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Neutrosophic Quadruple *BCK*/*BCI*-Algebras

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Abstract: The notion of a neutrosophic quadruple BCK/BCI-number is considered, and a neutrosophic quadruple BCK/BCI-algebra, which consists of neutrosophic quadruple BCK/BCI-numbers, is constructed. Several properties are investigated, and a (positive implicative) ideal in a neutrosophic quadruple BCK-algebra and a closed ideal in a neutrosophic quadruple BCI-algebra are studied. Given subsets A and B of a BCK/BCI-algebra, the set $NQ(A, B)$, which consists of neutrosophic quadruple BCK/BCI-numbers with a condition, is established. Conditions for the set $NQ(A, B)$ to be a (positive implicative) ideal of a neutrosophic quadruple BCK-algebra are provided, and conditions for the set $NQ(A, B)$ to be a (closed) ideal of a neutrosophic quadruple BCI-algebra are given. An example to show that the set $\{\tilde{0}\}$ is not a positive implicative ideal in a neutrosophic quadruple BCK-algebra is provided, and conditions for the set $\{\tilde{0}\}$ to be a positive implicative ideal in a neutrosophic quadruple BCK-algebra are then discussed.

Keywords: neutrosophic quadruple BCK/BCI-number; neutrosophic quadruple BCK/BCI-algebra; neutrosophic quadruple subalgebra; (positive implicative) neutrosophic quadruple ideal

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1. Introduction

The notion of a neutrosophic set was developed by Smarandache [1–3] and is a more general platform that extends the notions of classic sets, (intuitionistic) fuzzy sets, and interval valued (intuitionistic) fuzzy sets. Neutrosophic set theory is applied to a different field (see [4–8]). Neutrosophic algebraic structures in BCK/BCI-algebras are discussed in [9–16]. Neutrosophic quadruple algebraic structures and hyperstructures are discussed in [17,18].

In this paper, we will use neutrosophic quadruple numbers based on a set and construct neutrosophic quadruple BCK/BCI-algebras. We investigate several properties and consider ideals and positive implicative ideals in neutrosophic quadruple BCK-algebra, and closed ideals in neutrosophic quadruple BCI-algebra. Given subsets A and B of a neutrosophic quadruple BCK/BCI-algebra, we consider sets $NQ(A, B)$, which consist of neutrosophic quadruple BCK/BCI-numbers with a condition. We provide conditions for the set $NQ(A, B)$ to be a (positive implicative) ideal of a neutrosophic quadruple BCK-algebra and for the set $NQ(A, B)$ to be a (closed) ideal of a neutrosophic quadruple BCI-algebra. We give an example to show that the set $\{\tilde{0}\}$ is not a positive implicative ideal in a neutrosophic quadruple BCK-algebra, and we then consider conditions for the set $\{\tilde{0}\}$ to be a positive implicative ideal in a neutrosophic quadruple BCK-algebra.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by Iséki (see [19,20]).

By a BCI-algebra, we mean a set X with a special element 0 and a binary operation $*$ that satisfies the following conditions:

(I) $(\forall x,y,z \in X)\ (((x*y)*(x*z))*(z*y)=0)$;
(II) $(\forall x,y \in X)\ ((x*(x*y))*y=0)$;
(III) $(\forall x \in X)\ (x*x=0)$;
(IV) $(\forall x,y \in X)\ (x*y=0,\ y*x=0\ \Rightarrow\ x=y)$.

If a BCI-algebra X satisfies the identity

(V) $(\forall x \in X)\ (0*x=0)$,

then X is called a *BCK-algebra*. Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\ (x*0=x) \tag{1}$$
$$(\forall x,y,z \in X)\ (x \leq y\ \Rightarrow\ x*z \leq y*z,\ z*y \leq z*x) \tag{2}$$
$$(\forall x,y,z \in X)\ ((x*y)*z=(x*z)*y) \tag{3}$$
$$(\forall x,y,z \in X)\ ((x*z)*(y*z) \leq x*y) \tag{4}$$

where $x \leq y$ if and only if $x*y=0$. Any BCI-algebra X satisfies the following conditions (see [21]):

$$(\forall x,y \in X)(x*(x*(x*y))=x*y), \tag{5}$$
$$(\forall x,y \in X)(0*(x*y)=(0*x)*(0*y)). \tag{6}$$

A BCK-algebra X is said to be *positive implicative* if the following assertion is valid.

$$(\forall x,y,z \in X)\ ((x*z)*(y*z)=(x*y)*z)\,. \tag{7}$$

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x*y \in S$ for all $x,y \in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies

$$0 \in I, \tag{8}$$
$$(\forall x \in X)\ (\forall y \in I)\ (x*y \in I\ \Rightarrow\ x \in I)\,. \tag{9}$$

A subset I of a BCI-algebra X is called a *closed ideal* (see [21]) of X if it is an ideal of X which satisfies

$$(\forall x \in X)(x \in I\ \Rightarrow\ 0*x \in I). \tag{10}$$

A subset I of a BCK-algebra X is called a *positive implicative ideal* (see [22]) of X if it satisfies (8) and

$$(\forall x,y,z \in X)(((x*y)*z \in I,\ y*z \in I\ \Rightarrow\ x*z \in I)\,. \tag{11}$$

Observe that every positive implicative ideal is an ideal, but the converse is not true (see [22]). Note also that a BCK-algebra X is positive implicative if and only if every ideal of X is positive implicative (see [22]).

We refer the reader to the books [21,22] for further information regarding BCK/BCI-algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

3. Neutrosophic Quadruple *BCK/BCI*-Algebras

We consider neutrosophic quadruple numbers based on a set instead of real or complex numbers.

Definition 1. *Let X be a set. A neutrosophic quadruple X-number is an ordered quadruple (a, xT, yI, zF) where $a, x, y, z \in X$ and T, I, F have their usual neutrosophic logic meanings.*

The set of all neutrosophic quadruple X-numbers is denoted by $NQ(X)$, that is,

$$NQ(X) := \{(a, xT, yI, zF) \mid a, x, y, z \in X\},$$

and it is called the *neutrosophic quadruple set* based on X. If X is a BCK/BCI-algebra, a neutrosophic quadruple X-number is called a *neutrosophic quadruple BCK/BCI-number* and we say that $NQ(X)$ is the *neutrosophic quadruple BCK/BCI-set*.

Let X be a BCK/BCI-algebra. We define a binary operation $\odot$ on $NQ(X)$ by

$$(a, xT, yI, zF) \odot (b, uT, vI, wF) = (a * b, (x * u)T, (y * v)I, (z * w)F)$$

for all (a, xT, yI, zF), $(b, uT, vI, wF) \in NQ(X)$. Given $a_1, a_2, a_3, a_4 \in X$, the neutrosophic quadruple BCK/BCI-number (a_1, a_2T, a_3I, a_4F) is denoted by $\tilde{a}$, that is,

$$\tilde{a} = (a_1, a_2T, a_3I, a_4F),$$

and the zero neutrosophic quadruple BCK/BCI-number $(0, 0T, 0I, 0F)$ is denoted by $\tilde{0}$, that is,

$$\tilde{0} = (0, 0T, 0I, 0F).$$

We define an order relation "$\ll$" and the equality "$=$" on $NQ(X)$ as follows:

$$\tilde{x} \ll \tilde{y} \Leftrightarrow x_i \leq y_i \text{ for } i = 1, 2, 3, 4$$
$$\tilde{x} = \tilde{y} \Leftrightarrow x_i = y_i \text{ for } i = 1, 2, 3, 4$$

for all $\tilde{x}, \tilde{y} \in NQ(X)$. It is easy to verify that "$\ll$" is an equivalence relation on $NQ(X)$.

Theorem 1. *If X is a BCK/BCI-algebra, then $(NQ(X); \odot, \tilde{0})$ is a BCK/BCI-algebra.*

Proof. Let X be a BCI-algebra. For any $\tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)$, we have

$$\begin{aligned}
(\tilde{x} \odot \tilde{y}) \odot (\tilde{x} \odot \tilde{z}) &= (x_1 * y_1, (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) \\
&\qquad \odot (x_1 * z_1, (x_2 * z_2)T, (x_3 * z_3)I, (x_4 * z_4)F) \\
&= ((x_1 * y_1) * (x_1 * z_1), ((x_2 * y_2) * (x_2 * z_2))T, \\
&\qquad ((x_3 * y_3) * (x_3 * z_3))I, ((x_4 * y_4) * (x_4 * z_4))T) \\
&\ll (z_1 * y_1, (z_2 * y_2)T, (z_3 * y_3)I, (z_4 * y_4)F) \\
&= \tilde{z} \odot \tilde{y}
\end{aligned}$$

$$\begin{aligned}
\tilde{x} \odot (\tilde{x} \odot \tilde{y}) &= (x_1, x_2T, x_3I, x_4F) \odot (x_1 * y_1, (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) \\
&= (x_1 * (x_1 * y_1), (x_2 * (x_2 * y_2))T, (x_3 * (x_3 * y_3))I, (x_4 * (x_4 * y_4))F) \\
&\ll (y_1, y_2T, y_3I, y_4F) \\
&= \tilde{y}
\end{aligned}$$

$$\begin{aligned}
\tilde{x} \odot \tilde{x} &= (x_1, x_2T, x_3I, x_4F) \odot (x_1, x_2T, x_3I, x_4F) \\
&= (x_1 * x_1, (x_2 * x_2)T, (x_3 * x_3)I, (x_4 * x_4)F) \\
&= (0, 0T, 0I, 0F) = \tilde{0}.
\end{aligned}$$

Assume that $\tilde{x} \odot \tilde{y} = \tilde{0}$ and $\tilde{y} \odot \tilde{x} = \tilde{0}$. Then

$$(x_1 * y_1, (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) = (0, 0T, 0I, 0F)$$

and

$$(y_1 * x_1, (y_2 * x_2)T, (y_3 * x_3)I, (y_4 * x_4)F) = (0, 0T, 0I, 0F).$$

It follows that $x_1 * y_1 = 0 = y_1 * x_1$, $x_2 * y_2 = 0 = y_2 * x_2$, $x_3 * y_3 = 0 = y_3 * x_3$ and $x_4 * y_4 = 0 = y_4 * x_4$. Hence, $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$, and $x_4 = y_4$, which implies that

$$\tilde{x} = (x_1, x_2T, x_3I, x_4F) = (y_1, y_2T, y_3I, y_4F) = \tilde{y}.$$

Therefore, we know that $(NQ(X); \odot, \tilde{0})$ is a BCI-algebra. We call it the *neutrosophic quadruple BCI-algebra*. Moreover, if X is a BCK-algebra, then we have

$$\tilde{0} \odot \tilde{x} = (0 * x_1, (0 * x_2)T, (0 * x_3)I, (0 * x_4)F) = (0, 0T, 0I, 0F) = \tilde{0}.$$

Hence, $(NQ(X); \odot, \tilde{0})$ is a BCK-algebra. We call it the *neutrosophic quadruple BCK-algebra*. □

Example 1. *If $X = \{0, a\}$, then the neutrosophic quadruple set $NQ(X)$ is given as follows:*

$$NQ(X) = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}, \tilde{8}, \tilde{9}, \tilde{10}, \tilde{11}, \tilde{12}, \tilde{13}, \tilde{14}, \tilde{15}\}$$

where

$\tilde{0} = (0, 0T, 0I, 0F)$, $\tilde{1} = (0, 0T, 0I, aF)$, $\tilde{2} = (0, 0T, aI, 0F)$, $\tilde{3} = (0, 0T, aI, aF)$,
$\tilde{4} = (0, aT, 0I, 0F)$, $\tilde{5} = (0, aT, 0I, aF)$, $\tilde{6} = (0, aT, aI, 0F)$, $\tilde{7} = (0, aT, aI, aF)$,
$\tilde{8} = (a, 0T, 0I, 0F)$, $\tilde{9} = (a, 0T, 0I, aF)$, $\tilde{10} = (a, 0T, aI, 0F)$, $\tilde{11} = (a, 0T, aI, aF)$,
$\tilde{12} = (a, aT, 0I, 0F)$, $\tilde{13} = (a, aT, 0I, aF)$, $\tilde{14} = (a, aT, aI, 0F)$, *and* $\tilde{15} = (a, aT, aI, aF)$.

Consider a BCK-algebra $X = \{0, a\}$ with the binary operation $*$, which is given in Table 1.

Table 1. Cayley table for the binary operation "$*$".

$*$	**0**	***a***
0	0	0
a	a	0

Then $(NQ(X), \odot, \tilde{0})$ is a BCK-algebra in which the operation $\odot$ is given by Table 2.

Table 2. Cayley table for the binary operation "$\odot$".

$\odot$	$\tilde{0}$	$\tilde{1}$	$\tilde{2}$	$\tilde{3}$	$\tilde{4}$	$\tilde{5}$	$\tilde{6}$	$\tilde{7}$	$\tilde{8}$	$\tilde{9}$	$\tilde{10}$	$\tilde{11}$	$\tilde{12}$	$\tilde{13}$	$\tilde{14}$	$\tilde{15}$
$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$
$\tilde{1}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$
$\tilde{2}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{0}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{0}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{0}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{0}$
$\tilde{3}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$
$\tilde{4}$	$\tilde{4}$	$\tilde{4}$	$\tilde{4}$	$\tilde{4}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{4}$	$\tilde{4}$	$\tilde{4}$	$\tilde{4}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$
$\tilde{5}$	$\tilde{5}$	$\tilde{4}$	$\tilde{5}$	$\tilde{4}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{5}$	$\tilde{4}$	$\tilde{5}$	$\tilde{4}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$
$\tilde{6}$	$\tilde{6}$	$\tilde{6}$	$\tilde{4}$	$\tilde{4}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{0}$	$\tilde{6}$	$\tilde{6}$	$\tilde{4}$	$\tilde{4}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{0}$
$\tilde{7}$	$\tilde{7}$	$\tilde{6}$	$\tilde{5}$	$\tilde{4}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$	$\tilde{7}$	$\tilde{6}$	$\tilde{5}$	$\tilde{4}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$
$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$
$\tilde{9}$	$\tilde{9}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{9}$	$\tilde{8}$	$\tilde{9}$	$\tilde{8}$	$\tilde{9}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$
$\tilde{10}$	$\tilde{10}$	$\tilde{10}$	$\tilde{8}$	$\tilde{8}$	$\tilde{10}$	$\tilde{10}$	$\tilde{8}$	$\tilde{8}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{2}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{0}$

Table 2. *Cont*.

$\odot$	$\tilde{0}$	$\tilde{1}$	$\tilde{2}$	$\tilde{3}$	$\tilde{4}$	$\tilde{5}$	$\tilde{6}$	$\tilde{7}$	$\tilde{8}$	$\tilde{9}$	$\tilde{10}$	$\tilde{11}$	$\tilde{12}$	$\tilde{13}$	$\tilde{14}$	$\tilde{15}$
$\tilde{11}$	$\tilde{11}$	$\tilde{10}$	$\tilde{9}$	$\tilde{8}$	$\tilde{11}$	$\tilde{10}$	$\tilde{9}$	$\tilde{8}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$
$\tilde{12}$	$\tilde{12}$	$\tilde{12}$	$\tilde{12}$	$\tilde{12}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{8}$	$\tilde{4}$	$\tilde{4}$	$\tilde{4}$	$\tilde{4}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$
$\tilde{13}$	$\tilde{13}$	$\tilde{12}$	$\tilde{13}$	$\tilde{12}$	$\tilde{9}$	$\tilde{8}$	$\tilde{9}$	$\tilde{8}$	$\tilde{5}$	$\tilde{4}$	$\tilde{5}$	$\tilde{4}$	$\tilde{1}$	$\tilde{0}$	$\tilde{1}$	$\tilde{0}$
$\tilde{14}$	$\tilde{14}$	$\tilde{14}$	$\tilde{12}$	$\tilde{12}$	$\tilde{10}$	$\tilde{10}$	$\tilde{8}$	$\tilde{8}$	$\tilde{6}$	$\tilde{6}$	$\tilde{4}$	$\tilde{4}$	$\tilde{2}$	$\tilde{2}$	$\tilde{0}$	$\tilde{0}$
$\tilde{15}$	$\tilde{15}$	$\tilde{14}$	$\tilde{13}$	$\tilde{12}$	$\tilde{11}$	$\tilde{10}$	$\tilde{9}$	$\tilde{8}$	$\tilde{7}$	$\tilde{6}$	$\tilde{5}$	$\tilde{4}$	$\tilde{3}$	$\tilde{2}$	$\tilde{1}$	$\tilde{0}$

Theorem 2. *The neutrosophic quadruple set $NQ(X)$ based on a positive implicative BCK-algebra X is a positive implicative BCK-algebra.*

Proof. Let X be a positive implicative BCK-algebra. Then X is a BCK-algebra, so $(NQ(X); \odot, \tilde{0})$ is a BCK-algebra by Theorem 1. Let $\tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)$. Then

$$(x_i * z_i) * (y_i * z_i) = (x_i * y_i) * z_i$$

for all $i = 1, 2, 3, 4$ since $x_i, y_i, z_i \in X$ and X is a positive implicative BCK-algebra. Hence, $(\tilde{x} \odot \tilde{z}) \odot (\tilde{y} * \tilde{z}) = (\tilde{x} \odot \tilde{y}) \odot \tilde{z}$; therefore, $NQ(X)$ based on a positive implicative BCK-algebra X is a positive implicative BCK-algebra. □

Proposition 1. *The neutrosophic quadruple set $NQ(X)$ based on a positive implicative BCK-algebra X satisfies the following assertions.*

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in NQ(X))\ (\tilde{x} \odot \tilde{y} \ll \tilde{z} \Rightarrow \tilde{x} \odot \tilde{z} \ll \tilde{y} \odot \tilde{z}) \tag{12}$$

$$(\forall \tilde{x}, \tilde{y} \in NQ(X))\ (\tilde{x} \odot \tilde{y} \ll \tilde{y} \Rightarrow \tilde{x} \ll \tilde{y}). \tag{13}$$

Proof. Let $\tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)$. If $\tilde{x} \odot \tilde{y} \ll \tilde{z}$, then

$$\tilde{0} = (\tilde{x} \odot \tilde{y}) \odot \tilde{z} = (\tilde{x} \odot \tilde{z}) \odot (\tilde{y} \odot \tilde{z}),$$

so $\tilde{x} \odot \tilde{z} \ll \tilde{y} \odot \tilde{z}$. Assume that $\tilde{x} \odot \tilde{y} \ll \tilde{y}$. Using Equation (12) implies that

$$\tilde{x} \odot \tilde{y} \ll \tilde{y} \odot \tilde{y} = \tilde{0},$$

so $\tilde{x} \odot \tilde{y} = \tilde{0}$, i.e., $\tilde{x} \ll \tilde{y}$. □

Let X be a BCK/BCI-algebra. Given $a, b \in X$ and subsets A and B of X, consider the sets

$$NQ(a, B) := \{(a, aT, yI, zF) \in NQ(X) \mid y, z \in B\}$$

$$NQ(A, b) := \{(a, xT, bI, bF) \in NQ(X) \mid a, x \in A\}$$

$$NQ(A, B) := \{(a, xT, yI, zF) \in NQ(X) \mid a, x \in A; y, z \in B\}$$

$$NQ(A^*, B) := \bigcup_{a \in A} NQ(a, B)$$

$$NQ(A, B^*) := \bigcup_{b \in B} NQ(A, b)$$

and

$$NQ(A \cup B) := NQ(A, 0) \cup NQ(0, B).$$

The set $NQ(A, A)$ is denoted by $NQ(A)$.

Proposition 2. *Let X be a BCK/BCI-algebra. Given $a, b \in X$ and subsets A and B of X, we have*

(1) *$NQ(A^*, B)$ and $NQ(A, B^*)$ are subsets of $NQ(A, B)$.*
(1) *If $0 \in A \cap B$ then $NQ(A \cup B)$ is a subset of $NQ(A, B)$.*

Proof. Straightforward. □

Let X be a BCK/BCI-algebra. Given $a, b \in X$ and subalgebras A and B of X, $NQ(a, B)$ and $NQ(A, b)$ may not be subalgebras of $NQ(X)$ since

$$(a, aT, x_3I, x_4F) \odot (a, aT, u_3I, v_4F) = (0, 0T, (x_3 * u_3)I, (x_4 * v_4)F) \notin NQ(a, B)$$

and

$$(x_1, x_2T, bI, bF) \odot (u_1, u_2T, bI, bF) = (x_1 * u_1, (x_2 * u_2)T, 0I, 0F) \notin NQ(A, b)$$

for $(a, aT, x_3I, x_4F) \in NQ(a, B)$, $(a, aT, u_3I, v_4F) \in NQ(a, B)$, $(x_1, x_2T, bI, bF) \in NQ(A, b)$, and $(u_1, u_2T, bI, bF) \in NQ(A, b)$.

Theorem 3. *If A and B are subalgebras of a BCK/BCI-algebra X, then the set $NQ(A, B)$ is a subalgebra of $NQ(X)$, which is called a neutrosophic quadruple subalgebra.*

Proof. Assume that A and B are subalgebras of a BCK/BCI-algebra X. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$ and $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ be elements of $NQ(A, B)$. Then $x_1, x_2, y_1, y_2 \in A$ and $x_3, x_4, y_3, y_4 \in B$, which implies that $x_1 * y_1 \in A$, $x_2 * y_2 \in A$, $x_3 * y_3 \in B$, and $x_4 * y_4 \in B$. Hence,

$$\tilde{x} \odot \tilde{y} = (x_1 * y_1, (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) \in NQ(A, B),$$

so $NQ(A, B)$ is a subalgebra of $NQ(X)$. □

Theorem 4. *If A and B are ideals of a BCK/BCI-algebra X, then the set $NQ(A, B)$ is an ideal of $NQ(X)$, which is called a neutrosophic quadruple ideal.*

Proof. Assume that A and B are ideals of a BCK/BCI-algebra X. Obviously, $\tilde{0} \in NQ(A, B)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$ and $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ be elements of $NQ(X)$ such that $\tilde{x} \odot \tilde{y} \in NQ(A, B)$ and $\tilde{y} \in NQ(A, B)$. Then

$$\tilde{x} \odot \tilde{y} = (x_1 * y_1, (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) \in NQ(A, B),$$

so $x_1 * y_1 \in A$, $x_2 * y_2 \in A$, $x_3 * y_3 \in B$ and $x_4 * y_4 \in B$. Since $\tilde{y} \in NQ(A, B)$, we have $y_1, y_2 \in A$ and $y_3, y_4 \in B$. Since A and B are ideals of X, it follows that $x_1, x_2 \in A$ and $x_3, x_4 \in B$. Hence, $\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(A, B)$, so $NQ(A, B)$ is an ideal of $NQ(X)$. □

Since every ideal is a subalgebra in a BCK-algebra, we have the following corollary.

Corollary 1. *If A and B are ideals of a BCK-algebra X, then the set $NQ(A, B)$ is a subalgebra of $NQ(X)$.*

The following example shows that Corollary 1 is not true in a BCI-algebra.

Example 2. *Consider a BCI-algebra* $(\mathbb{Z}, -, 0)$. *If we take* $A = \mathbb{N}$ *and* $B = \mathbb{Z}$, *then* $NQ(A, B)$ *is an ideal of* $NQ(\mathbb{Z})$. *However, it is not a subalgebra of* $NQ(\mathbb{Z})$ *since*

$$(2, 3T, -5I, 6F) \odot (3, 5T, 6I, -7F) = (-1, -2T, -11I, 13F) \notin NQ(A, B)$$

for $(2, 3T, -5I, 6F), (3, 5T, 6I, -7F) \in NQ(A, B)$.

Theorem 5. *If A and B are closed ideals of a BCI-algebra X, then the set* $NQ(A, B)$ *is a closed ideal of* $NQ(X)$.

Proof. If A and B are closed ideals of a BCI-algebra X, then the set $NQ(A, B)$ is an ideal of $NQ(X)$ by Theorem 4. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(A, B)$. Then

$$\tilde{0} \odot \tilde{x} = (0 * x_1, (0 * x_2)T, (0 * x_3)I, (0 * x_4)F) \in NQ(A, B)$$

since $0 * x_1, 0 * x_2 \in A$ and $0 * x_3, 0 * x_4 \in B$. Therefore, $NQ(A, B)$ is a closed ideal of $NQ(X)$. □

Since every closed ideal of a BCI-algebra X is a subalgebra of X, we have the following corollary.

Corollary 2. *If A and B are closed ideals of a BCI-algebra X, then the set* $NQ(A, B)$ *is a subalgebra of* $NQ(X)$.

In the following example, we know that there exist ideals A and B in a BCI-algebra X such that $NQ(A, B)$ is not a closed ideal of $NQ(X)$.

Example 3. *Consider BCI-algebras* $(Y, *, 0)$ *and* $(\mathbb{Z}, -, 0)$. *Then* $X = Y \times \mathbb{Z}$ *is a BCI-algebra (see [21]). Let* $A = Y \times \mathbb{N}$ *and* $B = \{0\} \times \mathbb{N}$. *Then A and B are ideals of X, so* $NQ(A, B)$ *is an ideal of* $NQ(X)$ *by Theorem 4. Let* $((0, 0), (0, 1)T, (0, 2)I, (0, 3)F) \in NQ(A, B)$. *Then*

$$\begin{aligned} &((0,0),(0,0)T,(0,0)I,(0,0)F) \odot ((0,0),(0,1)T,(0,2)I,(0,3)F) \\ &= ((0,0),(0,-1)T,(0,-2)I,(0,-3)F) \notin NQ(A,B). \end{aligned}$$

Hence, $NQ(A, B)$ *is not a closed ideal of* $NQ(X)$.

We provide conditions wherethe set $NQ(A, B)$ is a closed ideal of $NQ(X)$.

Theorem 6. *Let A and B be ideals of a BCI-algebra X and let*

$$\Gamma := \{\tilde{a} \in NQ(X) \mid (\forall \tilde{x} \in NQ(X))(\tilde{x} \ll \tilde{a} \;\Rightarrow\; \tilde{x} = \tilde{a})\}.$$

Assume that, if $\Gamma \subseteq NQ(A, B)$, *then* $|\Gamma| < \infty$. *Then* $NQ(A, B)$ *is a closed ideal of* $NQ(X)$.

Proof. If A and B are ideals of X, then $NQ(A, B)$ is an ideal of $NQ(X)$ by Theorem 4. Let $\tilde{a} = (a_1, a_2T, a_3I, a_4F) \in NQ(A, B)$. For any $n \in \mathbb{N}$, denote $n(\tilde{a}) := \tilde{0} \odot (\tilde{0} \odot \tilde{a})^n$. Then $n(\tilde{a}) \in \Gamma$ and

$$\begin{aligned} n(\tilde{a}) &= (0 * (0 * a_1)^n, (0 * (0 * a_2)^n)T, (0 * (0 * a_3)^n)I, (0 * (0 * a_4)^n)F) \\ &= (0 * (0 * a_1^n), (0 * (0 * a_2^n))T, (0 * (0 * a_3^n))I, (0 * (0 * a_4^n))F) \\ &= \tilde{0} \odot (\tilde{0} \odot \tilde{a}^n). \end{aligned}$$

Hence,

$$\begin{aligned} n(\tilde{a}) \odot \tilde{a}^n &= (\tilde{0} \odot (\tilde{0} \odot \tilde{a}^n)) \odot \tilde{a}^n \\ &= (\tilde{0} \odot \tilde{a}^n) \odot (\tilde{0} \odot \tilde{a}^n) \\ &= \tilde{0} \in NQ(A, B), \end{aligned}$$

so $n(\tilde{a}) \in NQ(A,B)$, since $\tilde{a} \in NQ(A,B)$, and $NQ(A,B)$ is an ideal of $NQ(X)$. Since $|\Gamma| < \infty$, it follows that $k \in \mathbb{N}$ such that $n(\tilde{a}) = (n+k)(\tilde{a})$, that is, $n(\tilde{a}) = n(\tilde{a}) \odot (\tilde{0} \odot \tilde{a})^k$, and thus

$$\begin{aligned} k(\tilde{a}) &= \tilde{0} \odot (\tilde{0} \odot \tilde{a})^k \\ &= (n(\tilde{a}) \odot (\tilde{0} \odot \tilde{a})^k) \odot n(\tilde{a}) \\ &= n(\tilde{a}) \odot n(\tilde{a}) = \tilde{0}, \end{aligned}$$

i.e., $(k-1)(\tilde{a}) \odot (\tilde{0} \odot \tilde{a}) = \tilde{0}$. Since $\tilde{0} \odot \tilde{a} \in \Gamma$, it follows that $\tilde{0} \odot \tilde{a} = (k-1)(\tilde{a}) \in NQ(A,B)$. Therefore, $NQ(A,B)$ is a closed ideal of $NQ(X)$. □

Theorem 7. *Given two elements a and b in a BCI-algebra X, let*

$$A_a := \{x \in X \mid a * x = a\} \text{ and } B_b := \{x \in X \mid b * x = b\}. \tag{14}$$

Then $NQ(A_a, B_b)$ is a closed ideal of $NQ(X)$.

Proof. Since $a * 0 = a$ and $b * 0 = b$, we have $0 \in A_a \cap B_b$. Thus, $\tilde{0} \in NQ(A_a, B_b)$. If $x \in A_a$ and $y \in B_b$, then

$$0 * x = (a * x) * a = a * a = 0 \text{ and } 0 * y = (b * y) * b = b * b = 0. \tag{15}$$

Let $x, y, c, d \in X$ be such that $x, y * x \in A_a$ and $c, d * c \in B_b$. Then

$$(a * y) * a = 0 * y = (0 * y) * 0 = (0 * y) * (0 * x) = 0 * (y * x) = 0$$

and

$$(b * d) * b = 0 * d = (0 * d) * 0 = (0 * d) * (0 * c) = 0 * (d * c) = 0,$$

that is, $a * y \leq a$ and $b * d \leq b$. On the other hand,

$$a = a * (y * x) = (a * x) * (y * x) \leq a * y$$

and

$$b = b * (d * c) = (b * c) * (d * c) \leq b * d.$$

Thus, $a * y = a$ and $b * d = b$, i.e., $y \in A_a$ and $d \in B_b$. Hence, A_a and B_b are ideals of X, and $NQ(A_a, B_b)$ is therefore an ideal of $NQ(X)$ by Theorem 4. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(A_a, B_b)$. Then $x_1, x_2 \in A_a$, and $x_3, x_4 \in B_b$. It follows from Equation (15) that $0 * x_1 = 0 \in A_a$, $0 * x_2 = 0 \in A_a$, $0 * x_3 = 0 \in B_b$, and $0 * x_4 = 0 \in B_b$. Hence,

$$\tilde{0} \odot \tilde{x} = (0 * x_1, (0 * x_2)T, (0 * x_3)I, (0 * x_4)F) \in NQ(A_a, B_b).$$

Therefore, $NQ(A_a, B_b)$ is a closed ideal of $NQ(X)$. □

Proposition 3. *Let A and B be ideals of a BCK-algebra X. Then*

$$NQ(A) \cap NQ(B) = \{\tilde{0}\} \iff (\forall \tilde{x} \in NQ(A))(\forall \tilde{y} \in NQ(B))(\tilde{x} \odot \tilde{y} = \tilde{x}). \tag{16}$$

Proof. Note that $NQ(A)$ and $NQ(B)$ are ideals of $NQ(X)$. Assume that $NQ(A) \cap NQ(B) = \{\tilde{0}\}$. Let

$$\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(A) \text{ and } \tilde{y} = (y_1, y_2T, y_3I, y_4F) \in NQ(B).$$

Since $\tilde{x} \odot (\tilde{x} \odot \tilde{y}) \ll \tilde{x}$ and $\tilde{x} \odot (\tilde{x} \odot \tilde{y}) \ll \tilde{y}$, it follows that $\tilde{x} \odot (\tilde{x} \odot \tilde{y}) \in NQ(A) \cap NQ(B) = \{\tilde{0}\}$. Obviously, $(\tilde{x} \odot \tilde{y}) \odot \tilde{x} \in \{\tilde{0}\}$. Hence, $\tilde{x} \odot \tilde{y} = \tilde{x}$.

Conversely, suppose that $\tilde{x} \odot \tilde{y} = \tilde{x}$ for all $\tilde{x} \in NQ(A)$ and $\tilde{y} \in NQ(B)$. If $\tilde{z} \in NQ(A) \cap NQ(B)$, then $\tilde{z} \in NQ(A)$ and $\tilde{z} \in NQ(B)$, which is implied from the hypothesis that $\tilde{z} = \tilde{z} \odot \tilde{z} = \tilde{0}$. Hence $NQ(A) \cap NQ(B) = \{\tilde{0}\}$. □

Theorem 8. *Let A and B be subsets of a BCK-algebra X such that*

$$(\forall a, b \in A \cap B)(K(a, b) \subseteq A \cap B) \tag{17}$$

where $K(a, b) := \{x \in X \mid x * a \leq b\}$. Then the set $NQ(A, B)$ is an ideal of $NQ(X)$.

Proof. If $x \in A \cap B$, then $0 \in K(x, x)$ since $0 * x \leq x$. Hence, $0 \in A \cap B$ by Equation (17), so it is clear that $\tilde{0} \in NQ(A, B)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$ and $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ be elements of $NQ(X)$ such that $\tilde{x} \odot \tilde{y} \in NQ(A, B)$ and $\tilde{y} \in NQ(A, B)$. Then

$$\tilde{x} \odot \tilde{y} = (x_1 * y_1, (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) \in NQ(A, B),$$

so $x_1 * y_1 \in A$, $x_2 * y_2 \in A$, $x_3 * y_3 \in B$, and $x_4 * y_4 \in B$. Using (II), we have $x_1 \in K(x_1 * y_1, y_1) \subseteq A$, $x_2 \in K(x_2 * y_2, y_2) \subseteq A$, $x_3 \in K(x_3 * y_3, y_3) \subseteq B$, and $x_4 \in K(x_4 * y_4, y_4) \subseteq B$. This implies that $\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(A, B)$. Therefore, $NQ(A, B)$ is an ideal of $NQ(X)$. □

Corollary 3. *Let A and B be subsets of a BCK-algebra X such that*

$$(\forall a, x, y \in X)(x, y \in A \cap B,\ (a * x) * y = 0 \ \Rightarrow\ a \in A \cap B). \tag{18}$$

Then the set $NQ(A, B)$ is an ideal of $NQ(X)$.

Theorem 9. *Let A and B be nonempty subsets of a BCK-algebra X such that*

$$(\forall a, x, y \in X)(x, y \in A \text{ (or } B),\ a * x \leq y \ \Rightarrow\ a \in A \text{ (or } B)). \tag{19}$$

Then the set $NQ(A, B)$ is an ideal of $NQ(X)$.

Proof. Assume that the condition expressed by Equation (19) is valid for nonempty subsets A and B of X. Since $0 * x \leq x$ for any $x \in A$ (or B), we have $0 \in A$ (or B) by Equation (19). Hence, it is clear that $\tilde{0} \in NQ(A, B)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$ and $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ be elements of $NQ(X)$ such that $\tilde{x} \odot \tilde{y} \in NQ(A, B)$ and $\tilde{y} \in NQ(A, B)$. Then

$$\tilde{x} \odot \tilde{y} = (x_1 * y_1, (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) \in NQ(A, B),$$

so $x_1 * y_1 \in A$, $x_2 * y_2 \in A$, $x_3 * y_3 \in B$, and $x_4 * y_4 \in B$. Note that $x_i * (x_i * y_i) \leq y_i$ for $i = 1, 2, 3, 4$. It follows from Equation (19) that $x_1, x_2 \in A$ and $x_3, x_4 \in B$. Hence,

$$\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(A, B);$$

therefore, $NQ(A, B)$ is an ideal of $NQ(X)$. □

Theorem 10. *If A and B are positive implicative ideals of a BCK-algebra X, then the set $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$, which is called a positive implicative neutrosophic quadruple ideal.*

Proof. Assume that A and B are positive implicative ideals of a BCK-algebra X. Obviously, $\tilde{0} \in NQ(A,B)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$, and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ be elements of $NQ(X)$ such that $(\tilde{x} \odot \tilde{y}) \odot \tilde{z} \in NQ(A,B)$ and $\tilde{y} \odot \tilde{z} \in NQ(A,B)$. Then

$$(\tilde{x} \odot \tilde{y}) \odot \tilde{z} = ((x_1 * y_1) * z_1, ((x_2 * y_2) * z_2)T, \\ ((x_3 * y_3) * z_3)I, ((x_4 * y_4) * z_4)F) \in NQ(A,B),$$

and

$$\tilde{y} \odot \tilde{z} = (y_1 * z_1, (y_2 * z_2)T, (y_3 * z_3)I, (y_4 * z_4)F) \in NQ(A,B),$$

so $(x_1 * y_1) * z_1 \in A$, $(x_2 * y_2) * z_2 \in A$, $(x_3 * y_3) * z_3 \in B$, $(x_4 * y_4) * z_4 \in B$, $y_1 * z_1 \in A$, $y_2 * z_2 \in A$, $y_3 * z_3 \in B$, and $y_4 * z_4 \in B$. Since A and B are positive implicative ideals of X, it follows that $x_1 * z_1, x_2 * z_2 \in A$ and $x_3 * z_3, x_4 * z_4 \in B$. Hence,

$$\tilde{x} \odot \tilde{z} = (x_1 * z_1, (x_2 * z_2)T, (x_3 * z_3)I, (x_4 * z_4)F) \in NQ(A,B),$$

so $NQ(A,B)$ is a positive implicative ideal of $NQ(X)$. □

Theorem 11. *Let A and B be ideals of a BCK-algebra X such that*

$$(\forall x,y,z \in X)((x * y) * z \in A \text{ (or } B) \;\Rightarrow\; (x * z) * (y * z) \in A \text{ (or } B)). \tag{20}$$

Then $NQ(A,B)$ is a positive implicative ideal of $NQ(X)$.

Proof. Since A and B are ideals of X, it follows from Theorem 4 that $NQ(A,B)$ is an ideal of $NQ(X)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$, and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ be elements of $NQ(X)$ such that $(\tilde{x} \odot \tilde{y}) \odot \tilde{z} \in NQ(A,B)$ and $\tilde{y} \odot \tilde{z} \in NQ(A,B)$. Then

$$(\tilde{x} \odot \tilde{y}) \odot \tilde{z} = ((x_1 * y_1) * z_1, ((x_2 * y_2) * z_2)T, \\ ((x_3 * y_3) * z_3)I, ((x_4 * y_4) * z_4)F) \in NQ(A,B),$$

and

$$\tilde{y} \odot \tilde{z} = (y_1 * z_1, (y_2 * z_2)T, (y_3 * z_3)I, (y_4 * z_4)F) \in NQ(A,B),$$

so $(x_1 * y_1) * z_1 \in A$, $(x_2 * y_2) * z_2 \in A$, $(x_3 * y_3) * z_3 \in B$, $(x_4 * y_4) * z_4 \in B$, $y_1 * z_1 \in A$, $y_2 * z_2 \in A$, $y_3 * z_3 \in B$, and $y_4 * z_4 \in B$. It follows from Equation (20) that $(x_1 * z_1) * (y_1 * z_1) \in A$, $(x_2 * z_2) * (y_2 * z_2) \in A$, $(x_3 * z_3) * (y_3 * z_3) \in B$, and $(x_4 * z_4) * (y_4 * z_4) \in B$. Since A and B are ideals of X, we get $x_1 * z_1 \in A$, $x_2 * z_2 \in A$, $x_3 * z_3 \in B$, and $x_4 * z_4 \in B$. Hence,

$$\tilde{x} \odot \tilde{z} = (x_1 * z_1, (x_2 * z_2)T, (x_3 * z_3)I, (x_4 * z_4)F) \in NQ(A,B).$$

Therefore, $NQ(A,B)$ is a positive implicative ideal of $NQ(X)$. □

Corollary 4. *Let A and B be ideals of a BCK-algebra X such that*

$$(\forall x,y \in X)((x * y) * y \in A \text{ (or } B) \;\Rightarrow\; x * y \in A \text{ (or } B)). \tag{21}$$

Then $NQ(A,B)$ is a positive implicative ideal of $NQ(X)$.

Proof. If the condition expressed in Equation (21) is valid, then the condition expressed in Equation (20) is true. Hence, $NQ(A,B)$ is a positive implicative ideal of $NQ(X)$ by Theorem 11. □

Theorem 12. *Let A and B be subsets of a BCK-algebra X such that $0 \in A \cap B$ and*

$$((x * y) * y) * z \in A \text{ (or } B), \; z \in A \text{ (or } B) \;\Rightarrow\; x * y \in A \text{ (or } B) \tag{22}$$

for all $x, y, z \in X$. Then $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$.

Proof. Since $0 \in A \cap B$, it is clear that $\tilde{0} \in NQ(A, B)$. We first show that

$$(\forall x, y \in X)(x * y \in A \text{ (or } B), \; y \in A \text{ (or } B) \Rightarrow x \in A \text{ (or } B)). \tag{23}$$

Let $x, y \in X$ be such that $x * y \in A$ (or B) and $y \in A$ (or B). Then

$$((x * 0) * 0) * y = x * y \in A \text{ (or } B)$$

by Equation (1), which, based on Equations (1) and (22), implies that $x = x * 0 \in A$ (or B). Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$, and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ be elements of $NQ(X)$ such that $(\tilde{x} \odot \tilde{y}) \odot \tilde{z} \in NQ(A, B)$ and $\tilde{y} \odot \tilde{z} \in NQ(A, B)$. Then

$$\begin{aligned}(\tilde{x} \odot \tilde{y}) \odot \tilde{z} = ((x_1 * y_1) * z_1, &((x_2 * y_2) * z_2)T, \\ &((x_3 * y_3) * z_3)I, ((x_4 * y_4) * z_4)F) \in NQ(A, B),\end{aligned}$$

and

$$\tilde{y} \odot \tilde{z} = (y_1 * z_1, (y_2 * z_2)T, (y_3 * z_3)I, (y_4 * z_4)F) \in NQ(A, B),$$

so $(x_1 * y_1) * z_1 \in A$, $(x_2 * y_2) * z_2 \in A$, $(x_3 * y_3) * z_3 \in B$, $(x_4 * y_4) * z_4 \in B$, $y_1 * z_1 \in A$, $y_2 * z_2 \in A$, $y_3 * z_3 \in B$, and $y_4 * z_4 \in B$. Note that

$$(((x_i * z_i) * z_i) * (y_i * z_i)) * ((x_i * y_i) * z_i) = 0 \in A \text{ (or } B)$$

for $i = 1, 2, 3, 4$. Since $(x_i * y_i) * z_i \in A$ for $i = 1, 2$ and $(x_j * y_j) * z_j \in B$ for $j = 3, 4$, it follows from Equation (23) that $((x_i * z_i) * z_i) * (y_i * z_i) \in A$ for $i = 1, 2$, and $((x_j * z_j) * z_j) * (y_j * z_j) \in B$ for $j = 3, 4$. Moreover, since $y_i * z_i \in A$ for $i = 1, 2$, and $y_j * z_j \in B$ for $j = 3, 4$, we have $x_1 * z_1 \in A$, $x_2 * z_2 \in A$, $x_3 * z_3 \in B$, and $x_4 * z_4 \in B$ by Equation (22). Hence,

$$\tilde{x} \odot \tilde{z} = (x_1 * z_1, (x_2 * z_2)T, (x_3 * z_3)I, (x_4 * z_4)F) \in NQ(A, B).$$

Therefore, $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$. □

Theorem 13. *Let A and B be subsets of a BCK-algebra X such that $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$. Then the set*

$$\Omega_{\tilde{a}} := \{\tilde{x} \in NQ(X) \mid \tilde{x} \odot \tilde{a} \in NQ(A, B)\} \tag{24}$$

is an ideal of $NQ(X)$ for any $\tilde{a} \in NQ(X)$.

Proof. Obviously, $\tilde{0} \in \Omega_{\tilde{a}}$. Let $\tilde{x}, \tilde{y} \in NQ(X)$ be such that $\tilde{x} \odot \tilde{y} \in \Omega_{\tilde{a}}$ and $\tilde{y} \in \Omega_{\tilde{a}}$. Then $(\tilde{x} \odot \tilde{y}) \odot \tilde{a} \in NQ(A, B)$ and $\tilde{y} \odot \tilde{a} \in NQ(A, B)$. Since $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$, it follows from Equation (11) that $\tilde{x} \odot \tilde{a} \in NQ(A, B)$ and therefore that $\tilde{x} \in \Omega_{\tilde{a}}$. Hence, $\Omega_{\tilde{a}}$ is an ideal of $NQ(X)$. □

Combining Theorems 12 and 13, we have the following corollary.

Corollary 5. *If A and B are subsets of a BCK-algebra X satisfying $0 \in A \cap B$ and the condition expressed in Equation (22), then the set $\Omega_{\tilde{a}}$ in Equation (24) is an ideal of $NQ(X)$ for all $\tilde{a} \in NQ(X)$.*

Theorem 14. *For any subsets A and B of a BCK-algebra X, if the set $\Omega_{\tilde{a}}$ in Equation (24) is an ideal of $NQ(X)$ for all $\tilde{a} \in NQ(X)$, then $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$.*

Proof. Since $\tilde{0} \in \Omega_{\tilde{a}}$, we have $\tilde{0} = \tilde{0} \odot \tilde{a} \in NQ(A, B)$. Let $\tilde{x}$, $\tilde{y}$, $\tilde{z} \in NQ(X)$ be such that $(\tilde{x} \odot \tilde{y}) \odot \tilde{z} \in NQ(A, B)$ and $\tilde{y} \odot \tilde{z} \in NQ(A, B)$. Then $\tilde{x} \odot \tilde{y} \in \Omega_{\tilde{z}}$ and $\tilde{y} \in \Omega_{\tilde{z}}$. Since $\Omega_{\tilde{z}}$ is an ideal of $NQ(X)$, it follows that $\tilde{x} \in \Omega_{\tilde{z}}$. Hence, $\tilde{x} \odot \tilde{z} \in NQ(A, B)$. Therefore, $NQ(A, B)$ is a positive implicative ideal of $NQ(X)$. □

Theorem 15. *For any ideals A and B of a BCK-algebra X and for any $\tilde{a} \in NQ(X)$, if the set $\Omega_{\tilde{a}}$ in Equation (24) is an ideal of $NQ(X)$, then $NQ(X)$ is a positive implicative BCK-algebra.*

Proof. Let Ω be any ideal of $NQ(X)$. For any $\tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)$, assume that $(\tilde{x} \odot \tilde{y}) \odot \tilde{z} \in \Omega$ and $\tilde{y} \odot \tilde{z} \in \Omega$. Then $\tilde{x} \odot \tilde{y} \in \Omega_{\tilde{z}}$ and $\tilde{y} \in \Omega_{\tilde{z}}$. Since $\Omega_{\tilde{z}}$ is an ideal of $NQ(X)$, it follows that $\tilde{x} \in \Omega_{\tilde{z}}$. Hence, $\tilde{x} \odot \tilde{z} \in \Omega$, which shows that Ω is a positive implicative ideal of $NQ(X)$. Therefore, $NQ(X)$ is a positive implicative BCK-algebra. □

In general, the set $\{\tilde{0}\}$ is an ideal of any neutrosophic quadruple BCK-algebra $NQ(X)$, but it is not a positive implicative ideal of $NQ(X)$ as seen in the following example.

Example 4. *Consider a BCK-algebra $X = \{0, 1, 2\}$ with the binary operation $*$, which is given in Table 3.*

Table 3. Cayley table for the binary operation "$*$".

$*$	**0**	**1**	**2**
0	0	0	0
1	1	0	0
2	2	1	0

Then the neutrosophic quadruple BCK-algebra $NQ(X)$ has 81 elements. If we take $\tilde{a} = (2, 2T, 2I, 2F)$ and $\tilde{b} = (1, 1T, 1I, 1F)$ in $NQ(X)$, then

$$\begin{aligned}(\tilde{a} \odot \tilde{b}) \odot \tilde{b} &= ((2 * 1) * 1, ((2 * 1) * 1)T, ((2 * 1) * 1)I, ((2 * 1) * 1)F) \\ &= (1 * 1, (1 * 1)T, (1 * 1)I, (1 * 1)F) = (0, 0T, 0I, 0F) = \tilde{0},\end{aligned}$$

and $\tilde{b} \odot \tilde{b} = \tilde{0}$. However,

$$\tilde{a} \odot \tilde{b} = (2 * 1, (2 * 1)T, (2 * 1)I, (2 * 1)F) = (1, 1T, 1I, 1F) \neq \tilde{0}.$$

Hence, $\{\tilde{0}\}$ is not a positive implicative ideal of $NQ(X)$.

We now provide conditions for the set $\{\tilde{0}\}$ to be a positive implicative ideal in the neutrosophic quadruple BCK-algebra.

Theorem 16. *Let $NQ(X)$ be a neutrosophic quadruple BCK-algebra. If the set*

$$\Omega(\tilde{a}) := \{\tilde{x} \in NQ(X) \mid \tilde{x} \ll \tilde{a}\} \tag{25}$$

is an ideal of $NQ(X)$ for all $\tilde{a} \in NQ(X)$, then $\{\tilde{0}\}$ is a positive implicative ideal of $NQ(X)$.

Proof. We first show that

$$(\forall \tilde{x}, \tilde{y} \in NQ(X))((\tilde{x} \odot \tilde{y}) \odot \tilde{y} = \tilde{0} \ \Rightarrow \ \tilde{x} \odot \tilde{y} = \tilde{0}). \tag{26}$$

Assume that $(\tilde{x} \odot \tilde{y}) \odot \tilde{y} = \tilde{0}$ for all $\tilde{x}, \tilde{y} \in NQ(X)$. Then $\tilde{x} \odot \tilde{y} \ll \tilde{y}$, so $\tilde{x} \odot \tilde{y} \in \Omega(\tilde{y})$. Since $\tilde{y} \in \Omega(\tilde{y})$ and $\Omega(\tilde{y})$ is an ideal of $NQ(X)$, we have $\tilde{x} \in \Omega(\tilde{y})$. Thus, $\tilde{x} \ll \tilde{y}$, that is, $\tilde{x} \odot \tilde{y} = \tilde{0}$. Let $\tilde{u} := (\tilde{x} \odot \tilde{y}) \odot \tilde{y}$. Then

$$((\tilde{x} \odot \tilde{u}) \odot \tilde{y}) \odot \tilde{y} = ((\tilde{x} \odot \tilde{y}) \odot \tilde{y}) \odot \tilde{u} = \tilde{0},$$

which implies, based on Equations (3) and (26), that

$$(\tilde{x} \odot \tilde{y}) \odot ((\tilde{x} \odot \tilde{y}) \odot \tilde{y}) = (\tilde{x} \odot \tilde{y}) \odot \tilde{u} = (\tilde{x} \odot \tilde{u}) \odot \tilde{y} = \tilde{0},$$

that is, $\tilde{x} \odot \tilde{y} \ll (\tilde{x} \odot \tilde{y}) \odot \tilde{y}$. Since $(\tilde{x} \odot \tilde{y}) \odot \tilde{y} \ll \tilde{x} \odot \tilde{y}$, it follows that

$$(\tilde{x} \odot \tilde{y}) \odot \tilde{y} = \tilde{x} \odot \tilde{y}. \tag{27}$$

If we put $\tilde{y} = \tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))$ in Equation (27), then

$$\begin{aligned}
\tilde{x} \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) &= (\tilde{x} \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})))) \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) \\
&\ll (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) \\
&\ll (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{x} \odot \tilde{y}) \\
&= (\tilde{y} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x}) \\
&= ((\tilde{y} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{y} \odot \tilde{x}) \\
&\ll (\tilde{x} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&((\tilde{x} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{x} \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})))) \\
&= ((\tilde{x} \odot (\tilde{x} \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))))) \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x})) \\
&= ((\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x})) \\
&\ll (\tilde{y} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) \odot (\tilde{y} \odot \tilde{x})) = \tilde{0},
\end{aligned}$$

so $((\tilde{x} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{x} \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})))) = \tilde{0}$, that is,

$$((\tilde{x} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x})) \ll \tilde{x} \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))).$$

Hence,

$$\tilde{x} \odot (\tilde{x} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) = ((\tilde{x} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x})). \tag{28}$$

If we use $\tilde{y} \odot \tilde{x}$ instead of $\tilde{x}$ in Equation (28), then

$$\begin{aligned}
\tilde{y} \odot \tilde{x} &= (\tilde{y} \odot \tilde{x}) \odot \tilde{0} \\
&= (\tilde{y} \odot \tilde{x}) \odot ((\tilde{y} \odot \tilde{x}) \odot (\tilde{y} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})))) \\
&= ((\tilde{y} \odot \tilde{x}) \odot ((\tilde{y} \odot \tilde{x}) \odot \tilde{y})) \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \\
&= (\tilde{y} \odot \tilde{x}) \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})),
\end{aligned}$$

which, by taking $\tilde{x} = \tilde{y} \odot \tilde{x}$, implies that

$$\begin{aligned}\tilde{y} \odot (\tilde{y} \odot \tilde{x}) &= (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{y} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) \\ &= (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{y} \odot \tilde{x}).\end{aligned}$$

It follows that

$$\begin{aligned}(\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{x} \odot \tilde{y}) &= ((\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{x} \odot \tilde{y}) \\ &\ll (\tilde{x} \odot (\tilde{y} \odot \tilde{x})) \odot (\tilde{x} \odot \tilde{y}) \\ &= (\tilde{x} \odot (\tilde{x} \odot \tilde{y})) \odot (\tilde{y} \odot \tilde{x}),\end{aligned}$$

so,

$$\begin{aligned}\tilde{y} \odot \tilde{x} &= (\tilde{y} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) \odot \tilde{0} \\ &= (\tilde{y} \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x}))) \odot ((\tilde{y} \odot \tilde{x}) \odot \tilde{y}) \\ &\ll ((\tilde{y} \odot \tilde{x}) \odot ((\tilde{y} \odot \tilde{x}) \odot \tilde{y})) \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \\ &= (\tilde{y} \odot \tilde{x}) \odot (\tilde{y} \odot (\tilde{y} \odot \tilde{x})) \\ &\ll (\tilde{y} \odot \tilde{x}) \odot \tilde{x}.\end{aligned}$$

Since $(\tilde{y} \odot \tilde{x}) \odot \tilde{x} \ll \tilde{y} \odot \tilde{x}$, it follows that

$$(\tilde{y} \odot \tilde{x}) \odot \tilde{x} = \tilde{y} \odot \tilde{x}. \tag{29}$$

Based on Equation (29), it follows that

$$\begin{aligned}&((\tilde{x} \odot \tilde{z}) * (\tilde{y} \odot \tilde{z})) \odot ((\tilde{x} \odot \tilde{y}) \odot \tilde{z}) \\ &= (((\tilde{x} \odot \tilde{z}) \odot \tilde{z}) \odot (\tilde{y} \odot \tilde{z})) \odot ((\tilde{x} \odot \tilde{y}) \odot \tilde{z}) \\ &\ll ((\tilde{x} \odot \tilde{z}) \odot \tilde{y}) \odot ((\tilde{x} \odot \tilde{y}) \odot \tilde{z}) \\ &= \tilde{0},\end{aligned}$$

that is, $(\tilde{x} \odot \tilde{z}) * (\tilde{y} \odot \tilde{z}) \ll (\tilde{x} \odot \tilde{y}) \odot \tilde{z}$. Note that

$$\begin{aligned}&((\tilde{x} \odot \tilde{y}) \odot \tilde{z}) \odot ((x \odot \tilde{z}) \odot (\tilde{y} \odot \tilde{z})) \\ &= ((\tilde{x} \odot \tilde{y}) \odot \tilde{z}) \odot ((x \odot (\tilde{y} \odot \tilde{z})) \odot \tilde{z}) \\ &\ll (\tilde{x} \odot \tilde{y}) \odot (\tilde{x} \odot (\tilde{y} \odot \tilde{z})) \\ &\ll (\tilde{y} \odot \tilde{z}) \odot \tilde{y} = \tilde{0},\end{aligned}$$

which shows that $(\tilde{x} \odot \tilde{y}) \odot \tilde{z} \ll (\tilde{x} \odot \tilde{z}) \odot (\tilde{y} \odot \tilde{z})$. Hence, $(\tilde{x} \odot \tilde{y}) \odot \tilde{z} = (\tilde{x} \odot \tilde{z}) \odot (\tilde{y} \odot \tilde{z})$. Therefore, $NQ(X)$ is a positive implicative, so $\{\tilde{0}\}$ is a positive implicative ideal of $NQ(X)$. □

4. Conclusions

We have considered a neutrosophic quadruple BCK/BCI-number on a set and established neutrosophic quadruple BCK/BCI-algebras, which consist of neutrosophic quadruple BCK/BCI-numbers. We have investigated several properties and considered ideal theory in a neutrosophic quadruple BCK-algebra and a closed ideal in a neutrosophic quadruple BCI-algebra. Using subsets A and B of a neutrosophic quadruple BCK/BCI-algebra, we have considered sets $NQ(A, B)$, which consist of neutrosophic quadruple BCK/BCI-numbers with a condition. We have provided conditions for the set $NQ(A, B)$ to be a (positive implicative) ideal of a neutrosophic quadruple BCK-algebra, and the set $NQ(A, B)$ to be a (closed) ideal of a neutrosophic quadruple BCI-algebra. We have provided an example to show that the set $\{\tilde{0}\}$ is not a positive implicative ideal in a neutrosophic quadruple BCK-algebra, and we have considered conditions for the set $\{\tilde{0}\}$ to be a positive implicative ideal in a neutrosophic quadruple BCK-algebra.

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