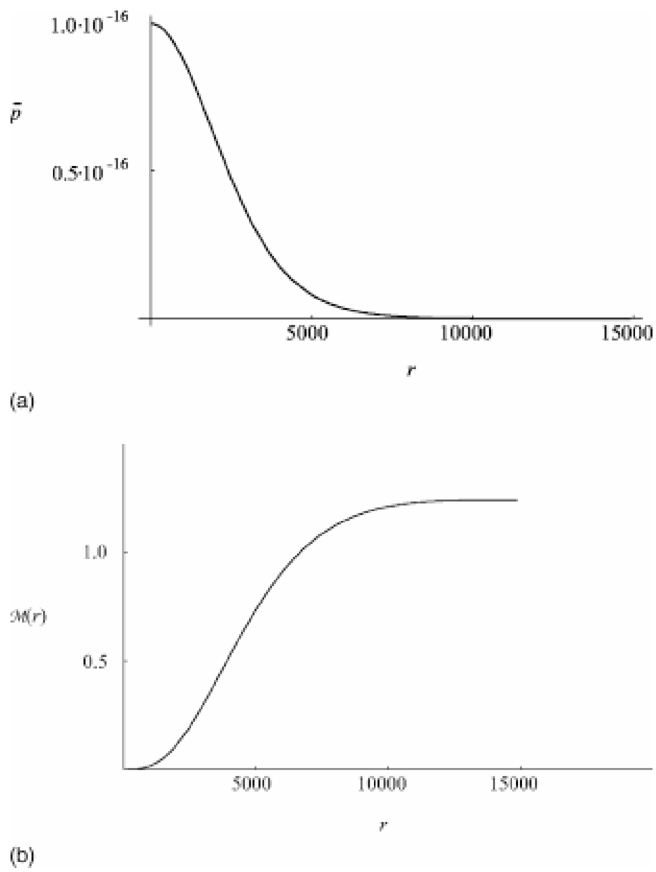

Fig. 3. The quantities (a) $\overline{p}(r)$ and (b) $\mathcal{M}(r)$ (in $M_\odot$) versus radius r (in km) for white dwarfs using the relativistic electron Fermi gas model. These curves are obtained from the integration of Eqs. (23) and (26) with the polytropic equation of state of Eq. (14). This is the case for the central pressure $\overline{p}(0)=10^{-16}$.

If we substitute the values of the physical constants in Eq. (17), we find

$$K_{\mathrm{nonrel}}=3.309\times10^{-23}\,\mathrm{cm}^2/\mathrm{ergs}^{2/3}. \tag{34}$$

After some experimentation, we choose the constant

$$\alpha=0.05 \ \ \mathrm{km}, \quad (k_F \ll m_e), \tag{35}$$

which then fixes

$$\epsilon_0=2.488\times10^{37} \ \ \mathrm{ergs/cm}^3=0.013\,92 M_\odot c^2/\mathrm{km}^3. \tag{36}$$

Note that this value of ϵ_0 is much smaller than our choice for the relativistic case. The other constant we need is from Eq. (27),

$$\beta=0.005\,924 \ \ \mathrm{km}^{-3}, \tag{37}$$

which, unlike the relativistic case, is not larger than α, but is smaller.

When we first ran our code for this case, we (inadvertently) tried a value of $\overline{p}(0)=10^{-12}$. This value gave a star with radius $R=5310$ km and mass $M=3.131$. This mass is *bigger* than the largest mass of 1.243 that we found for the relativistic equation of state. What did we do wrong? What happened (students can write their program so this trap can be avoided) is that the choice $\overline{p}(0)=10^{-12}$ violates the assumption that $k_F \ll m_e$. One really needs values such that $\overline{p}(0)<4\times10^{-15}$. That is, the value $\overline{p}(0)=10^{-16}$ for the case that we plotted in Fig. 3 is not really appropriate for a calculation using a relativistic polytrope for the equation of state.

Table II. Radius R (in km) and mass M (in $M_\odot$) for white dwarfs with a nonrelativistic electron Fermi gas equation of state.

$\overline{p}(0)$	R	M
10^{-15}	10 620	0.3941
10^{-16}	13 360	0.1974

The results for the nonrelativistic case for the last two values of $\overline{p}(0)$ in Table I are shown in Table II. It is instructive to compare the differences in the two tables. The masses are, of course, smaller, as expected, and now they vary with $\overline{p}(0)$. Figure 4 shows the pressure distribution for the latter case, which is to be compared with the corresponding graph in Fig. 3. Note that this nonrelativistic pressure curve does not have the peculiar long flat tail found using the relativistic equation of state (Fig. 3).

By this time students should realize that neither of these polytropes is very physical, at least not for all cases. The nonrelativistic assumption certainly does not work for central pressures $\overline{p}(0)>10^{-14}$, that is, for the more massive (and more common) white dwarfs. On the other hand, the relativistic equation of state certainly should not work when the pressure becomes small, that is, in the outer regions of the star (where it eventually goes to zero at the star's radius). Can we find an equation of state to cover the whole range of pressures?

We have not actually found such an equation for white dwarfs, but the program would be similar to that discussed in the following for the full neutron star. Given the transcendental expressions for the energy and pressure that generate the curve shown in Fig. 2, Eqs. (10) and (13), it also should be possible to find a fit (using, for example, the intrinsic fitting function of Mathematica) such as

$$\overline{\epsilon}(\overline{p})=A_{\mathrm{NR}}\overline{p}^{3/5}+A_{\mathrm{R}}\overline{p}^{3/4}. \tag{38}$$

The second term dominates at high pressures (the relativistic case), but the first term takes over for low pressures when the $k_F \gg m_e$ assumption does not hold. (Setting the two terms equal and solving for $\overline{p}$, gives the value of $\overline{p}$ when special relativity starts to be important.) This expression for $\overline{\epsilon}(\overline{p})$ could then be used in place of the $\overline{p}^{1/\gamma}$ factors on the right-hand sides of the structure equations. Proceed to solve nu-

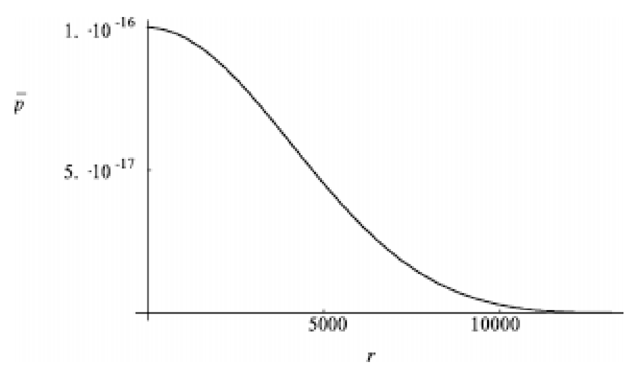

Fig. 4. $\overline{p}(r)$ for a white dwarf using the nonrelativistic electron Fermi gas model, Eq. (17), with central pressure $\overline{p}(0)=10^{-16}$.

Table III. Radius R (in km) and mass M (in $M_\odot$) for pure neutron stars with a nonrelativistic Fermi gas equation of state.

$\bar{p}(0)$	R (Newton)	M (Newton)	R (GR)	M (GR)
10^{-4}	16.5	0.7747	15.25	0.6026
10^{-5}	20.8	0.3881	20.00	0.3495
10^{-6}	26.3	0.1944	25.75	0.1864

merically as before. We leave this as an exercise for interested students.

IV. PURE NEUTRON STAR, FERMI GAS EQUATION OF STATE

In this case we must include the general relativistic contributions represented by the three dimensionless factors in the TOV equation, Eq. (5). One of the first programming problems that comes to mind is how one deals numerically with the (apparent) singularities in these factors at $r=0$.

Also, as for the white dwarfs, there is a question of what to use for the equation of state. In this section we show what can be done for pure neutron stars, once again using a Fermi gas model for a neutron gas instead of an electron gas. Such a model, however, is unrealistic for two reasons. First, a real neutron star consists not just of neutrons, but contains a small fraction of protons and electrons (to inhibit the neutrons from decaying into protons and electrons by their weak interactions). Second, the Fermi gas model ignores the strong nucleon–nucleon interactions, which give important contributions to the energy density. Each of these points will be dealt with in Secs. V and VI, respectively.

A. The nonrelativistic case, $k_F \ll m_n$

For a pure neutron star Fermi gas equation of state we can proceed much as in the white dwarf case, substituting the neutron mass m_n for the electron mass m_e in the equations in Sec. III. When $k_F \ll m_n$, we again find a polytrope with $\gamma = 5/3$. (Another exercise for the student.) From Eq. (17) we have

$$K_{\rm nonrel} = \frac{\hbar^2}{15\pi^2 m_n}\left(\frac{3\pi^2 Z}{A m_n c^2}\right)^{5/3} = 6.483\times 10^{-26}\,{\rm cm}^2/{\rm ergs}^{2/3}. \tag{39}$$

If we choose $\alpha = 1$ km, we find the scaling factor from Eq. (25) to be

$$\epsilon_0 = 1.603\times 10^{38}\ \ {\rm ergs/cm}^3 = 0.089\,69 M_\odot c^2/{\rm km}^3. \tag{40}$$

Furthermore, from Eqs. (21) and (27),

$$\bar{K} = 1.914 \quad \text{and} \quad \beta = 0.7636\ \ {\rm km}^{-3}. \tag{41}$$

Note that, in this case, the constants α and β are of similar magnitude.

To estimate the average energy density of a typical neutron star (mass$=M_\odot$, $R=10$ km), we expect that a good starting value for the central pressure $\bar{p}(0)$ is order 10^{-4} or less. Our program for this situation is essentially the same as the one for nonrelativistic white dwarfs, but with appropriate changes of the distance scale. It gives the results shown in Table III. Note that the general relativistic effects are small, but not negligible, for this nonrelativistic equation of state. As in the white dwarf case, these are smaller mass stars. We see that as the mass becomes smaller, the gravitational attraction is less and thus the star extends out to larger radii.

B. The relativistic case, $k_F \gg m_n$

In this case there is again a polytropic equation of state, but with $\gamma = 1$. In fact, $p = \epsilon/3$, a well-known result for a relativistic gas. The conversion to dimensionless quantities becomes very simple in this case with relations such as $K = \bar{K} = 1/3$. It is still useful to factor out an ϵ_0, which in our program we take to have a value of 1.6×10^{38} ergs/cm^3, as suggested by the value in Sec. IV A. Then, if we choose

$$\alpha = 3R_0 = 4.428\ \ {\rm km}, \tag{42}$$

we find

$$\beta = 3.374\ \ {\rm km}^{-3}. \tag{43}$$

We expect central pressures $\bar{p}(0)$ in this case to be greater than 10^{-4}. Other than these changes, our program is similar to the previous one, with care taken to avoid exponents such as $1/(\gamma-1)$.

Running our code gives, at first glance, enormous radii, values of R greater than 50 km. We can imagine the student looking frantically for a program bug that is not there. In fact, what really happens is that for this equation of state, the loop on $\bar{r}$ runs through its entire range, because the pressure $\bar{p}(r)$ never passes through zero. [A plot of $\bar{p}(r)$ looks quite similar, except for the distance scale, to that shown in Fig. 3, where $\gamma = 4/5$.] It only falls monotonically toward zero, becoming ever smaller. By the time students recognizes this behavior, they will probably also have realized that the relativistic gas equation of state is inappropriate for such small pressures. Something better should be done (as in Sec. IV C).

It turns out that the structure equations for $\gamma = 1$ are sufficiently simple that an analytic solution for $p(r)$ can be found, which corroborates the above remarks about not having a zero at a finite R. A suggestion for the student is to try a power law ansatz.

C. The Fermi gas equation of state for arbitrary relativity

To avoid the trap of the relativistic gas, we should find an equation of state for the noninteracting neutron Fermi gas that works for all values of the relativity parameter $x = k_F/m_n c$. By taking a hint from the two polytropes, we can try to fit the energy density as a function of pressure, each given as a transcendental function of k_F, with the form

$$\bar{\epsilon}(p) = A_{\rm NR}\bar{p}^{3/5} + A_{\rm R}\bar{p}. \tag{44}$$

For low pressures the nonrelativistic first term dominates over the second. [The power in the relativistic term is changed from that in Eq. (38).] It again is useful to factor out an ϵ_0 from both ϵ and p. In this case, it is more natural to define ϵ_0 as

$$\epsilon_0 = \frac{m_n^4 c^5}{(3\pi^2\hbar)^3} = 5.346\times 10^{36}\,{\rm ergs/cm}^3$$
$$= 0.003\,006 M_\odot c^2/{\rm km}^3. \tag{45}$$

Mathematica can easily create a table of exact $\bar{\epsilon}$ and $\bar{p}$ values as a function of k_F. The dimensionless values of A

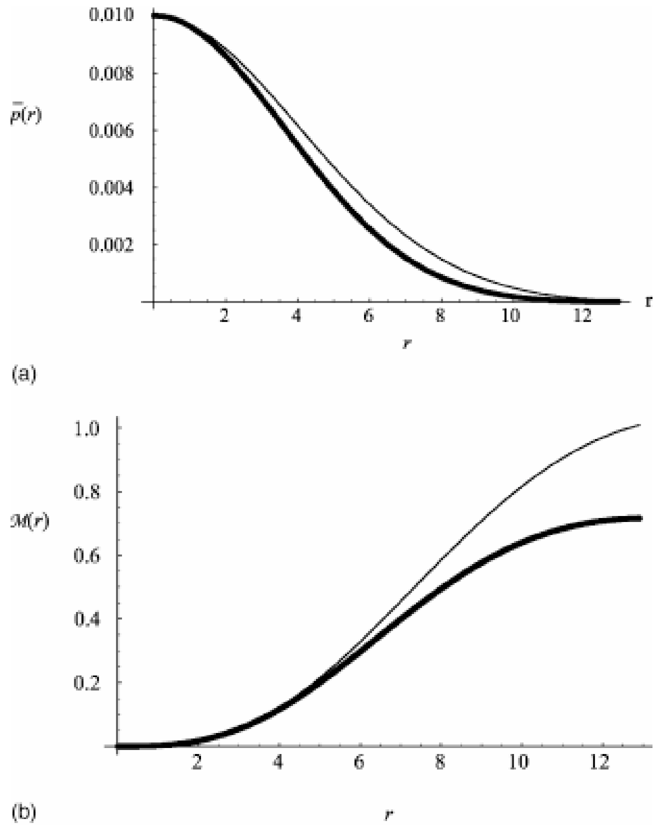

Fig. 5. The quantities (a) $\overline{p}(r)$ and (b) $\mathcal{M}(r)$ (in $M_\odot$) versus r (in km) for a pure neutron star with central pressure $\overline{p}(0)=0.01$, using the Fermi gas equation of state fit valid for all values of k_F, Eq. (44). The thin curves are results from the classical Newtonian structure equations, while the thick ones include general relativistic corrections.

can then be fit using Mathematica's intrinsic fitting function. From our efforts we found, to an accuracy of better than 1% over most of the range of k_F, that,[22]

$$A_{\mathrm{NR}}=2.4216 \quad \text{and} \quad A_{\mathrm{R}}=2.8663. \tag{46}$$

We used the fitted functional form for $\overline{\epsilon}$ of Eq. (44) in a Mathematica program similar to that for the neutron star based on the nonrelativistic equation of state. With the ϵ_0 of Eq. (45) and the choice $\alpha=R_0=1.476$ km, we obtained $\beta=0.037\,78$. Our results for a starting value of $\overline{p}(0)=0.01$, clearly in the relativistic regime, are

$$R=15.0, \quad M=1.037 \quad \text{(Newtonian equations)}, \tag{47}$$

$$R=13.4 \quad M=0.717. \quad \text{(full TOV equation)}. \tag{48}$$

For this more massive star, the general relativistic effects are significant (as should be expected from the size of GM/c^2R, and are about 10% in this case). Figure 5 displays the differences.

It is instructive to calculate M and R for a range of $\overline{p}(0)$ values. We display in Fig. 6 a (parametric) plot of M and R as a function of the central pressure. The low-mass/large-radius stars are to the right in the graph and correspond to small starting values of $\overline{p}(0)$. As the central pressure increases, the total mass that the star can support increases. And, the bigger the star mass, the bigger the gravitational attraction, which draws in the periphery of the star, making stars with smaller radii. That is, increasing $\overline{p}(0)$ corresponds to ''climbing the hill,'' moving upward and to the left in the diagram.

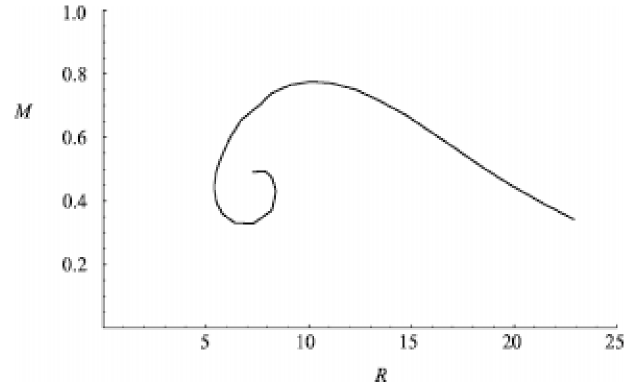

Fig. 6. The mass M (in $M_\odot$) and radius R (in km) for pure neutron stars, using a Fermi gas equation of state. The stars of low mass and large radius are solutions of the TOV equations for small values of central pressure $\overline{p}(0)$. The stars to the right of the maximum at $R=11$ are stable, while those to the left will suffer gravitational collapse.

At about $\overline{p}(0)=0.03$, the star reaches the top of the hill, achieving a maximum mass of about $0.8M_\odot$ at a radius of $R\approx 11$ km. This maximum value of M and its R agree with Oppenheimer and Volkov's seminal 1939 result for a Fermi gas equation of state.[5]

What about the solutions in Fig. 6 that are ''over the hill,'' that is, to the left of the maximum? It turns out that these stars are unstable against gravitational collapse into a black hole. The question of stability, however, is complicated,[23] perhaps too difficult for students at this level. The fact that things collapse to the left of the maximum, however, means that we probably should not worry too much about the peculiar tail on the $M-R$ curve in Fig. 6. It appears to be an artifact for very large values of $\overline{p}(0)$, as also is seen in other calculations, even though it is especially prominent for this Fermi gas equation of state.

D. Why is there a maximum mass?

One can argue on general grounds that cold compact objects such as white dwarfs and neutron stars must possess a limiting mass beyond which stable hydrostatic configurations are not possible. This limiting mass often is called the maximum mass of the object and was briefly mentioned at the end of Sec. II B and that the discussion relating to Fig. 6. In what follows, we outline the general argument.

The thermal component of the pressure in cold stars is by definition negligible. Thus, variations in both the energy density and pressure are caused only by changes in the density. Given this simple observation, let us examine why we expect a maximum mass in the Newtonian case.

An increase in the density results in a proportional increase in the energy density. This increase results in a corresponding increase in the gravitational attraction. To balance this increase, we require that the increment in pressure be large enough. However, the rate of change of the pressure with respect to the energy density is related to the speed of sound (see Sec. VI C). In a purely Newtonian world, this speed is in principle unbounded. However, the speed of all propagating signals cannot exceed the speed of light. This limit is a bound on the pressure increment associated with changes in density.

Once we accept this bound, we can safely conclude that all cold compact objects will eventually run into the situation in which any increase in density will result in an additional gravitational attraction that cannot be compensated by a corresponding increment in pressure, leading naturally to the existence of a limiting mass for the star.

When we include general relativistic corrections, as discussed in Sec. II B, they act to ''amplify'' gravity. Thus we can expect the maximum mass to occur at a somewhat lower mass than in the Newtonian case.

V. NEUTRON STARS WITH PROTONS AND ELECTRONS, FERMI GAS EQUATION OF STATE

As mentioned at the beginning of Sec. IV, neutron stars are not made only of neutrons, but also must include a small fraction of protons and electrons. The reason is that a free neutron will undergo a weak decay,

$$n \rightarrow p + e^- + \overline{\nu}_e , \tag{49}$$

with a lifetime of about 15 minutes. So, there must be something that prevents this decay for a star, and that is the presence of the protons and electrons.

The decay products have low energies $(m_n - m_p - m_e = 0.778\ \mathrm{MeV})$, with most of that energy being carried away by the light electron and (nearly massless) neutrino.[24] If all the available low-energy levels for the decay proton are already filled by the protons already present, then the Pauli exclusion principle takes over and prevents the decay from taking place.

The same might be said about the presence of the electrons, but in any case the electrons must be present within the star to cancel the positive charge of the protons. A neutron star is electrically neutral. We saw earlier that the number density of a particle species is fixed in terms of that particle's Fermi momentum [see Eq. (7)]. Thus equal numbers of electrons and protons implies that

$$k_{F,p} = k_{F,e} . \tag{50}$$

In addition to charge neutrality, we also require weak interaction equilibrium, that is, as many neutron decays [Eq. (49)] taking place as electron capture reactions, $p + e^- \rightarrow n + \nu_e$. This equilibrium can be expressed in terms of the chemical potentials for the three species,

$$\mu_n = \mu_p + \mu_e . \tag{51}$$

We already defined the chemical equilibrium for a particle after Eq. (12),

$$\mu_i(k_{F,i}) = \frac{d\epsilon}{dn_i} = (k_{F,i}^2 + m_i^2)^{1/2} \quad (i = n,p,e), \tag{52}$$

where, for the time being, we have set $c=1$ to simplify the equations somewhat. (The student is urged to prove the right-hand equality.) From Eqs. (50), (51), and (52), we can find a constraint that determines $k_{F,p}$ for a given $k_{F,n}$,

$$(k_{F,n}^2 + m_n^2)^{1/2} - (k_{F,p}^2 + m_p^2)^{1/2} - (k_{F,p}^2 + m_e^2)^{1/2} = 0. \tag{53}$$

Although an ambitious student can probably solve Eq. (53) for $k_{F,p}$ as a function of $k_{F,n}$, we were lazy and let Mathematica do it, and found

$$k_{F,p}(k_{F,n}) = \frac{[(k_{F,n}^2 + m_n^2 - m_e^2)^2 - 2m_p^2(k_{F,n}^2 + m_n^2 + m_e^2) + m_p^4]^{1/2}}{2(k_{F,n}^2 + m_n^2)^{1/2}} \tag{54}$$

$$\approx \frac{k_{F,n}^2 + m_n^2 - m_p^2}{2(k_{F,n}^2 + m_n^2)^{1/2}} \quad \text{for} \quad \frac{m_e}{k_{F,n}} \rightarrow 0. \tag{55}$$

The total energy density is the sum of the individual energy densities,

$$\epsilon_{\mathrm{tot}} = \sum_{i=n,p,e} \epsilon_i , \tag{56}$$

where

$$\epsilon_i(k_{F,i}) = \int_0^{k_{F,i}} (k^2 + m_i)^{1/2} k^2\, dk = \epsilon_0 \overline{\epsilon_i}(x_i, y_i), \tag{57}$$

and, as before,[25]

$$\epsilon_0 = m_n^4 / 3\pi^2 \hbar^3, \tag{58}$$

$$\overline{\epsilon_i}(x_i, y_i) = \int_0^{x_i} (u^2 + y_i^2)^{1/2} u^2\, du, \tag{59}$$

$$x_i = k_{F,i}/m_i , \quad y_i = m_i/m_n . \tag{60}$$

The corresponding total pressure is

$$p_{\mathrm{tot}} = \sum_{i=n,p,e} p_i , \tag{61}$$

$$p_i(k_{F,i}) = \int_0^{k_{F,i}} (k^2 + m_i)^{-1/2} k^4\, dk = \epsilon_0 \overline{p_i}(x_i, y_i), \tag{62}$$

$$\overline{p_i}(x_i, y_i) = \int_0^{x_i} (u^2 + y_i^2)^{-1/2} u^4\, du. \tag{63}$$

By using Mathematica the (dimensionless) integrals can be expressed in terms of log and $\sinh^{-1}$ functions of x_i and y_i. We can then generate a table of $\overline{\epsilon}_{\mathrm{tot}}$ versus $\overline{p}_{\mathrm{tot}}$ values for an appropriate range of $k_{F,n}$'s. These values can be fitted to the same sum of two terms as in Eq. (44). We found the coefficients to be

$$A_{\mathrm{NR}} = 2.572 \quad \text{and} \quad A_{\mathrm{R}} = 2.891. \tag{64}$$

These coefficients are not much changed from those in Eq. (46) for the pure neutron star. Therefore, we expect that the M versus R diagram for this more realistic Fermi gas model would not be much different from that in Fig. 6.

VI. INTRODUCING NUCLEAR INTERACTIONS

Nucleon–nucleon interactions can be included in the equation of state (they are important) by constructing a simple model for the nuclear potential that reproduces the general features of (normal) nuclear matter. In so doing we were much guided by Ref. 7.

We will use MeV and fm (10^{-13} cm) as the energy and distance units for much of this section, converting back to $M_\odot$ and km later. We also will continue setting $c=1$. In this regard, the important number to remember for making conversions is $\hbar c = 197.3$ MeV fm. We also will neglect the mass difference between protons and neutrons, and label their masses as m_N.

The von Weizäcker mass formula[26] for nuclides with Z protons and N neutrons gives, for normal symmetric nuclear matter ($A = N + Z$ with $N = Z$), an equilibrium number density n_0 of 0.16 nucleons/fm^3. For this value of n_0 the Fermi momentum is $k_F^0 = 263$ MeV/c [see Eq. (7)]. This momentum is sufficiently small compared with $m_N = 939$ MeV/c^2 to allow a nonrelativistic treatment of normal nuclear matter. At this density, the average binding energy per nucleon, BE $= E/A - m_N$, is -16 MeV.

The equilibrium density and the binding energy per nucleon are two physical quantities we definitely want our nuclear potential to respect, but there are two more that we will need to fix the parameters of the model.

We choose one of these as the nuclear compressibility, K_0, to be defined below. The magnitude of this quantity is not that well established, but is in the range of 200 to 400 MeV. The other is the symmetry energy term, which, when $Z = 0$, contributes about 30 MeV of energy above the symmetric matter minimum at n_0. (This quantity might really be better described as an asymmetry parameter, because it accounts for the energy that arises when $N \neq Z$.)

A. Symmetric nuclear matter

We defer the case when $N \neq Z$, which is our main interest in this paper, to Sec. VI B. Here we concentrate on obtaining a good (enough) equation of state for nuclear matter when $N = Z$, or, equivalently, when the proton and neutron number densities are equal, $n_n = n_p$. The total nucleon density $n = n_n + n_p$.

We need to relate the three nuclear quantities, n_0, BE, and K_0 to the energy density for symmetric nuclear matter, $\epsilon(n)$. Here $n = n(k_F)$ is the nuclear density (at and away from n_0). We will not worry in this section about the electrons that are present, because, as was seen in Sec. V, their contribution is small. The energy density now will include the nuclear potential, $V(n)$, which we will model in terms of two simple functions with three parameters that are fitted to reproduce the above three nuclear quantities. [The fourth quantity, the symmetry energy, will be used in Sec. VI B to fix a term in the potential that is proportional to $(N-Z)/A$.]

First, the average energy per nucleon, E/A, for symmetric nuclear matter is related to ϵ by

$$E(n)/A = \epsilon(n)/n, \tag{65}$$

which includes the rest mass energy, m_N, and has units of MeV. As a function of n, $E(n)/A - m_N$ has a minimum at $n = n_0$ with a depth BE $= -16$ MeV. This minimum occurs when

$$\frac{d}{dn}\left(\frac{E(n)}{A}\right) = \frac{d}{dn}\left(\frac{\epsilon(n)}{n}\right) = 0 \quad \text{at } n = n_0. \tag{66}$$

Equation (66) is one constraint on the parameters of $V(n)$. Another, of course, is the binding energy,

$$\frac{\epsilon(n)}{n} - m_N = \text{BE} \quad \text{at } n = n_0. \tag{67}$$

The positive curvature at the bottom of this valley is related to the nuclear (in)compressibility by[27]

$$K(n) = 9\,\frac{dp(n)}{dn} = 9\left[n^2 \frac{d^2}{dn^2}\left(\frac{\epsilon}{n}\right) + 2n\,\frac{d}{dn}\left(\frac{\epsilon}{n}\right)\right], \tag{68}$$

where we have used Eq. (12), which defines the pressure in terms of the energy density. At $n = n_0$ this quantity equals K_0. (The factor of 9 is a historical artifact from the convention originally defining K_0.) (Question: why does one *not* have to calculate the pressure at $n = n_0$?)

We will model $\epsilon(n)$ for the $N = Z$ part of the potential by[7]

$$\frac{\epsilon(n)}{n} = m_N + \frac{3}{5}\,\frac{\hbar^2 k_F^2}{2m_N} + \frac{A}{2}u + \frac{B}{\sigma+1}u^{\sigma}, \tag{69}$$

where $u = n/n_0$ and σ are dimensionless and A and B have units of MeV. The first term represents the rest mass energy and the second the average kinetic energy per nucleon. [These two terms are dominant in the nonrelativistic limit of the nucleonic version of Eq. (10).] For $k_F(n_0) = k_F^0$ we will abbreviate the kinetic energy term as $\langle E_F^0 \rangle$, which evaluates to 22.1 MeV. The kinetic energy term in Eq. (69) can be better written as $\langle E_F^0 \rangle u^{2/3}$.

From the three constraints, Eqs. (66)–(68), and the fact that $u = 1$ at $n = n_0$, we obtain three equations for the parameters A, B, and σ,

$$\langle E_F^0 \rangle + \frac{A}{2} + \frac{B}{\sigma+1} = \text{BE}, \tag{70}$$

$$\frac{2}{3}\langle E_F^0 \rangle + \frac{A}{2} + \frac{B\sigma}{\sigma+1} = 0, \tag{71}$$

$$\frac{10}{9}\langle E_F^0 \rangle + A + B\sigma = \frac{K_0}{9}. \tag{72}$$

By solving these three equations (which we found easier to do by hand than with Mathematica), we found

$$\sigma = \frac{K_0 + 2\langle E_F^0 \rangle}{3\langle E_F^0 \rangle - 9\,\text{BE}}, \tag{73}$$

$$B = \frac{\sigma+1}{\sigma-1}\left[\frac{1}{3}\langle E_F^0 \rangle - \text{BE}\right], \tag{74}$$

$$A = \text{BE} - \frac{5}{3}\langle E_F^0 \rangle - B. \tag{75}$$

For $K_0 = 400$ MeV (which is perhaps a high value),

$$A = -122.2 \ \text{MeV}, \quad B = 65.39 \ \text{MeV}, \quad \sigma = 2.112. \tag{76}$$

Note that $\sigma > 1$, which violates a basic principle of physics called "causality," a point that we will discuss in the following.

The student can try other values of K_0 to see how the parameters A, B, and σ change. More interesting is to see how the interplay between the A and B terms gives the valley at $n = n_0$. Figure 7 shows $E/A - m_N$ as a function of n using the parameters of Eq. (76). We hope students notice the funny little positive bump in this plot near $n = 0$ and sort out the reason for its occurrence.

Given $\epsilon(n)$ from Eq. (69), we can find the pressure using Eq. (12),

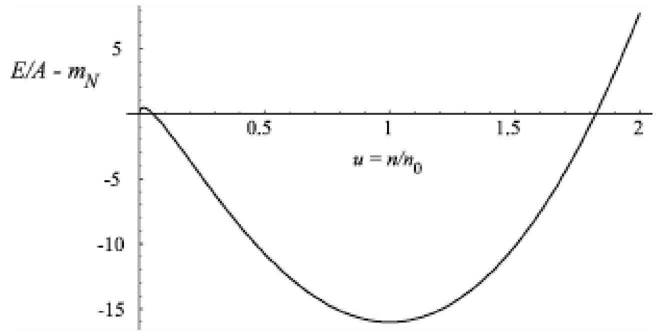

Fig. 7. The average energy per nucleon, less its rest mass, as a function of $u=n/n_0$ (in MeV). The position of the minimum is at $n=n_0=0.16\ \mathrm{fm}^{-3}$, its depth there is BE $=-16$ MeV, and its curvature (second derivative) corresponds to the nuclear compressibility $K_0=400$ MeV.

$$p(n)=n^2\frac{d}{dn}\left(\frac{\epsilon}{n}\right)=n_0\left[\frac{2}{3}\langle E_F^0\rangle u^{5/3}+\frac{A}{2}u^2+\frac{B\sigma}{\sigma+1}u^{\sigma+1}\right]. \tag{77}$$

For the parameters of Eq. (76) the dependence of $p(n)$ on n is shown in Fig. 8. On first seeing this graph, students should wonder why $p(u=1)=p(n_0)=0$. Also, what is the meaning of the negative values for pressure below $u=1$? (Hint: what is "cavitation"?)

If this $N=Z$ case were all we had for the nuclear equation of state, a plot of $\epsilon(n)$ versus $p(n)$ would only make sense for $n\geqslant n_0$. Such a plot looks much like a parabola opening to the right for the range $0<u<3$. At very large values of u, however, $\epsilon\approx p/3$, as it should for a relativistic nucleon gas (see Sec. IV B). We will not pursue this symmetric nuclear matter equation of state further because our interest is in the case when $N\gg Z$.[28]

B. Nonsymmetric nuclear matter

We continue following Ref. 7 closely. Let us represent the neutron and proton densities in terms of the parameter α as

$$n_n=\frac{1+\alpha}{2}n, \quad n_p=\frac{1-\alpha}{2}n. \tag{78}$$

This α is not to be confused with the constant defined in Eq. (24). For pure neutron matter $\alpha=1$. Note that

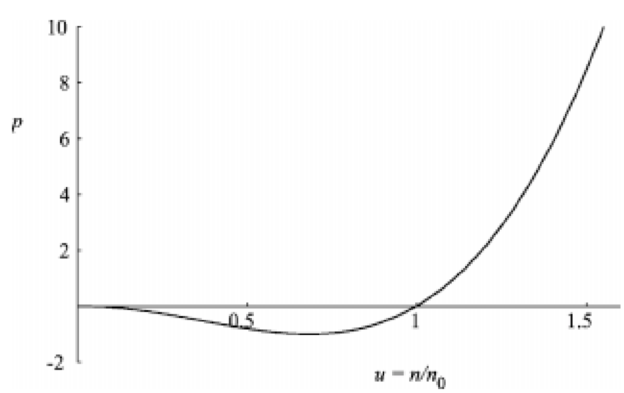

Fig. 8. The pressure for symmetric nuclear matter as a function of $u=n/n_0$. The student should ask what it means when the pressure is negative and why it is 0 at $u=1$.

$$\alpha=\frac{n_n-n_p}{n}=\frac{N-Z}{A}, \tag{79}$$

so we can expect that the isospin-symmetry-breaking interaction is proportional to α (or some power of it). An alternative notation is in terms of the fraction of protons in the star,

$$x=\frac{n_p}{n}=\frac{1-\alpha}{2}. \tag{80}$$

We now consider how the energy density changes from the symmetric case discussed above, where $\alpha=0$ (or $x=1/2$).

First, there are contributions to the kinetic energy part of ϵ from both neutrons and protons,

$$\epsilon_{\mathrm{KE}}(n,\alpha)=\frac{3}{5}\frac{k_{F,n}^2}{2m_N}n_n+\frac{3}{5}\frac{k_{F,p}^2}{2m_N}n_p$$
$$=n\langle E_F\rangle\frac{1}{2}[(1+\alpha)^{5/3}+(1-\alpha)^{5/3}], \tag{81}$$

where

$$\langle E_F\rangle=\frac{3}{5}\frac{\hbar^2}{2m_N}\left(\frac{3\pi^2 n}{2}\right)^{2/3} \tag{82}$$

is the mean kinetic energy of symmetric nuclear matter at density n. For $n=n_0$ we note that $\langle E_F\rangle=3\langle E_F^0\rangle/5$ [see Eq. (69)]. For nonsymmetric matter, $\alpha\neq 0$, the excess kinetic energy is

$$\Delta\epsilon_{\mathrm{KE}}(n,\alpha)=\epsilon_{\mathrm{KE}}(n,\alpha)-\epsilon_{\mathrm{KE}}(n,0)$$
$$=n\langle E_F\rangle\{\tfrac{1}{2}[(1+\alpha)^{5/3}+(1-\alpha)^{5/3}]-1\}$$
$$=n\langle E_F\rangle\{2^{2/3}[(1-x)^{5/3}+x^{5/3}]-1\}. \tag{83}$$

For pure neutron matter, $\alpha=1$, and

$$\Delta\epsilon_{\mathrm{KE}}(n,\alpha)=n\langle E_F\rangle(2^{2/3}-1). \tag{84}$$

It is useful to expand Eq. (84) to leading order in α,

$$\Delta\epsilon_{\mathrm{KE}}(n,\alpha)=n\langle E_F\rangle\frac{5}{9}\alpha^2\left(1+\frac{\alpha^2}{27}+\cdots\right)$$
$$=nE_F\frac{\alpha^2}{3}\left(1+\frac{\alpha^2}{27}+\cdots\right). \tag{85}$$

Keeping terms to order α^2 is evidently good enough for most purposes. For pure neutron matter, the energy per particle (which is ϵ/n) at normal density is $\Delta\epsilon_{\mathrm{KE}}(n_0,1)/n_0\approx 13$ MeV, more than a third of the total bulk symmetry energy of 30 MeV, our fourth nuclear parameter.

Thus the potential energy contribution to the bulk symmetry energy must be ≈ 20 MeV. Let us assume the quadratic approximation in α also works well enough for this potential contribution and write the total energy per particle as

$$E(n,\alpha)=E(n,0)+\alpha^2 S(n). \tag{86}$$

The isospin-symmetry breaking is proportional to α^2, which reflects (roughly) the pairwise nature of the nuclear interactions.

We will assume $S(u)$, $u=n/n_0$, has the form

$$S(u)=(2^{2/3}-1)\,\tfrac{3}{5}\langle E_F^0\rangle(u^{2/3}-F(u))+S_0F(u). \tag{87}$$

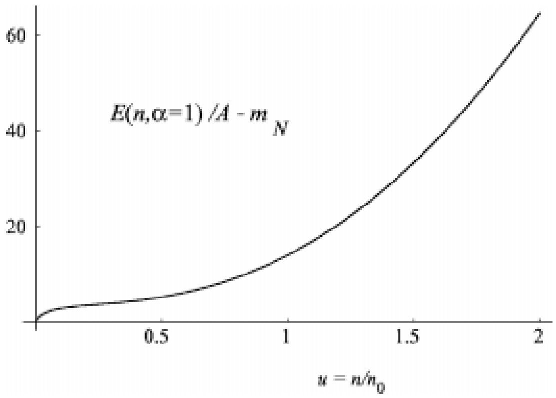

Fig. 9. The average energy per neutron (less its rest mass), in MeV, for pure neutron matter, as a function of $u=n/n_0$. The parameters for this curve are for a nuclear compressibility K_0 of 400 MeV.

Here $S_0=30$ MeV is the bulk symmetry energy parameter. The function $F(u)$ must satisfy $F(1)=1$ [so that $S(u=1)=S_0$] and $F(0)=0$ [so that $S(u=0)=0$; no matter means no energy]. Besides these two constraints, there is, from what we presently know, much freedom in what we may choose for $F(u)$. We will make the simplest possible choice here, namely,

$$F(u)=u, \tag{88}$$

but we encourage students to try other forms that satisfy the conditions on $F(u)$, such as $\sqrt{u}$, to see what difference it makes.

Figure 9 shows the energy per particle for pure neutron matter, $E(n,1)-m_N$, as a function of u for the parameters of Eq. (76) and $S_0=30$ MeV. In contrast with the $\alpha=0$ plot in Fig. 7, $E(n,1)\geqslant 0$ and is monotonically increasing. The plot looks almost quadratic as a function of u. The dominant term at large u goes like u^{σ} with $\sigma=2.112$ (for this case). However, we might have expected a linear increase instead. We will return to this point in Sec. VI C.

Given the energy density, $\epsilon(n,\alpha)=n_0 uE(n,\alpha)$, the corresponding pressure is, from Eq. (12),

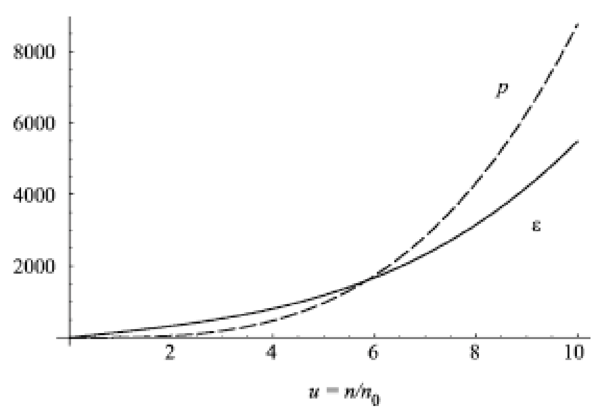

Fig. 10. The pressure (dashed curve) and energy density (solid) for pure neutron matter, as a function of $u=n/n_0$. The units for the y axis are MeV/fm^3. This curve uses parameters based on a nuclear compressibility $K_0=400$ MeV.

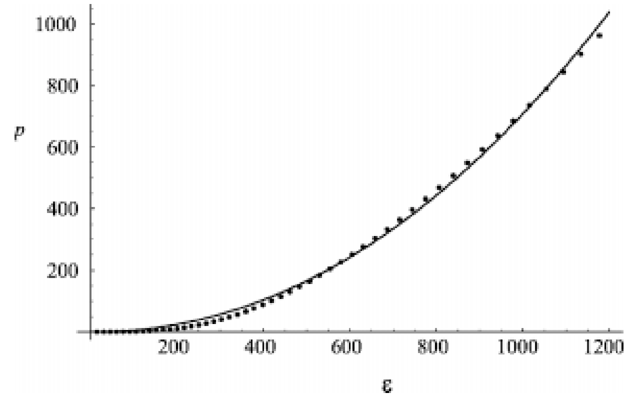

Fig. 11. The equation of state for pure neutron matter ($\alpha=1$), that is, the dependence of pressure versus energy density. The units for both axes are MeV/fm^3, and the nuclear compressibility in this case is $K_0=400$ MeV. The points are values calculated directly from Eq. (86), multiplied by n, and Eq. (89), while the solid curve is a fit to these points given in Eqs. (90) and (91).

$$p(n,x)=u\frac{d}{du}\epsilon(n,\alpha)-\epsilon(n,\alpha)$$
$$=p(n,0)+n_0\alpha^2\left[\frac{2^{2/3}-1}{5}\langle E_F^0\rangle(2u^{5/3}-3u^2)+S_0u^2\right], \tag{89}$$

where $p(n,0)$ is defined by Eq. (77). Figure 10 shows the dependence of the pure neutron $p(n,1)$ and $\epsilon(n,1)$ on $u=n/n_0$, ranging from 0 to 10 times normal nuclear density. Both functions increase smoothly and monotonically from $u=0$. We hope students will wonder why the pressure becomes greater than the energy density around $u=6$. Why doesn't it go like a relativistic nucleon gas, $p=\epsilon/3$? (Hint: check the assumptions.)

We can now look at the equation of state, that is, the dependence of p on ϵ (the points in Fig. 11). The pressure is smooth, non-negative, and monotonically increasing as a function of ϵ. In fact it looks almost quadratic over this energy range ($0\leqslant u\leqslant 5$). This behavior suggests that it might be reasonable to see if we can make a simple, polytropic fit. If we assume the form

$$p(\epsilon)=\kappa_0\epsilon^{\gamma}, \tag{90}$$

we find the fit shown in Fig. 11 as the solid curve with

$$\kappa_0=3.548\times 10^{-4} \quad \text{and} \quad \gamma=2.1, \tag{91}$$

where κ_0 has appropriate units so that p and ϵ are in MeV/fm^3. (We simply guessed and set $\gamma=2.1$.)

This polytrope can now be used in solving the TOV equation for a pure neutron star with nuclear interactions. Alternatively, we might solve for the structure by using the functional forms from Eq. (86), multiplied by n, and Eq. (89) directly. We defer that for a bit, because it would be a good idea to first find an equation of state that does not violate causality, a basic tenet of special relativity.

C. Does the speed of sound exceed that of light?

What is the speed of sound in nuclear matter? If we start from the elementary formula for the square of the speed of sound in terms of the bulk modulus,[29] we can show that

$$\left(\frac{c_s}{c}\right)^2 = \frac{B}{\rho c^2} = \frac{dp}{d\epsilon} = \frac{dp/dn}{d\epsilon/dn}. \tag{92}$$

To satisfy relativistic causality, we must require that the sound speed does not exceed that of light, which could happen when the density becomes very large, that is, when $u \to \infty$. For the simple model of nuclear interactions presented in Sec. V, the dominant terms at large u in p and ϵ are those going like $u^{\sigma+1}$. Thus, from Eq. (86), multiplied by n, and Eq. (89), we see that

$$\left(\frac{c_s}{c}\right)^2 = \frac{dp/dn}{d\epsilon/dn} \to \sigma = 2.11 \tag{93}$$

for the parameters of Eq. (76), and indeed for any set of parameters with K_0 greater than about 180 MeV.

We can recover causality by assuring that both $\epsilon(u)$ and $p(u)$ grow no faster than u^2. There must still be an interplay between the A and B terms in the nuclear potential, but one simple way of recovering causality is to modify the B term by introducing a fourth parameter C so that, for symmetric nuclear matter ($\alpha=0$),

$$V_{\text{nuc}}(u,0) = \frac{A}{2}u + \frac{B}{\sigma+1}\,\frac{u^\sigma}{1+Cu^{\sigma-1}}. \tag{94}$$

We can choose C small enough so that the effect of the denominator only becomes appreciable for very large u. The presence of the denominator would modify and complicate the constraint equations for A, B, and σ from those given in Eqs. (70)–(72). However, for small C, which can be chosen as we wish, the values of the other parameters should not be much changed from those, say, in Eq. (76). Thus, with a little bit of trial and error, we can simply readjust the A, B, and σ values to put the minimum of $E/A - m_N$ at the right position (n_0) and depth (BE), hoping that the resulting value of the (poorly known) compressibility K_0 remains sensible.

In our calculations we choose $C=0.2$ and start the search by hand with the $K_0=400$ MeV parameters in Eq. (76). We found that by fiddling only with B and σ, we could fit n_0 and B with only small changes,

$$B = 65.39 \to 83.8\ \text{MeV}, \quad \sigma = 2.11 \to 2.37. \tag{95}$$

For these new values of B and σ, A changes from -122.2 MeV to -136.7 MeV, and K_0 from 400 to 363.2 MeV. That is, it remains a reasonable nuclear model.

We can now proceed as in Sec. V to obtain $\epsilon(n,\alpha)$, $p(n,\alpha)$, and the equation of state, $p(\epsilon,\alpha)$. The results are not much different from those shown in Fig. 11. This time we decided to live with a quadratic fit for the equation of state for pure neutron matter, and found

$$p(\epsilon,1) = \kappa_0 \epsilon^2 \quad \text{and} \quad \kappa_0 = 4.012 \times 10^{-4}. \tag{96}$$

This result is not much different from before, Eq. (91). Somewhat more useful for solving the TOV equation is to express ϵ in terms of p,

$$\epsilon(p) = (p/\kappa_0))^{1/2}. \tag{97}$$

D. Pure neutron star with nuclear interactions

With all this groundwork, students can now proceed to solve the TOV equations as before for a pure neutron star, using the fit for $\epsilon(p)$ found in Sec. VI C. It is, once again, useful to convert first from units of MeV/fm^3 to ergs/cm^3, then to $M_\odot$/km^3 and, finally, to dimensionless $\bar{p}$ and $\bar{\epsilon}$.

$$\bar{\epsilon}(\bar{p}) = (\kappa_0 \epsilon_0)^{-1/2} \bar{p}^{1/2} = A_0 \bar{p}^{1/2}, \quad A_0 = 0.8642, \tag{98}$$

where this time we have defined

$$\epsilon_0 = \frac{m_n^4 c^5}{3\pi^2 \hbar^3}. \tag{99}$$

The constant α that occurs on the right-hand side of the TOV equation, Eq. (23), is $\alpha = A_0 R_0 = 1.276$ km. The constant for the mass equation, Eq. (26), is $\beta = 0.032\,65$, again in units of 1/km^3.

Now proceeding as before, we can solve the coupled TOV equations for $\bar{p}(r)$ and $\bar{\mathcal{M}}(r)$ for various initial central pressures, $\bar{p}(0)$. The plots of the solutions are very similar to those for the Fermi gas equation of state, Fig. 5.

More interesting is to solve for a range of initial $\bar{p}(0)$'s, generating, as before, a mass M versus radius R plot that now includes nucleon–nucleon interactions (Fig. 12). The effect of the nuclear potential is enormous as seen by comparing with the Fermi gas model predictions for M versus R shown in Fig. 6. The maximum star mass is now about $2.3 M_\odot$, rather than $0.8 M_\odot$. The radius for this maximum mass star is about 13.5 km, somewhat larger than the Fermi gas model radius of 11 km. The large value of maximum M is a reflection of the large value of the nuclear (in)compressibility $K_0 = 363$ MeV. The more incompressible something is, the more mass it can support. If we had fit to a smaller value of K_0, we would have obtained a smaller maximum mass.

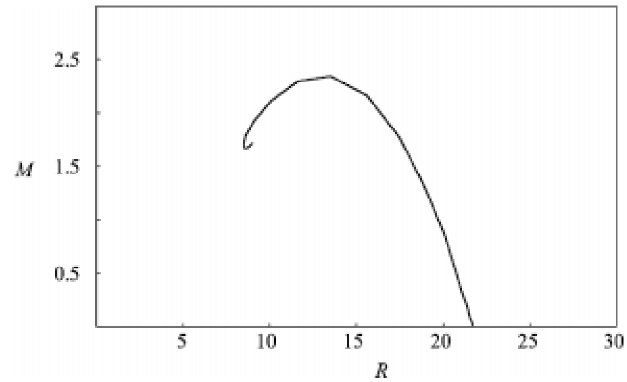

Fig. 12. The mass M (in $M_\odot$) and radius R (in km) for pure neutron stars using an equation of state that contains nucleon–nucleon interactions. Only those stars to the right of the maximum are stable against gravitational collapse. Compare this graph with that in Fig. 6 which is based on a non-interacting Fermi gas model for the equation of state.

E. What about a cosmological constant?

We do not know if there is a cosmological constant, but there are definite indications that much of our universe is something called "dark energy."[30] This conclusion comes about because we have recently learned that something is causing the universe to be accelerating instead of slowing down (as would be expected after the Big Bang).

One way to interpret this dark energy is in terms of Einstein's cosmological constant, which contributes a term

$\Lambda g_{\mu\nu}$ to the right-hand side of Einstein's field equation, the basic equation of general relativity. The most natural value for Λ would be zero, but that may not be the way the world is. If Λ is nonzero, it is nonetheless surprisingly small. What would the effect of a nonzero cosmological constant be on the structure of a neutron star? It turns out that the only modification to the TOV equation[31] is the correction factor

$$\left[1+\frac{4\pi r^3 p(r)}{\mathcal{M}(r)c^2}\right] \rightarrow \left[1+\frac{4\pi r^3 p(r)}{\mathcal{M}(r)c^2}-\frac{\Lambda r^3}{2G\mathcal{M}(r)}\right]. \quad (100)$$

We encourage students to first understand the units of Λ and then to see what values of Λ might affect the structure of a typical neutron star.

VII. CONCLUSIONS

The materials we have described would be suitable as an undergraduate thesis or special topics course accessible to junior or senior physics majors. It is a topic rich in the subjects students will have covered in their courses, ranging from thermodynamics to statistical mechanics to nuclear physics. The major emphasis of such a project would be to construct a (simple) equation of state. The latter is needed to solve the nonlinear structure equations. The numerical solution of these equations would develop the students' computational skills. Along the way, they also will learn some of the lore regarding degenerate stars, for example, white dwarfs and neutron stars. And, in the latter case, they also will come to appreciate the relative importance of special and general relativity.

ACKNOWLEDGMENTS

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[1]A. Hewish, S. J. Bell, J. D. H. Pilkington, P. F. Scott, and R. A. Collins, ''Observation of a rapidly pulsating radio source,'' Nature (London) **217**, 709–713 (1968).

[2]It is widely believed that neutron stars were proposed by Lev Landau in 1932, very soon after the neutron was discovered (although we are not aware of any documented evidence for this belief). In 1934 Fritz Zwicky and Walter Baade speculated that they might be formed in Type II supernova explosions, which is now generally accepted as true.

[3]An online catalog of pulsars can be found at ⟨http:/pulsar.princeton.edu⟩

[4]An intermediate-level online tutorial on the physics of pulsars can be found at ⟨http://www.jb.man.ac.uk/research/pulsar⟩. This tutorial follows the book by Andrew G. Lyne and Francis Graham-Smith, *Pulsar Astronomy*, 2nd ed. (Cambridge University Press, Cambridge, MA, 1998).

[5]R. C. Tolman, ''Static solutions of Einstein's field equations for spheres of fluid,'' Phys. Rev. **55**, 364–373 (1939); J. R. Oppenheimer and G. M. Volkov, ''On massive neutron cores,'' *ibid.* **55**, 374–381 (1939).

[6]Steven Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), Chap. 11.

[7]M. Prakash, ''Equation of state,'' in *The Nuclear Equation of State*, edited by A. Ausari and L. Satpathy (World Scientific, Singapore, 1996).

[8]See, for example, M. A. Preston, *Physics of the Nucleus* (Addison-Wesley, Reading, MA, 1962), Sec. 9-3. A more recent, fuller discussion of nuclear compressibility is given by J. P. Blaizot, ''Nuclear compressibilities,'' Phys. Rep. **64**, 171 (1980).

[9]If you are a mentor for such a student, you may want to see some of the Mathematica and MathCad files we developed. Send email to the first author convincing him that you are such a mentor, and he will direct you to a web site from which they can be downloaded. The idea behind this misdirection is that students will learn more by doing the programming themselves.

[10]R. Balian and J.-P. Blaizot, ''Stars and statistical physics: A teaching experience,'' Am. J. Phys. **67**, 1189–1206 (1999).

[11]S. L. Shapiro and S. A. Teukolsky, *Black Holes, White Dwarfs and Neutron Stars: The Physics of Compact Objects* (Wiley-Interscience, New York, 1983).

[12]We apologize to readers who are enthusiasts about SI units, but the first author was raised on CGS units. Also, much of today's astrophysical literature still uses CGS units. Besides, we strongly believe that by the time physics students are at this level, they should be comfortable in switching from one system of units to another.

[13]A discussion of how to solve these equations (using conventional programming languages) is given in S. Koonin, *Computational Physics* (Benjamin-Cummings, New York, 1986).

[14]For more details on white dwarfs, NASA provides a useful web page at ⟨http://imagine.gsfc.nasa.gov/docs/science/knowl1/dwarfs.html⟩

[15]This maximum mass of $1.4M_\odot$ is usually referred to as the Chandrasekhar limit. See S. Chandrasekhar's 1983 Nobel Prize lecture, ⟨http://www.nobel.se/physics/laureates/1983/⟩. For more detail see his treatise, *An Introduction to the Study of Stellar Structure* (Dover, New York, 1939).

[16]See, for example, C. Kittel and H. Kroemer, *Thermal Physics* (W. H. Freeman, San Francisco, CA, 1980). This result is from their modern physics course that students should review if they do not remember it.

[17]Mathematica is product of Wolfram Research, ⟨http://www.wolfram.com⟩, and its use is described by S. Wolfram, in *The Mathematica Book*, 4th ed. (Cambridge University Press, Cambridge, 1999). Whenever we use the phrase ''using Mathematica,'' we really mean using whatever package one has available or is familiar with, be it Maple, MathCad, or whatever. We did almost all of the numerical/symbolic work that we describe in this paper in Mathematica, but some of its notebooks were duplicated in MathCad.

[18]Enough of these explicit flags! Most of the equations from here on present challenges for the student to work through.

[19]For the Newtonian case, a polytropic equation of state also allows for a somewhat more analytic solution in terms of Lane-Emden functions. See Ref. 6, Sec. 11.3, or C. Flynn, *Lectures on Stellar Physics*, ⟨http://www.astro.utu.fi/~cflynn/Stars/⟩, especially Lectures 4 and 5.

[20]Despite the appearance of the $4\pi\epsilon_0$, the astute student will not be lulled into thinking that this factor has anything to do with a Coulomb potential or the dielectric constant of the vacuum.

[21]Note that the right-hand side of Eq. (23) is negative (for positive $\bar{p}$), so $\bar{p}(r)$ must fall monotonically from $\bar{p}(0)$.

[22]This fit is least accurate ($\approx 2\%$) at very low values of k_F. However, this is where the pure neutron approximation itself is least accurate. The surface of a neutron star is likely made of elements like iron. A fictional account of what life might be like on such a surface can be found in Robert Forward, *Dragon's Egg*, first published in 1981 by Del Rey Publishing and republished in 2000.

[23]See Ref. 6, Sec. 11.2.

[24]Because it is almost noninteracting with nuclear matter, a neutrino tends to escape from the neutron star. This escape is the major cooling mechanism as the neutron star is being formed in a supernova explosion. George Gamow named this mechanism the URCA process, the name of a Brazilian casino where people lost a lot of money.

[25]Does the student know how to put all the factors of c back into ϵ_0 so as to rewrite this equation in CGS units?

[26]See, for example, J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (Wiley, New York, 1952), Chap. 6, Sec. 2.

[27]The reason for the ''(in)'' is because a materials physicist might rather define compressibility as $\chi=-(1/V)(\partial V/\partial p)=-(1/n)(dp/dn)^{-1}$.

[28]People interested in RHIC physics might want to, however. (RHIC stands for Relativistic Heavy Ion Collider, an accelerator at the Brookhaven National Laboratory which is studying reactions like Au nuclei striking each other at center of mass energies around 200 GeV/nucleon.)

[29]See, for example, Hugh Young, *University Physics*, 8th ed. (Addison-Wesley, Reading MA, 1992), Sec. 19-5.

[30]See, for example, P. J. E. Peebles and Bharat Ratra, ''The cosmological constant and dark energy,'' Rev. Mod. Phys. **75**, 559–606 (2003).

[31]W. Y. Pauchy Hwang (private communication).