

## Neutrosophic Set and Neutrosophic Topological Spaces

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**Abstract:** Neutrosophy has been introduced by Smarandache [7, 8] as a new branch of philosophy. The purpose of this paper is to construct a new set theory called the neutrosophic set. After given the fundamental definitions of neutrosophic set operations, we obtain several properties, and discussed the relationship between neutrosophic sets and others. Finally, we extend the concepts of fuzzy topological space [4], and intuitionistic fuzzy topological space [5, 6] to the case of neutrosophic sets. Possible application to superstrings and  $\zeta^\infty$  space-time are touched upon.

**Keywords:** Fuzzy topology; fuzzy set; neutrosophic set; neutrosophic topology

### I. Introduction

The fuzzy set was introduced by Zadeh [9] in 1965, where each element had a degree of membership. The intuitionistic fuzzy set (Ifs for short) on a universe  $X$  was introduced by K. Atanassov [1, 2, 3] in 1983 as a generalization of fuzzy set, where besides the degree of membership and the degree of non-membership of each element. After the introduction of the neutrosophic set concept [7, 8]. In recent years neutrosophic algebraic structures have been investigated. Neutrosophy has laid the foundation for a whole family of new mathematical theories generalizing both their classical and fuzzy counterparts, such as a neutrosophic set theory.

### II. Terminologies

We recollect some relevant basic preliminaries, and in particular, the work of Smarandache in [7, 8], and Atanassov in [1, 2, 3]. Smarandache introduced the neutrosophic components  $T, I, F$  which represent the membership, indeterminacy, and non-membership values respectively, where  $]0^-, 1^+[$  is nonstandard unit interval.

**2.1 Definition.** [3,4]

Let  $T, I, F$  be real standard or nonstandard subsets of  $]0^-, 1^+[$ , with

$$\text{Sup}_T = t_{\text{sup}}, \text{inf}_T = t_{\text{inf}}$$

$$\text{Sup}_I = i_{\text{sup}}, \text{inf}_I = i_{\text{inf}}$$

$$\text{Sup}_F = f_{\text{sup}}, \text{inf}_F = f_{\text{inf}}$$

$$n\text{-sup} = t_{\text{sup}} + i_{\text{sup}} + f_{\text{sup}}$$

$$n\text{-inf} = t_{\text{inf}} + i_{\text{inf}} + f_{\text{inf}},$$

$T, I, F$  are called neutrosophic components

### III. Neutrosophic Sets and Its Operations

We shall now consider some possible definitions for basic concepts of the neutrosophic set and its operations.

#### 3.1 Definition

Let  $X$  be a non-empty fixed set. A neutrosophic set ( $NS$  for short)  $A$  is an object having the form  $A = \{ \langle x, \mu_A(x), \sigma_A(x), \gamma_A(x) \rangle : x \in X \}$  Where  $\mu_A(x), \sigma_A(x)$  and  $\gamma_A(x)$  which represent the degree of membership function (namely  $\mu_A(x)$ ), the degree of indeterminacy (namely  $\sigma_A(x)$ ), and the degree of non-membership (namely  $\gamma_A(x)$ ) respectively of each element  $x \in X$  to the set  $A$ .

#### 3.1 Remark

A neutrosophic  $A = \{ \langle x, \mu_A(x), \sigma_A(x), \gamma_A(x) \rangle : x \in X \}$  can be identified to an ordered triple  $\langle \mu_A, \sigma_A, \gamma_A \rangle$  in  $]0^-, 1^+[$  on  $X$ .

**3.2 Remark**

For the sake of simplicity, we shall use the symbol  $A = \langle x, \mu_A, \sigma_A, \gamma_A \rangle$  for the

$$NS A = \{ \langle x, \mu_A(x), \sigma_A(x), \gamma_A(x) \rangle : x \in X \}$$

**3.1 Example**

Every IFS  $A$  a non-empty set  $X$  is obviously on  $NS$  having the form

$$A = \{ \langle x, \mu_A(x), 1 - (\mu_A(x) + \gamma_A(x)), \gamma_A(x) \rangle : x \in X \}$$

Since our main purpose is to construct the tools for developing neutrosophic set and neutrosophic topology, we must introduce the  $NSS$   $0_N$  and  $1_N$  in  $X$  as follows:

$0_N$  may be defined as:

$$(0_1) 0_N = \{ \langle x, 0, 0, 1 \rangle : x \in X \}$$

$$(0_2) 0_N = \{ \langle x, 0, 1, 1 \rangle : x \in X \}$$

$$(0_3) 0_N = \{ \langle x, 0, 1, 0 \rangle : x \in X \}$$

$$(0_4) 0_N = \{ \langle x, 0, 0, 0 \rangle : x \in X \}$$

$1_N$  may be defined as:

$$(1_1) 1_N = \{ \langle x, 1, 0, 0 \rangle : x \in X \}$$

$$(1_2) 1_N = \{ \langle x, 1, 0, 1 \rangle : x \in X \}$$

$$(1_3) 1_N = \{ \langle x, 1, 1, 0 \rangle : x \in X \}$$

$$(1_4) 1_N = \{ \langle x, 1, 1, 1 \rangle : x \in X \}$$

**3.2 Definition**

Let  $A = \langle \mu_A, \sigma_A, \gamma_A \rangle$  a  $NS$  on  $X$ , then the complement of the set  $A$  ( $C(A)$ , for short) may be defined as three kinds of complements

$$(C_1) C(A) = \{ \langle x, 1 - \mu_A(x), 1 - \sigma_A(x), 1 - \gamma_A(x) \rangle : x \in X \},$$

$$(C_2) C(A) = \{ \langle x, \gamma_A, \sigma_A(x), \mu_A(x) \rangle : x \in X \},$$

$$(C_3) C(A) = \{ \langle x, \gamma_A, 1 - \sigma_A(x), \mu_A(x) \rangle : x \in X \}.$$

One can define several relations and operations between  $NSS$  follows:

**3.3 Definition**

Let  $X$  be a non-empty set, and  $NSS$   $A$  and  $B$  in the form  $A = \langle x, \mu_A(x), \sigma_A(x), \gamma_A(x) \rangle,$

$B = \langle x, \mu_B(x), \sigma_B(x), \gamma_B(x) \rangle,$  then we may consider two possible definitions for subsets ( $A \subseteq B$ )

( $A \subseteq B$ ) may be defined as

$$(1) A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x), \gamma_A(x) \geq \gamma_B(x) \text{ and } \sigma_A(x) \leq \sigma_B(x) \quad \forall x \in X$$

$$(2) A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x), \gamma_A(x) \geq \gamma_B(x) \text{ and } \sigma_A(x) \geq \sigma_B(x)$$

**3.1 Proposition**

For any neutrosophic set  $A$  the following are holds

$$(1) 0_N \subseteq A, \quad 0_N \subseteq 0_N$$

$$(2) A \subseteq 1_N, \quad 1_N \subseteq 1_N$$

**3.4. Definition**

Let  $X$  be a non-empty set, and  $A = \langle x, \mu_A(x), \gamma_A(x), \sigma_A(x) \rangle, B = \langle x, \mu_B(x), \sigma_B(x), \gamma_B(x) \rangle$  are  $NSS$ . Then

(1)  $A \cap B$  may be defined as:

$$(I_1) A \cap B = \langle x, \mu_A(x), \mu_B(x), \sigma_A(x), \sigma_B(x), \gamma_A(x), \gamma_B(x) \rangle$$

$$(I_2) A \cap B = \langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \gamma_A(x) \vee \gamma_B(x) \rangle$$

$$(I_3) A \cap B = \langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \gamma_A(x) \vee \gamma_B(x) \rangle$$

(2)  $A \cup B$  may be defined as:

$$(U_1) \quad A \cup B = \langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \gamma_A(x) \wedge \gamma_B(x) \rangle$$

$$(U_2) \quad A \cup B = \langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \gamma_A(x) \wedge \gamma_B(x) \rangle$$

$$(3) \quad [A] = \langle x, \mu_A(x), \sigma_A(x), 1 - \mu_A(x) \rangle$$

$$(4) \quad \langle x \rangle = \langle x, 1 - \gamma_A(x), \sigma_A(x), \gamma_A(x) \rangle$$

We can easily generalize the operations of intersection and union in definition 3.4 to arbitrary family of *NSS* as follow:

### 3.5 Definition

Let  $\{A_j : j \in J\}$  be an arbitrary family of *NSS* in  $X$ , then

(1)  $\cap A_j$  may be defined as:

$$(i) \quad \cap A_j = \langle x, \bigwedge_{j \in J} \mu_{A_j}(x), \bigwedge_{j \in J} \sigma_{A_j}(x), \bigvee_{j \in J} \gamma_{A_j}(x) \rangle$$

$$(ii) \quad \cap A_j = \langle x, \bigwedge_{j \in J} \mu_{A_j}(x), \bigvee_{j \in J} \sigma_{A_j}(x), \bigvee_{j \in J} \gamma_{A_j}(x) \rangle$$

(2)  $\cup A_j$  may be defined as:

$$(i) \quad \cup A_j = \langle x, \bigvee_{j \in J} \mu_{A_j}, \bigwedge_{j \in J} \sigma_{A_j}, \bigwedge_{j \in J} \gamma_{A_j} \rangle,$$

$$(ii) \quad \cup A_j = \langle x, \bigvee_{j \in J} \mu_{A_j}, \bigvee_{j \in J} \sigma_{A_j}, \bigwedge_{j \in J} \gamma_{A_j} \rangle.$$

### 3.6. Definition

Let  $A$  and  $B$  are neutrosophic sets then

$A|B$  may be defined as

$$A|B = \langle x, \mu_A \wedge \mu_B, \sigma_A(x) \sigma_B(x), \gamma_A \vee \gamma_B(x) \rangle$$

### 3.2. Proposition

For all  $A, B$  two neutrosophic sets then the following are true

$$(1) \quad C(A \cap B) = C(A) \cup C(B)$$

$$(2) \quad C(A \cup B) = C(A) \cap C(B)$$

## 1. NEUTROSOPHIC TOPOLOGICAL SPACES

Here we extend the concepts of fuzzy topological space [4], and intuitionistic fuzzy topological space [5, 7] to the case of neutrosophic sets.

### 4.1 Definition

A neutrosophic topology (*NT* for short) on a non empty set  $X$  is a family  $\tau$  of neutrosophic subsets in  $X$  satisfying the following axioms

$$(NT_1) \quad \emptyset_N, 1_N \in \tau,$$

$$(NT_2) \quad G_1 \cap G_2 \in \tau \text{ for any } G_1, G_2 \in \tau,$$

$$(NT_3) \quad \bigcup G_i \in \tau \quad \forall \{G_i : i \in J\} \subseteq \tau$$

In this case the pair  $(X, \tau)$  is called a neutrosophic topological space (*NTS* for short) and any neutrosophic set in  $\tau$  is known as neutrosophic open set (*NOS* for short) in  $X$ . The elements of  $\tau$  are called open neutrosophic sets, A neutrosophic set  $F$  is closed if and only if  $C(F)$  is neutrosophic open.

### 4.1 Example

Any fuzzy topological space  $(X, \tau_0)$  in the sense of Chang is obviously a *NTS* in the form  $\tau = \{A : \mu_A \in \tau_0\}$  wherever we identify a fuzzy set in  $X$  whose membership function is  $\mu_A$  with its counterpart.

**4.1. Remark** Neutrosophic topological spaces are very natural generalizations of fuzzy topological spaces allow more general functions to be members of fuzzy topology.

### 4.3 Example

Let  $X = \{x\}$  and

$$A = \{(x, 0.5, 0.5, 0.4) : x \in X\}$$

$$B = \{(x, 0.4, 0.6, 0.8) : x \in X\}$$

$$D = \{ \langle x, 0.5, 0.6, 0.4 \rangle : x \in X \}$$

$$C = \{ \langle x, 0.4, 0.5, 0.8 \rangle : x \in X \}$$

Then the family  $\tau = \{O_n, I_n, A, B, C, D\}$  of  $\mathcal{N}Ss$  in  $X$  is neutrosophic topology on  $X$ .

#### 4.4 Example

Let  $(X, \tau_0)$  be a fuzzy topological space in changes sense such that  $\tau_0$  is not indiscrete suppose now that

$\tau_0 = \{0_N, 1_N\} \cup \{V_j : j \in J\}$  then we can construct two  $NTSS$  on  $X$  as follows

a)  $\tau_0 = \{0_N, 1_N\} \cup \{ \langle x, V_j, \sigma(x), 0 \rangle : j \in J \}$ .

b)  $\tau_0 = \{0_N, 1_N\} \cup \{ \langle x, V_j, 0, \sigma(x), 1 - V_j \rangle : j \in J \}$ .

#### 4.1 Proposition

Let  $(X, \tau)$  be an  $NTS$  on  $X$ , then we can also construct several  $NTSS$  on  $X$  in the following way:

a)  $\tau_{o,1} = \{ [ ]G : G \in \tau \}$ ,

b)  $\tau_{o,2} = \{ \langle \rangle G : G \in \tau \}$ .

#### Proof

a)  $(NT_1)$  and  $(NT_2)$  are easy.

$(NT_3)$  Let  $\{ [ ]G_j : j \in J, G_j \in \tau \} \subseteq \tau_{o,1}$ . Since

$\cup G_j = \{ \langle x, \vee \mu_{G_j}, \vee \sigma_{G_j}, \wedge \gamma_{G_j} \rangle \text{ or } \langle x, \vee \mu_{G_j}, \wedge \sigma_{G_j}, \wedge \gamma_{G_j} \rangle \text{ or } \langle x, \vee \mu_{G_j}, \wedge \sigma_{G_j}, \vee \gamma_{G_j} \rangle \} \in \tau$ , we have

$$\cup ([ ]G_j) = \{ \langle x, \vee \mu_{G_j}, \vee \sigma_{G_j}, \wedge (1 - \mu_{G_j}) \rangle \text{ or } \langle x, \vee \mu_{G_j}, \vee \sigma_{G_j}, (1 - \vee \mu_{G_j}) \rangle \} \in \tau_{o,1}$$

b) This similar to (a)

#### 4.2 Definition

Let  $(X, \tau_1), (X, \tau_2)$  be two neutrosophic topological spaces on  $X$ . Then  $\tau_1$  is said be contained in  $\tau_2$  (in symbols  $\tau_1 \subseteq \tau_2$ ) if  $G \in \tau_2$  for each  $G \in \tau_1$ . In this case, we also say that  $\tau_1$  is coarser than  $\tau_2$ .

#### 4.2 Proposition

Let  $\{ \tau_j : j \in J \}$  be a family of  $NTSS$  on  $X$ . Then  $\cap \tau_j$  is a neutrosophic topology on  $X$ . Furthermore,  $\cap \tau_j$  is the coarsest  $NT$  on  $X$  containing all  $\tau_j$ , s

**Proof.** Obvious

#### 4.3 Definition

The complement of  $A$  ( $C(A)$  for short) of  $NOS$ .  $A$  is called a neutrosophic closed set ( $NCS$  for short) in  $X$ .

Now, we define neutrosophic closure and interior operations in neutrosophic topological spaces:

#### 4.4 Definition

Let  $(X, \tau)$  be  $NTS$  and  $A = \langle x, \mu_A(x), \gamma_A(x), \sigma_A(x) \rangle$  be a  $NS$  in  $X$ .

Then the neutrosophic closer and neutrosophic interior of  $A$  are defined by

$$NCl(A) = \cap \{ K : K \text{ is an NCS in } X \text{ and } A \subseteq K \}$$

$$NInt(A) = \cup \{ G : G \text{ is an NOS in } X \text{ and } G \subseteq A \}$$

It can be also shown that  $NCl(A)$  is  $NCS$  and  $NInt(A)$  is a  $NOS$  in  $X$

a)  $A$  is in  $X$  if and only if  $NCl(A)$ .

b)  $A$  is  $NCS$  in  $X$  if and only if  $NInt(A) = A$ .

#### 4.2 Proposition

For any neutrosophic set  $A$  in  $(X, \tau)$  we have

(a)  $NCl(C(A)) = C(NInt(A))$ ,

(b)  $NInt(C(A)) = C(NCl(A))$ .

#### Proof.

a) Let  $A = \langle x, \mu_A, \sigma_A, \nu_A \rangle : x \in X$  and suppose that the family of neutrosophic subsets contained in  $A$  are indexed by the family if  $NSS$  contained in  $A$  are indexed by the

family  $A = \{ \langle x, \mu_{G_i}, \sigma_{G_i}, \nu_{G_i} \rangle : i \in J \}$ . Then we see that  $NInt(A) = \{ \langle x, \vee \mu_{G_i}, \vee \sigma_{G_i}, \wedge \nu_{G_i} \rangle \}$  and hence  $C(NInt(A)) = \{ \langle x, \wedge \mu_{G_i}, \vee \sigma_{G_i}, \vee \nu_{G_i} \rangle \}$ . Since  $C(A)$  and  $\mu_{G_i} \leq \mu_A$  and  $\nu_{G_i} \geq \nu_A$  for each  $i \in J$ , we obtaining  $C(A)$ . i.e.  $NCl(C(A)) = \{ \langle x, \wedge \nu_{G_i}, \vee \sigma_{G_i}, \vee \mu_{G_i} \rangle \}$ . Hence  $NCl(C(A)) = C(NInt(A))$ , follows immediately

b) This is analogous to (a).

### 4.3 Proposition

Let  $(x, \tau)$  be a *NTS* and  $A, B$  be two neutrosophic sets in  $X$ . Then the following properties hold:

- (a)  $NInt(A) \subseteq A$ ,
- (b)  $A \subseteq NCl(A)$ ,
- (c)  $A \subseteq B \Rightarrow NInt(A) \subseteq NInt(B)$ ,
- (d)  $A \subseteq B \Rightarrow NCl(A) \subseteq NCl(B)$ ,
- (e)  $NInt(NInt(A)) = NInt(A) \wedge NInt(B)$ ,
- (f)  $NCl(A \cup B) = NCl(A) \vee NCl(B)$ ,
- (g)  $NInt(1_N) = 1_N$ ,
- (h)  $NCl(0_N) = 0_N$ .

**Proof** (a), (b) and (e) are obvious (c) follows from (a) and Definitions.

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