NEVANLINNA THEORY AND HOLOMORPHIC MAPPINGS BETWEEN ALGEBRAIC VARIETIES

BY

PHILLIP GRIFFITHS and JAMES KING [1]

Harvard University, Cambridge, Mass. 02138, USA and M.I.T., Cambridge, Mass. 02139, USA

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Introduction

Let A, V be smooth algebraic varieties with V *projective* (and therefore compact). We wish to study holomorphic mappings

$$A \xrightarrow{f} V. \tag{1}$$

The most important case is when A is affine, and is thus representable as an algebraic subvariety in $\mathbf{C}^N$, and we shall make this assumption throughout. Then the mapping f is generally *not* an algebraic mapping, but may well have an essential singularity at infinity in A. *Nevanlinna theory*, or the *theory of value distributions*, studies the position of the image $f(A)$ relative to the algebraic subvarieties of V. Given an algebraic subvariety $Z \subset V$, we set $Z_f = f^{-1}(Z)$ and assume throughout that

$$\operatorname{codim}_x(Z_f) = \operatorname{codim}_{f(x)}(Z)$$

at all points $x \in A$. There are two basic questions with which we shall deal:

(A) Can we find an *upper bound* on the size of Z_f in terms of Z and the "growth" of the mapping f?

(B) Can we find a *lower bound* on the size of Z_f, again in terms of Z and the growth of the mapping?

We are able to give a reasonably satisfactory answer to (A) in case codim $(Z)=1$ and to (B) in case codim $(\mathbf{Z})=1$ and the image $f(A)$ contains an open set in V.

Let us explain this in more detail. The affine algebraic character of A enters in that A possesses a *special exhaustion function* (cf. § 2); i.e., an exhaustion function

$$A \xrightarrow{\tau} \mathbf{R} \cup \{-\infty\} \tag{2}$$

which satisfies

$$\begin{cases} \tau \text{ is proper} \\ dd^c\tau \geq 0 \\ (dd^c\tau)^{m-1} \neq 0 \text{ but } (dd^c\tau)^m = 0 \text{ where } \dim_{\mathbf{C}} A = m.\,(^1) \end{cases} \tag{3}$$

We set $A[r]=\{x \in A: \tau(x) \leq r\}$, and for an analytic subvariety $W \subset A$ define

$$\begin{cases} n(W,t) = \displaystyle\int_{W[t]} (dd^c\tau)^d \quad (d = \dim_{\mathbf{C}} W) \\ N(W,r) = \displaystyle\int_0^r n(W,t)\frac{dt}{t} \quad \text{(counting function).} \end{cases} \tag{4}$$

(The reason for logarithmically averaging $n(W,t)$ is the usual one arising from *Jensen's theorem.*) We may think of the counting function $N(W,r)$ as measuring the *growth* of W; e.g., it follows from a theorem of Stoll [17] (which is proved below in § 4 in case codim $(W)=1$) that

$$W \text{ is algebraic} \Leftrightarrow N(W,r) = O(\log r).$$

Suppose now that $\{Z_\lambda\}_{\lambda \in \Lambda}$ is an algebraic family of algebraic subvarieties $Z_\lambda \subset V$ (think of the Z_λ as being linear spaces in $\mathbf{P}^N$, in which case the parameter space Λ is a Grassmannian). Suppose that $d\lambda$ is a smooth measure on Λ, and define the *average* or *order function* for f and $\{Z_\lambda\}_{\lambda\in\Lambda}$ by

$$T(r) = \int_{\lambda \in \Lambda} N(f^{-1}(Z_\lambda), r)\, d\lambda. \tag{5}$$

The First Main Theorem (F.M.T.) expresses $N(f^{-1}(Z_\lambda), r)$ in terms of $T(r)$, and leads to an inequality

$$N(f^{-1}(Z_\lambda), r) \leq T(r) + S(r,\lambda) + O(1) \tag{6}$$

(1) A by-product of the construction is a short and elementary proof of *Chow's theorem*; this is also given in § 2.

in case the Z_λ are complete intersections of positive divisors (cf. § 5). The *remainder term* $S(r, \lambda)$ is non-negative, and for divisors the condition $(dd^c\tau)^m = 0$ in (3) gives $S(r, \lambda) \equiv 0$. In this case (6) reduces to a Nevanlinna inequality

$$N(f^{-1}(Z_\lambda), r) \leqslant T(r) + O(1), \tag{7}$$

which bounds the growth of any $f^{-1}(Z_\lambda)$ by the average growth. Such inequalities are entirely lacking when codim $(Z_\lambda) > 1$ [7], and finding a suitable method for studying the size of $f^{-1}(Z_\lambda)$ remains as one of the most important problems in general Nevanlinna theory.

Our F.M.T. is similar to that of many other authors; cf. Stoll [18] for a very general result as well as a history of the subject. One novelty here is our systematic use of the local theory of currents and of "blowings up" to reduce the F.M.T. to a fairly simple and essentially local result, even in the presence of singularities (cf. § 1). Another new feature is the isolation of special exhaustion functions which account for the "parabolic character" of affine algebraic varieties.

Concerning problem (B) of finding a lower bound on $N(f^{-1}(Z_\lambda), r)$, we first prove an *equidistribution in measure* result (§ 5 (c)) following Chern, Stoll, and Wu (cf. [18] and the references cited there). This states that, under the condition

$$\int_{\lambda \in \Lambda} S(r, \lambda)\, d\lambda = o(T(r)) \tag{8}$$

in (6), the image $f(A)$ meets almost all Z_λ in the measure-theoretic sense. In the case of divisors, $S(r, \lambda) \equiv 0$ so that (8) is trivially satisfied, and then we have a *Casorati-Weierstrass* type theorem for complex manifolds having special exhaustion functions.

Our deeper results occur when the Z_λ are divisors and the image $f(A)$ contains an open subset of V (*Note:* it does not follow from this that $\overline{f(A)} = V$, as illustrated by the Fatou-Bieberbach example [3]). In this case we use the method of *singular volume forms* (§ 6(a)) introduced in [6] to obtain a Second Main Theorem (S.M.T.) of the form (§ 6(b))

$$T^{\neq}(r) + N_1(r) \leqslant N(f^{-1}(Z_\lambda), r) + \log \frac{d^2 T^{\neq}(r)}{dr^2} + O(\log r) \tag{9}$$

under the assumptions that (i) the divisor Z_λ has *simple normal crossings* (cf. § 0 for the definition), and (ii)

$$c(Z_\lambda) > c(K_V^*) \tag{10}$$

where K_V^* is the anti-canonical divisor and $c(D)$ denotes the *Chern class* of a divisor D.

In (9), $T^{\#}(r)$ is an increasing convex function of $\log r$ which is closely related to the order function $T(r)$ in (5), and the other term

$$N_1(r) = N(R, r)$$

where $R \subset A$ is the *ramification divisor* of f. It is pretty clear that (9) gives a lower bound on $N(f^{-1}(Z_\lambda), r)$, and when this is made precise we obtain a *defect relation* of the following sort: Define the *Nevanlinna defect*

$$\delta(Z_\lambda) = 1 - \overline{\lim_{r \to \infty}} \frac{N(f^{-1}(Z_\lambda), r)}{T(r)}. \tag{11}$$

Then $0 \leqslant \delta(Z_\lambda) \leqslant 1$ because of (7), and $\delta(Z_\lambda) = 1$ if $f(A)$ does not meet Z_λ. Then, under the above assumptions,

$$\delta(Z_\lambda) \leqslant \frac{c(K_V^*)}{c(Z_\lambda)} + \varkappa \tag{12}$$

where $\varkappa = 0$ in case $A = \mathbf{C}^N$ or f is transcendental. As a corollary to (12), we have a *big Picard theorem*. In case Z_λ has simple normal crossings and $c(Z_\lambda) > c(K_V^*)$, any holomorphic mapping $A \to V - Z_\lambda$ such that $f(A)$ contains an open set is necessarily rational. (1)

For $V = \mathbf{P}^1$ and $Z_\lambda = \{0, 1, \infty\}$ we obtain the usual big Picard theorem, and in case $\dim_{\mathbf{C}} V = \dim_{\mathbf{C}} A$ and $c(K_V) > 0$ (so that we may take Z_λ to be empty), we obtain the main results in [11].

It should be remarked that our big Picard theorems are presented globally on the domain space, in that they state that a holomorphic mapping $f: A \to V$ between algebraic varieties is, under suitable conditions, necessarily rational. The corresponding local statement is that a holomorphic mapping $f: M - S \to V$ defined on the complement of an analytic subvariety S of a complex manifold M extends meromorphically across S, and these results will be proved in the Appendix. The reason for stating our results globally in the main text is to emphasize the strongly geometric flavor of the Nevanlinna theory.

In addition to finding an upper bound on $N(f^{-1}(Z_\lambda), r)$ when codim $(Z_\lambda) > 1$, the other most important outstanding general problem in Nevanlinna theory is to obtain lower bounds (or defect relations) on the counting functions $N(f^{-1}(Z_\lambda), r)$ when Z_λ is a divisor but where the image $f(A)$ may not contain an open set. In addition to the Ahlfors defect relation [1] for

$$\mathbf{C} \xrightarrow{f} \mathbf{P}^n,$$

there has been some recent progress on this question by M. Green [10]. Since it is *always* the case that

(1) Our terminology regarding Picard theorems is the following: A little Picard theorem means that a holomorphic mapping is degenerate, and a big Picard theorem means that a holomorphic mapping has an inessential singularity.

$$\int_{\lambda \in \Lambda} \delta(Z_\lambda)\, d\lambda = 0, \tag{13}$$

it at least makes sense to look for a general defect relation.

To conclude this introduction, we want to discuss a little the problem of finding applications of Nevanlinna theory. The global study of holomorphic mappings certainly has great formal elegance and intrinsic beauty, but as mentioned by Ahlfors in the introduction to [1] and by Wu in [21], has suffered a lack of applications. This state of affairs seems to be improving, and indeed one of our main points in this paper has been to emphasize some applications of Nevanlinna theory.

In § 4 we have used the F.M.T. to give a simple proof of Stoll's theorem [17] that a divisor D in $\mathbf{C}^n$ is algebraic $\Leftrightarrow$

$$\frac{v(D[r])}{r^{2n-2}} = O(1)$$

where $v(D[r])$ is the Euclidean volume of $D \cap \{z: \|z\| \leqslant r\}$. This proof is in fact similar to Stoll's original proof, but we are able to avoid his use of degenerate elliptic equations by directly estimating the remainder term in the F.M.T. (this is the only case we know where such an estimate has been possible).

In § 9 (b) we have used a S.M.T. to prove an analogue of the recent extension theorem of Kwack (cf. [11] for a proof and further reference). Our result states that if V is a quasi-projective, *negatively curved* algebraic variety having a *bounded ample line bundle* (cf. § 9 (b) for the definitions), then any holomorphic mapping $f: A \to V$ from an algebric variety A into V is necessarily rational. Our hypotheses are easily verified in case $V = X/\Gamma$ is the quotient of a bounded symmetric domain by an arithmetic group [2], and so we obtain a rather conceptual and easy proof of the result of Borel [5] that any holomorphic mapping $f: A \to X/\Gamma$ is rational. This theorem has been extremely useful in algebraic geometry; e.g., it was recently used by Deligne to verify the Riemann hypothesis for $K3$ surfaces.

In § 9 (a) we have used the method of singular volume forms to derive a generalization of R. Nevanlinna's "lemma on the logarithmic derivative" [16]. Here the philosophy is that estimates are possible using metrics, or volume forms, whose curvature is negative but not necessarily bounded away from zero. Such estimates are rather delicate, and we hope to utilize them in studying holomorphic curves in general algebraic varieties.

Finally, still regarding applications of value distribution theory we should like to call attention to a recent paper of Kodaira [14] in which, among other things, he uses Nevanlinna theory to study analytic surfaces which contain $\mathbf{C}^2$ as an open set. In a related development, Iitaka (not yet published) has used Nevanlinna theory to partially classify algebraic varieties of dimension 3 whose universal convering is $\mathbf{C}^3$.

As a general source of "big Picard theorems" and their applications, we suggest the excellent recent monograph *Hyperbolic manifolds and holomorphic mappings,* Marcel Dekker, New York (1970) by S. Kobayashi, which, among other things, contains the original proof of Kwack's theorem along with many interesting examples and open questions.

0. Notations and terminology

(a) Divisors and line bundles

Let M be a complex manifold. Given an open set $U \subset M$, we shall denote by $\mathcal{M}(U)$ the field of meromorphic functions in U, by $\mathcal{O}(U)$ the ring of holomorphic functions in U, and by $\mathcal{O}^*(U)$ the nowhere vanishing functions in $\mathcal{O}(U)$. Given a meromorphic function $\alpha \in \mathcal{M}(U)$, the divisor (α) is well-defined. A *divisor* D on M has the property that

$$D \cap U = (\alpha) \quad (\alpha \in \mathcal{M}(U))$$

for sufficiently small open sets U on M. Equivalently, a divisor is a locally finite sum of irreducible analytic hypersurfaces on M with integer coefficients. The divisor is *effective* if locally $D \cap U = (\alpha)$ for a *holomorphic* function $\alpha \in \mathcal{O}(U)$. Two divisors D_1, D_2 are *linearly equivalent* if $D_1 - D_2 = (\alpha)$ is the divisor of a global meromorphic function α on M. We shall denote by $|D|$ the complete linear system of effective divisors linearly equivalent to a fixed effective divisor D.

Suppose now that M is compact so that we have Poincaré duality between $H_q(M, \mathbf{Z})$ and $H^{2m-q}(M, \mathbf{Z})$. A divisor D on M carries a fundamental homology class

$$\{D\} \in H_{2m-2}(M, \mathbf{Z}) \cong H^2(M, \mathbf{Z}).$$

We may consider $\{D\}$ as an element in $H^2_{\mathrm{DR}}(M, \mathbf{R})$, the de Rham cohomology group of closed C^∞ differential forms modulo exact ones. Then the divisor D is said to be *positive,* written $D > 0$, if $\{D\}$ is represented by a closed, positive (1, 1) form ω. Thus locally

$$\omega = \frac{\sqrt{-1}}{2\pi} \sum_{i,j} g_{i\bar{j}}\, dz_i \wedge d\bar{z}_j$$

where the Hermitian matrix $(g_{i\bar{j}})$ is positive definite. In this way there is induced a partial linear ordering on the set of divisors on M.

We want to have a method for localizing the above considerations, and for this we will use the theory of *line bundles.* A line bundle is defined to be a holomorphic vector bundle

$$L \to M$$

with fibre $\mathbf{C}$. Relative to a suitably small covering $\{U_i\}$ of M, there will be trivializations

$$L\,|\,U_i \cong \mathbf{C} \times U_i$$

which then lead in the usual way to the transition functions $\alpha_{ij} \in O^*(U_i \cap U_j)$ for L. These transition functions obey the cocycle rule $\alpha_{ij}\,\alpha_{jk} = \alpha_{ik}$ in $U_i \cap U_j \cap U_k$, and indeed it is well-known that the group of isomorphism classes of line bundles on M is just the Čech cohomology group $H^1(M, O^*)$. The vector space of holomorphic cross sections $H^0(V, L)$ is given by those collections of functions $\sigma = \{\sigma_i\}$ where $\sigma_i \in O(U_i)$ and

$$\sigma_i = \alpha_{ij}\,\sigma_j$$

in $U_i \cap U_j$. For each cross-section σ the divisor D_σ given by $D_\sigma \cap U_i = (\sigma_i)$ is well-defined, and any two such divisors are linearly equivalent. We shall denote by $|L|$ the complete linear system of effective divisors D_σ for $\sigma \in H^0(V, L)$. Clearly $|L| \cong P(H^0(V, L))$, the projective space of lines in the vector space $H^0(V, L)$.

Let D be a divisor on M. Then $D \cap U_i = (\alpha_i)$ and the ratios

$$\alpha_i/\alpha_j = \alpha_{ij} \in O^*(U_i \cap U_j)$$

give transition functions for a line bundle $[D] \to M$. Moreover, if D is effective, then there is a holomorphic section $\sigma \in H^0(V, [D])$ such that $D = D_\sigma$. The mapping $D \to [D]$ is a homorphism from the group of divisors on M to the group of line bundles, and we obviously have the relation

$$|D| = |[D]|$$

for any effective divisor D.

Returning to our consideration of line bundles, the coboundary map

$$H^1(M, O^*) \xrightarrow{\ \delta\ } H^2(M, \mathbf{Z})$$

arising from the cohomology sequence of the exponential sheaf sequence $0 \to \mathbf{Z} \to O \to O^* \to 1$ allows us to define the *Chern class* $c(L) = \delta(\{\alpha_{ij}\})$ for any line bundle $L \to M$. We wish to give a prescription for computing $c(L)$ in the de Rham group $H^2_{\mathrm{DR}}(M, \mathbf{R})$. For this recall that a metric in L is given by positive C^∞ functions ϱ_i in U_i which satisfy $\varrho_i = |\alpha_{ij}|^2\,\varrho_j$ in $U_i \cap U_j$. Thus, if $\sigma = \{\sigma_i\}$ is a section of L, then the length function

$$|\sigma|^2 = \frac{|\sigma_i|^2}{\varrho_i} \tag{0.1}$$

is well-defined on M. The closed $(1, 1)$ form ω given by

$$\omega\,|\,U_i = dd^c \log(\varrho_i) \tag{0.2}$$

is globally defined and represents the Chern class $c(L)$ in $H^2_{\mathrm{DR}}(M, \mathbf{R})$. We call ω the *curvature* *form* (for the metric $\{\varrho_i\}$) for the line bundle $L \to M$. If $\{\varrho_i'\}$ is another metric leading to its curvature form ω', then the difference

$$\omega - \omega' = dd^c \varphi \tag{0.3}$$

where φ is a global C^∞ function on M.

If M is a compact Kähler manifold, then every closed (1, 1) form ω in the cohomology class $c(L) \in H^2_{\mathrm{DR}}(M, \mathbf{R})$ is a curvature form for a suitable metric in $L \to M$. In particular, any two representatives of $c(L)$ in $H^2_{\mathrm{DR}}(M, \mathbf{R})$ will satisfy (0.3). We shall say that the line bundle $L \to M$ is *positive*, written $L > 0$, if there is a metric in L whose curvature form is a positive-definite (1, 1) form.

Now suppose again that D is a divisor on M with corresponding line bundle $[D]$. Then we have the equality

$$\{D\} = c([D]) \in H^2(M, \mathbf{Z})$$

between the homology class of D and the Chern class of $[D]$. Moreover, the divisor D is positive if, and only if, the line bundle $[D]$ is positive. Thus, between the divisors and line bundles we have a complete dictionary:

$$\begin{gathered} D \leftrightarrow [D] \\ |D| \leftrightarrow |[D]| \\ \{D\} \leftrightarrow c([D]) \\ D > 0 \Leftrightarrow [D] > 0. \end{gathered}$$

As mentioned above, the reason for introducing the line bundles is that it affords us a good technique for localizing and utilizing metric methods in the study of divisors. Moreover, the theory of line bundles is contravariant in a very convenient way. Thus, given a holomorphic map $f: N \to M$ and a line bundle $L \to M$, there is an induced line bundle $L_f \to N$. Moreover, there is a homomorphism

$$\sigma \to \sigma_f$$

from $H^0(M, L)$ to $H^0(N, L_f)$, and the relation

$$(\sigma_f) = f^{-1}(D) \underset{\text{defn.}}{=} D_f$$

holds valid. Finally, a metric in $L \to M$ induces a metric in $L_f \to N$, and the curvature forms are contravariant so that the curvature form ω_f for L_f is the pull-back of the curvature form ω for L. In summary then, the theory of line bundles both localizes and functorializes the study of divisors on a complex manifold.

One last notation is that a divisor D on M is said to have *normal crossings* if locally D is given by an equation

$$z_1 \dots z_k = 0$$

where $(z_1, \ldots, z_n)$ are local holomorphic coordinates on M. If moreover each irreducible component of D is smooth, then we shall say that D has *simple normal crossings*. In case $M = \mathbf{P}^m$ is complex projective space and $D = H_1 + \ldots + H_N$ is linear combination of hyperplanes, then D has normal crossings if, and only if, the hyperplanes H_μ $(\mu = 1, \ldots, N)$ are in *general position*.

(b) The canonical bundle and volume forms

Let M be a complex manifold and $\{U_i\}$ a covering of M by coordinate neighborhoods with holomorphic coordinates $z_i = (z_i^1, \ldots, z_i^n)$ in U_i. Then the Jacobian determinants

$$\varkappa_{ij} = \det\left(\frac{\partial z_j^\alpha}{\partial z_i^\beta}\right)$$

define the canonical bundle $\varkappa_M \to M$. The holomorphic cross-sections of this bundle are the globally defined holomorphic n-forms on M.

A *volume form* Ψ on M is a C^∞ and everywhere positive (n, n) form. Using the notation

$$\Phi_i = \frac{\sqrt{-1}}{2\pi}(dz_i^1 \wedge d\bar{z}_i^1) \wedge \ldots \wedge \frac{\sqrt{-1}}{2\pi}(dz_i^n \wedge d\bar{z}_i^n),$$

a volume form has the local representation

$$\Psi = \varrho_i \Phi_i \tag{0.4}$$

where ϱ_i is a positive C^∞ function. The transition rule in $U_i \cap U_j$ is

$$\varrho_i = |\varkappa_{ij}|^2 \varrho_j,$$

so that a volume form is the same as a metric in the canonical bundle. The curvature form is, in this case, called the *Ricci form* and denoted by $\operatorname{Ric}\Psi$. Thus, in U_i,

$$\operatorname{Ric}\Psi = dd^c \log \varrho_i. \tag{0.5}$$

The conditions

$$\begin{aligned} &\operatorname{Ric}\Psi > 0 \\ &(\operatorname{Ric}\Psi)^n \geqslant c\Psi \quad (c > 0) \end{aligned} \tag{0.6}$$

will play a decisive role for us. Geometrically, they may be thought of as saying that "the canonical bundle has positive curvature which is bounded from below." To explain this, suppose that M is a Riemann surface. Using the correspondence

$$\varrho \frac{\sqrt{-1}}{2\pi}(dz \wedge d\bar{z}) \leftrightarrow \varrho\, dz \otimes d\bar{z},$$

we see that a volume form is the same as a Hermitian metric on M. Furthermore, the Ricci form

$$\operatorname{Ric}\left\{\varrho \frac{\sqrt{-1}}{2\pi}(dz \wedge d\bar{z})\right\} = -\varkappa\left\{\varrho \frac{\sqrt{-1}}{4\pi}(dz \wedge d\bar{z})\right\} \tag{0.7}$$

where $\varkappa = -(1/\varrho)/(\partial^2 \log \varrho)/(\partial z\, \partial \bar{z})$ is the *Gaussian curvature* of the Hermitian metric $\varrho\, dz \otimes d\bar{z}$. We see then that (0.6) is equivalent to

$$\varkappa \leqslant -c_1 < 0,$$

the Gaussian curvature should be *negative* and bounded from above. We have chosen our signs in the definition of Ric Ψ so as to avoid carrying a $(-1)^m$ sign throughout.

The theory of volume forms is contravariant. If M and N are complex manifolds of the same dimension and $f\colon M \to N$ is a holomorphic mapping, then for a volume form Ψ on N the pull-back $\Psi_f = f^*\Psi$ is a *pseudo-volume form* on M. This means that Ψ_f is positive outside an analytic subvariety of M (in this case, outside the ramification divisor of f).

(c) Differential forms and currents

(Lelong [15].) On a complex manifold M we denote by $A^{p,q}(M)$ the vector space of C^∞ differential forms of type (p,q) and by $A_c^{p,q}(M)$ the forms with compact support. Providing $A_c^{m-p,m-q}(M)$ with the Schwartz topology, the dual space $C^{p,q}(M)$ is the space of currents of type (p,q) on M. Given a current T and a form φ, we shall denote by $T(\varphi)$ the value of T on φ. The graded vector space of currents

$$C^*(M) = \bigoplus_{p,q} C^{p,q}(M)$$

forms a module over the differential forms $A^*(M) = \bigoplus_{p,q} A^{p,q}(M)$ by the rule

$$\varphi \wedge T(\eta) = T(\varphi \wedge \eta)$$

where $\varphi \in A^*(M)$, $T \in C^*(M)$ and $\eta \in A_c^*(M)$.

We shall use the notations

$$\begin{cases} d = \partial + \bar{\partial}; \\ d^c = \dfrac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial); \\ dd^c = \left(\dfrac{\sqrt{-1}}{2\pi}\right)\partial\bar{\partial}. \end{cases} \tag{0.8}$$

The factor $1/4\pi$ is put in front of d^c to eliminate the need for keeping track of universal constants, such as the area of the unit sphere in $\mathbf{C}^m$, in our computations. As usual, all differential operators act on the currents by rules of the form

$$\partial T(\varphi) = T(\partial\varphi).$$

The action of $A^*(M)$ is compatible with these rules.

A current $T \in C^{p,p}(M)$ is real if $T = \bar{T}$, closed if $dT = 0$, and *positive* if

$$(\sqrt{-1})^{p(p-1)/2}\, T(\eta \wedge \bar{\eta}) \geqslant 0$$

for all $\eta \in A_c^{m-p,0}(M)$. In case $p = 1$, we may locally write $T \in C^{1,1}(M)$ as

$$T = \frac{\sqrt{-1}}{2\pi} \sum_{i,j} t_{ij}\, dz_i \wedge d\bar{z}_j$$

where the t_{ij} may be identified with distributions according to the rule

$$(-1)^{i+j+m-1}\, t_{ij}(\alpha) = T(\alpha\, dz_1 \ldots d\hat{z}_i \ldots dz_n \wedge d\bar{z}_1 \ldots \widehat{d\bar{z}_j} \ldots dz_m).$$

Then T is real and positive if, the distributions

$$T(\lambda) = \sum_{i,j} t_{ij} \lambda_i \bar{\lambda}_j \quad (\lambda_j \in \mathbf{C})$$

are non-negative on positive functions. In this case, by taking monotone limits we may extend the domain of definition of $T(\lambda)$ from the C^∞ functions to a suitable class of functions in $L^1(\mathrm{loc}, M)$ which are integrable for the positive Radon measure

$$\alpha \to T(\lambda)(\alpha)$$

initially defined on the C^∞ functions. A similar discussion applies to positive currents of type (p, p).

For any positive current T, each of the distributions t_{ij} is a Radon measure; in addition each t_{ij} is absolutely continuous with respect to the diagonal measure $\sum_i t_{ii}$ [15].

The principal examples of currents we shall utilize are the following three:

(i) A form $\psi \in A^{p,q}(M)$ may be considered as a current by the rule

$$\psi(\varphi) = \int_M \psi \wedge \varphi \quad (\varphi \in A_c^{m-p,m-q}(M)). \tag{0.9}$$

By Stokes' theorem, $d\psi$ in the sense of currents agrees with $d\psi$ in the sense of differential forms. Moreover, the $A^*(M)$-module structure on $C^*(M)$ induces the usual exterior multiplication on the subspace $A^*(M)$ of $C^*(M)$.

(ii) An analytic subvariety Z of M of pure codimension q defines a current $Z \in C^{q,q}(M)$ by the formula [15]

$$Z(\varphi) = \int_{Z_{\text{reg}}} \varphi \quad (\varphi \in A_c^{m-q,m-q}(M)). \tag{0.10}$$

This current is real, closed and positive. By linearity, any analytic cycle on M also defines a current.

Note on multiplicities. We say that the current Z is a variety with multiplicities if there is a variety $|Z|$ and an integer-valued function $n(z)$ on $\operatorname{Reg}|Z|$ which is locally-constant on this manifold. Then Z is the pair $(|Z|, n)$ and $Z(\varphi) = \int_Z n(z)\,\varphi$. It is clear that $dZ = d^cZ = 0$.

Now given holomorphic functions $f_1, \ldots, f_r$ and $|Z| = \{f_1 = \ldots = f_r = 0\}$, there is an *integer multiplicity* $\operatorname{mult}_z Z$ defined algebraically at each z and which is locally constant on $\operatorname{Reg}|Z|$, [8]. This is what we will mean by saying $Z = \{f_1 = \ldots = f_r = 0\}$ with algebraic multiplicities. Multiplicities on the set $\operatorname{sing}|Z|$ will be ignored since $\operatorname{sing}|Z|$ is a set of measure 0.

(iii) We shall denote by $L^1_{(p,q)}(\text{loc}, M)$ the vector space of (p, q) forms whose coefficients are locally L^1 functions on M. Each $\psi \in L^1_{(p,q)}(\text{loc}, M)$ defines a current by the formula (0.9) above. In the cases we shall consider, ψ will be C^∞ outside an analytic subset S of M. Moreover, ψ will have singularities of a fairly precise type along S, and $d\psi$ in the sense of differential forms on $M - S$ will again be locally L^1 on all of M. It will usually *not* be the case, however, that $d\psi$ in the sense of currents agrees with $d\psi$ in the sense of differental forms. This is because the singularities of ψ will cause trouble in Stokes' theorem, and we will have an equation of the type

$$\left\{\begin{matrix} d\psi \text{ in the} \\ \text{sense of} \\ \text{currents} \end{matrix}\right\} = \left\{\begin{matrix} d\psi \text{ in the} \\ \text{sense of} \\ \text{forms} \end{matrix}\right\} + \left\{\begin{matrix} \text{current} \\ \text{supported} \\ \text{on } S. \end{matrix}\right\} \tag{0.11}$$

The relation (0.11) will be the basis of all our integral formulas.

1. Differential forms, currents and analytic cycles

(a) The Poincaré equation

Let U be an open set in a complex manifold of dimension n and let $\alpha \in \mathcal{M}(U)$ be a meromorphic function on U. Denote by $D = (\alpha)$ the divisor of α. Then both D and $\log|\alpha|^2$ define currents as described in § 0 (c). We wish to show that

$$D = dd^c \log|\alpha|^2;$$

this is a kind of residue formula as will become apparent from the proof. In fact, we will prove the following stronger result, which will be useful in the next two sections:

(1.1) PROPOSITION. *For α and $D=(\alpha)$ as above, let $X\subset U$ be a purely k-dimensional subvariety such that* $\dim(X\cap D)=k-1$. *Then*

(1.2) $dd^c(X\wedge \log|\alpha|^2)=X\cdot D$ (*Poincaré equation*).

Remark. What this means is that, for any $\varphi\in A_c^{k-1,k-1}(U)$, we have

$$\int_X \log|\alpha|^2 dd^c\varphi=\int_{X\cdot D}\varphi \tag{1.3}$$

where the integral on the left always converges, and where $X.D$ is the usual intersection of analytic varieties. If $X=U$, then we have

$$\int_U \log|\alpha|^2 dd^c\varphi=\int_D\varphi.$$

To prove the result, we need this lemma, whose proof will be given toward the end of this section:

(1.4) LEMMA. *If α is not constant on any component of X, $\log|\alpha|^2$ is locally L^1 on X, or equivalently $\int_X\log|\alpha|^2\mu$ is defined for all $\mu\in A_c^{k,k}(U)$. Also, $dd^c(X\wedge\log|\alpha|^2)$ is a positive current.*

Proof of proposition. Since both sides of the equation are linear, we may use a partition of unity to localize the problem. Initially we may choose U small enough that α is a quotient of holomorphic functions α_1/α_2. Since $\log|\alpha_1/\alpha_2|^2=\log|\alpha_1|^2-\log|\alpha_2|^2$ and $(\alpha_1/\alpha_2)=(\alpha_1)-(\alpha_2)$ we may assume that α is holomorphic in U.

First, let us assume that both X and $X\cap D$ are nonsingular; by localizing further, we can choose coordinates $(w_1,\ldots,w_k)$ on X such that $X\cap D=\{w_k=0\}$. In this case the restriction of α to X equals βw_k^r, where β is a holomorphic function which never vanishes on U. Thus on X, $X\cdot D=r(w_k)$.

Furthermore, since $\log|\alpha|^2=\log|\beta|^2+r\log|w_k|^2$ and $dd^c\log|\beta|^2=0$, it suffices to show the proposition assuming $\alpha=w_k$. For $\varphi\in A_c^{k-1,k-1}(U)$,

$$\int_X\log|w_k|^2dd^c\varphi=\lim_{\varepsilon\to 0}\int_{X_\varepsilon}\log|w_k|^2dd^c\varphi \tag{1.5}$$

where $X_\varepsilon=\{x\in X=|w_k(x)|\geqslant\varepsilon\}$. Thus $\partial X_\varepsilon=-S_\varepsilon$, where $S_\varepsilon=\{x\in X:|w_k(x)|=\varepsilon\}$ is oriented with its normal in the direction of increasing $|w_k|$. Then by Stokes' theorem

$$\int_{X_\varepsilon} \log|w_k|^2 dd^c\varphi = -\int_{X_\varepsilon} d\log|w_k|^2 \wedge d^c\varphi - \int_{S_\varepsilon} \log|w_k|^2 d^c\varphi. \tag{1.6}$$

Since $d\log|w_k|^2 \wedge d^c\varphi = -d^c\log|w_k|^2 \wedge d\varphi$ and $dd^c\log|w_k|^2 = 0$ on X_ε,

$$-\int_{X_\varepsilon} d\log|w_k|^2 \wedge d^c\varphi = -\int_{X_\varepsilon} d(d^c\log|w_k|^2 \wedge \varphi) = \int_{S_\varepsilon} d^c\log|w_k|^2 \wedge \varphi. \tag{1.7}$$

Now clearly $$\int_{S_\varepsilon} \log|w_k|^2 d^c\varphi = (2\log\varepsilon)\int_{S_\varepsilon} d^c\varphi \to 0 \text{ as } \varepsilon \to 0.$$

Furthermore, if we write $w_k = re^{i\theta}$, $d^c\log|w_k|^2 = (2\pi)^{-1}d\theta$. Thus

$$\int_{S_\varepsilon} d^c\log|w_k|^2 \wedge \varphi \to \int_{\{w_k=0\}} \varphi. \tag{1.8}$$

This completes the proof of the nonsingular case.

Next we show that it suffices to prove (1.2) on the complement of a small analytic set. More precisely, if $Y \subset U$ is a subvariety of dimension $< k-1$ and if the restrictions of the currents $X \cdot D$ and $dd^c(X \wedge \log|\alpha|^2)$ to $U - Y$ are equal, then (1.2) holds.

One approach is to cite a theorem. Both of these currents are so-called flat currents and it can be proved that two such currents of real dimension l, which differ only on a set of real dimension $l-2$, are in fact equal (see [13] and [9]).

We can actually prove this here, however, since $T = X \cdot D$ and $T' = dd^c(X \wedge \log|\alpha|^2)$ are positive. Choosing coordinates $(z_1, \ldots, z_n)$ near any point in U, it suffices to show that $T \wedge \omega_{\mathrm{I}} = T' \wedge \omega_{\mathrm{I}}$, where $\omega_{\mathrm{I}} = (i/2)^{k-1} dz_{i_1} \wedge d\bar z_{i_1} \wedge \ldots \wedge dz_{i_{k-1}} \wedge d\bar z_{i_{k-1}}$ for every $(k-1)$-tuple I. For in the notation of § 0, it follows that $T_{\mathrm{I}\bar{\mathrm{J}}} = T'_{\mathrm{I}\bar{\mathrm{J}}}$. Thus Y is a set of $T'_{\mathrm{I}\bar{\mathrm{J}}}$ measure zero since $Y \cap X \cap D \subset X \cap D$ is a set of $2k-2$ measure zero. Consequently the $T'_{\mathrm{I}\bar{\mathrm{J}}}$ measure of Y is also zero since by positivity this measure is absolutely continuous with respect to $\sum_i T'_{\mathrm{I}\bar{\mathrm{J}}}$.

To show $T \wedge \omega_{\mathrm{I}} = T' \wedge \omega_{\mathrm{I}}$, let $\pi_{\mathrm{I}} = U \to \mathbf{C}^{k-1}$ be the coordinate mapping $z \to (z_{i_1}, \ldots, z_{i_{k-1}})$. For any $\varphi \in A_c^{0,0}(U)$, $T \wedge \omega_{\mathrm{I}}(\varphi) = T \wedge \varphi(\pi_{\mathrm{I}}^*\omega) = (\pi_{\mathrm{I}*}T \wedge \varphi)(\omega)$, where ω is the volume form on $\mathbf{C}^{k-1}$. Similarly $T' \wedge \omega_{\mathrm{I}}(\varphi) = (\pi_{\mathrm{I}*}T' \wedge \varphi)(\omega)$. Now both $\mu = \pi_{\mathrm{I}*}T \wedge \varphi$ and $\mu' = \pi_{\mathrm{I}*}T' \wedge \varphi \in C^{0,0}(\mathbf{C}^{k-1})$. The current μ is the current defined by the continuous function $\sum_{y \in X \cap \pi_{\mathrm{I}}^{-1}(x)} \varphi(y)$ (each y counted with suitable multiplicity). If we knew that μ' was also given by an L'_{loc} function we would be through, since the two currents agree on the complement of $\pi_{\mathrm{I}}(Y)$, which is a set of measure zero. We can show that $\mu' \in L^1_{\mathrm{loc}}$ by the Radon Nikodyn theorem, i.e., we show that μ' is absolutely continuous with respect to Lebesgue $2k-2$ measure on $\mathbf{C}^{k-1}$. Let E be a set of Lebesgue measure zero, then $\mu'(E) = \int_{X \cap \pi_{\mathrm{I}}^{-1}(E)} \log|\alpha|^2 dd^c\varphi \wedge \omega_{\mathrm{I}}$.

But this integral is zero because $\pi_I^{-1}(E) \cap \operatorname{Reg}(X)$ has $2k$ measure zero since π has maximal rank on Y except for a set of $2k$ measure zero (a subvariety of X of lower dimension). Thus the extension part of the proof is complete.

Lastly, we wish to show equality except on a subvariety $Y \subset U$ of dimension $< k-1$. Observe that if the variety X is normal then the nonsingular case applies except on the singular locus of X which has dimension $< k-1$ ([15]). If X is not normal there is a unique normalization $\tilde{X}$ and a finite proper map $\varrho: \tilde{X} \to X$ which is one-to-one over the regular locus of X. Localize so that ϱ extends to a map $\tilde{U} \to U$, for $\tilde{X} \subset \tilde{U}$. Then $\int_X \log|\alpha|^2 dd^c\varphi = \int_{\tilde{X}} \log|\alpha \circ \varrho|^2 dd^c\varrho^*\varphi = \int_{\tilde{X}\cdot(\alpha\circ\varrho)} \varrho^*\varphi$ since (1.1) holds for $\tilde{X}$. But $Z = \tilde{X} \cdot (\alpha \circ \varrho)$ is a variety of dimension $k-1$ and $\varrho: Z \to X \cap D$ is a finite map. Thus this last integral equals $\int_{``X\cap D"} \varphi$, where "$X \cap D$" is $X \cap D$ counted with the appropriate number of multiplicities. It can be verified that these multiplicities define $X \cdot D$ as it is defined by local algebra (on the regular points of $X \cdot D$ which is all that effects integration). See [13].

Proof of Lemma 1.4. Let $\pi: X \to \Delta \subset \mathbf{C}^k$ be a proper finite holomorphic map of degree d, i.e., a finite branched cover; we may assume that $\log|\alpha|^2 < 0$ on X. Then if $\varphi \in A_c^{kk}(\Delta)$, $\int_X \log|\alpha|^2 \pi^*\varphi = \pi_*(X \wedge \log|\alpha|^2)\varphi$, where $\pi_*(X \wedge \log|\alpha|^2)$ is the function $\zeta(x) = \sum_{y \in X \cap \pi^{-1}(x)} \log|\alpha(y)|^2$. On $\Delta - \pi(X \cap D)$, $dd^c\zeta = \pi_*(X \wedge dd^c \log|\alpha|^2) = 0$ so ζ is a smooth pluriharmonic function. Since $\zeta = -\infty$ on $\pi(X \cap D)$ this shows that ζ is plurisubharmonic [15] and hence locally L^1 on $\mathbf{C}^k$. On X we see that $\zeta \circ \pi \leqslant \log|\alpha|^2$. Since $\int_X (\zeta \circ \pi)\, \pi^*\varphi = d\int_\Delta \zeta\varphi$ is finite, so is $\int \log|\alpha|^2 \pi^*\varphi$.

Now we may assume $X \subset U \subset \mathbf{C}^n$ with coordinates chosen so that each coordinate projection $\pi_I: X \to \pi_I(U) \subset \mathbf{C}^k$ is as above. Then $\int_X \log|\alpha|^2 d\,(\text{volume}) = (1/k!) \int_X \log|\alpha|^2 \omega$, where $\omega = \sum \omega_I$ and $\omega_I = \pi_I^* \varphi$, where φ is the volume form on $\mathbf{C}^k$. This proves the first part of the lemma.

The second part is immediate since there is a monotone decreasing sequence $\zeta_1 \geqslant \zeta_2 \geqslant \dots$ of smooth plurisubharmonic functions converging to $\log|\alpha|^2$ (let $\zeta_r = \log(|\alpha|^2 + 1/r)$. For any positive form $\varphi \in A_c^{k-1,k-1}(U)$,

$$0 \leqslant \int_X dd^c\zeta_r \wedge \varphi = \int_X \zeta_r \wedge dd^c\varphi \to \int_X \log|\alpha|^2 \wedge dd^c\varphi = dd^c(X \wedge \log|\alpha|^2)(\varphi)$$

by the monotone convergence theorem. Q.E.D.

(b) The Poincaré equation for vector-valued functions

We now wish to establish a Poincaré formula for more than one function. First we define forms that play the role analogous to that of $\log|z|^2$ in the one-variable case. If $(z_1, \dots, z_r)$ are linear coordinates in $\mathbf{C}^r$ let

$$(1.9) \qquad \begin{cases} \theta_l = (dd^c \log \|z\|^2)^l \\ \Theta_l = \log \|z\|^2 \theta_l. \end{cases}$$

If $F = (f_1, \ldots, f_r)\colon U \to \mathbf{C}^r$ is a holomorphic mapping of a complex manifold U, then $F^*\theta_l = (dd^c \log \|f\|^2)^l$ and $F^*\Theta_l = \log \|f\|^2 \, F^*\theta^l$.

(1.10) PROPOSITION (*Poincaré-Martinelli formula*). *Let U be a complex n-manifold and $F\colon U \to \mathbf{C}^r$ be a holomorphic map. The forms $F^*\theta_l$ and $F^*\Theta^l$ are in $L^1_{(l,l)}(U, \mathrm{loc})$ for all l. If $W = F^{-1}(0)$ has dimension $n-r$, then $dd^c F^*\Theta_{l-1} = \theta_l$ for $l < r$ and $dd^c F^*\Theta_{r-1} = W$ where W is counted with the appropriate algebraic multiplicities, i.e., for $\varphi \in A_c^{n-r,\, n-r}(U)$*

$$(1.10) \qquad \int_U F^* \Theta_{r-1} \wedge dd^c \varphi = \int_W \varphi.$$

Remark. The proof will show that if $X \subset U$ is a k-dimensional subvariety and the dimension of $X \wedge W$ is $k-r$, then

$$dd^c (X \wedge F^* \Theta_{r-1}) = X \cdot W.$$

Before beginning the proof we will study the forms θ_l and Θ_l further by blowing up the origin in $\mathbf{C}^r$ to get a manifold $\hat{\mathbf{C}}^r$. If $(z_1, \ldots, z_r)$ are linear coordinates in $\mathbf{C}^r$ and $[w_1, \ldots, w_r]$ are homogeneous coordinates in $\mathbf{P}^{r-1}$, $\hat{\mathbf{C}}^r \subset \mathbf{C}^r \times \mathbf{P}^{r-1}$ is defined by $w_i z_j - w_j z_i = 0$, $(1 \leqslant i, j \leqslant n)$. The first coordinate projection gives a proper map $\pi\colon \hat{\mathbf{C}}^r \to \mathbf{C}^r$. If $E = \pi^{-1}(0)$, $\pi\colon \hat{\mathbf{C}}^r - E \to \mathbf{C}^r - \{0\}$ is a biholomorphism and the divisor E is $\{0\} \times \mathbf{P}^{r-1}$. In fact, the second coordinate projection $\varrho\colon \hat{\mathbf{C}} \to \mathbf{P}^{r-1}$ gives $\hat{\mathbf{C}}^r$ the structure of a holomorphic line bundle.

If $U_i = \{w_i \neq 0\} \subset \mathbf{P}^{r-1}$ and $\hat{\mathbf{C}}^r_i = \hat{\mathbf{C}}^r \cap \mathbf{C}^r \times U_i$, local coordinates on $\hat{\mathbf{C}}^r_i$ are given by $(u_{li}, \ldots, u_{i-1,i}, z_i, u_{i+1,i}, \ldots, u_{ri})$ where $u_{ji} = w_j/w_i$. In these coordinates the map $\pi = \hat{\mathbf{C}}^r_i \to \mathbf{C}^r$ is given by $\pi(u_{1i}, \ldots, z_i, \ldots, u_{ri}) = (u_{1i} z_i, \ldots, z_i, \ldots, u_{ri} z_i)$.

Now in $\hat{\mathbf{C}}^r_i$,

$$(1.11) \qquad \pi^* \log \|z\|^2 = \log |z_i|^2 + \log \big(1 + \sum_{j \neq i} |u_{ji}|^2\big)$$

where the second term is evidently a C^∞ function. Then

$$(1.12) \qquad \pi^* dd^c \log \|z\|^2 = dd^c \log \big(1 + \sum_{j \neq i} |u_{ji}|^2\big) = \varrho^* \omega$$

where ω is the usual Kähler form on $\mathbf{P}^{r-1}$.

Proof of Proposition (1.10). Now suppose we are given $F\colon U \to \mathbf{C}_r$. Let Γ be the graph of $F = \{(x, F(x))\} \subset U \times \mathbf{C}^r$. Let $\hat{\Gamma} \subset U \times \hat{\mathbf{C}}^r$ be the closure of $\pi^{-1}\{\Gamma - W \times 0\}$ and $\hat{W} = \hat{\Gamma} \cdot (0 \times \mathbf{P}^{r-1})$; these are varieties of dimension n and $n-1$, respectively, and the following diagram is commutative (identifying U with Γ):

$$
(1.13)\qquad
\begin{array}{ccccccc}
\hat{W} & \overset{i}{\subset} & \hat{\Gamma} & \overset{j}{\subset} & U \times \hat{\mathbf{C}}^r & \xrightarrow{\varrho} & \mathbf{P}^{r-1} \\
\downarrow \pi & & \downarrow \pi & & \downarrow \pi & & \\
W & \overset{i}{\subset} & \Gamma & \overset{j}{\subset} & U \times \mathbf{C}^r. & &
\end{array}
$$

Furthermore, each π is proper, $\pi: \hat{\Gamma} - \hat{W} \to \Gamma - W$ is biholomorphic and $\pi^{-1}(w) = w \times \mathbf{P}^{r-1}$ for each $w \in W$.

Now we wish to show that $\int_U F^*\Theta_{r-1} dd^c \varphi = \int_W \varphi$. The left hand integral equals $\int_\Gamma p_2^* \Theta_{r-1} \wedge dd^c p_1^* \varphi$ where p_1, p_2 are the projections of $U \times \mathbf{C}^r$ onto U, $\mathbf{C}^r$, respectively. This in turn equals $\int_{\hat{\Gamma}} \pi^* p_2^* \Theta_{r-1} \wedge dd^r \pi^* p_1^* \varphi$ since π is a map of degree 1.

Now by (1.12) in $U \times \mathbf{C}_i^r$,

$$\pi^* p_2^* \Theta_{r-1} = (\log |z_i|^2 + \log (1 + \sum_{j \neq i} |u_{ij}|^2)) (\varrho^* \omega)^{r-1}.$$

Furthermore, $dd^c (\log |1 + \sum |u_{ji}|^2) \varrho^* \omega^{*-1} = \varrho^* \omega^r = 0$ since $\mathbf{P}^{r-1}$ has no r-forms. Thus our integral becomes $\int_{\hat{\Gamma}} \log |z_i|^2 dd^c (\varrho^* \omega^{r-1} \wedge \pi^* p_1^* \varphi)$ which equals

$$\int_{\hat{\Gamma} \cdot (0 \times \mathbf{P}^{r-1})} \varrho^* \omega^{r-1} \wedge \pi^* p_1^* \varphi = \int_{\hat{W}} \varrho^* \omega^{r-1} \wedge \pi^* p_1^* \varphi = \int_W \varphi$$

since $\int_{\mathbf{P}^{r-1}} \omega^{r-1} = 1$ and each fiber of $\pi = W \to \hat{W}$ is $\mathbf{P}^{r-1}$. Strictly speaking we have only shown here that $dd^c F^* \Theta_{r-1} =$ "W", that is integration over W with some multiplicity. That this is the correct algebraic multiplicity is easily shown once enough properties of the algebraic multiplicity are established [13].

To show the rest of the proposition, we observe that both $F^*\Theta_l$ and $F^*\theta_l$ are $L^1_{(l,l)}(U, \text{loc})$ because $\pi^* p_1^* F \Theta_l$ and $\pi^* p_1^* F^* \theta_l$ are $L^1_{(l,l)}(U, \text{loc})$ on $\hat{\Gamma}$ by Lemma 1.4. Now check that $dd^c F^* \Theta_{l-1} = F^* \theta_l$ for $l < r$ by the same method observing at the last step that $\int_{\hat{W}} \varrho^* \omega^{l-1} \wedge \pi^* p_1^* \varphi = 0$ if $l < r$ (the integrand is a form which involves more than $2n - 2r$ coordinates from the base).

(c) Globalization of the Poincaré and Martinelli equations

Using the notation and terminology of § 0(a) we let M be a complex manifold, $L \to M$ a line bundle having a metric with curvature form ω, and $\sigma \in H^0(M, L)$ a holomorphic cross-section with divisor D. The function $\log |\sigma|^2 \in L^1(M, \text{loc})$ and the global version of (1.1) is:

(1.14) PROPOSITION. *On M we have the equation of currents,*

$$dd^c \log |\sigma|^2 = D - \omega.$$

Proof. This follows immediately from (0.1), (0.2), and (1.1).

This proposition says that D and ω are cohomologous. More precisely, we can take the cohomology of the *complex of currents:*

$$\dots \to C^r(M) \xrightarrow{d} C^{r+1}(M) \to \dots$$

in analogy with the de Rham cohomology arising from the complex of C^∞ forms:

$$\dots \to A^r(M) \xrightarrow{d} A^{r+1}(M) \to \dots.$$

Standard arguments involving the smoothing of currents show that

$$H^*_C(M, \mathbf{R}) = H^*_{DR}(M, \mathbf{R}),$$

and by de Rham's theorem both yield the usual cohomology. Thus the proposition says that the cohomology class of D is $c(L)$ in $H^2(M, \mathbf{R})$. This may also be interpreted to say that, viewing D as a chain, the homology class represented by D in $H_{2n-2}(M, \mathbf{R})$ is the Poincaré dual of $c(L)$.

Since intersection in homology is the dual of cup product in cohomology (wedge product in de Rham cohomology) the following proposition is not surprising.

(1.15) PROPOSITION. *If $\sigma_1, \dots, \sigma_r$ are holomorphic sections of the line bundle $L \to M$ with curvature form ω, and if the divisors D_{σ_i} intersect in a variety of codimension r, then*

$$\omega^r - D_{\sigma_1} \cdot D_{\sigma_2} \dots D_{\sigma_r} = dd^c\Lambda$$

as currents, where the locally L^1 form

$$\Lambda = \log \frac{1}{\|\sigma\|^2} \left(\sum_{k=1}^{r-1} \omega_0^{r-1-k}\, \omega^k \right),$$

with $\omega_0 = \omega + dd^c \log \|\sigma\|^2 = \omega + dd^c \log (\sum_{i=1}^r |\sigma_i|^2)$. Furthermore, if $\omega \geq 0$ *and* $\|\sigma\| \leq 1$, *then* $\Lambda \geq 0$.

Proof. If σ_i is given in local coordinates by s_i and the metric by the function a_α, then $\|\sigma\|^2 = (1/a_\alpha)\,(|s_1|^2 + \dots + |s_r|^2)$. Thus locally $\omega_0^p = s^*\theta_p$; also, $\log \|\sigma\|^{-2} = \log |a_\alpha|^2 - \log \|s\|^2$. In these local coordinates,

$$dd^c\Lambda = \sum_{k=1}^{r-1} (s^*\theta_{r-1-k}\, \omega^{k+1} - dd^c(s^*\Theta_{r-1-k} \wedge \omega^k)) = \omega^r - D_{\sigma_1} \dots D_{\sigma_r}$$

by the Martinelli equation (1.10). Since $s^*\theta_l \geq 0$, $\omega \geq 0$ and $\log \|\sigma\|^{-2} \geq 0$ together imply $\Lambda \geq 0$.

(d) Lelong numbers

Let U be an open set containing the closed R-ball $\mathbf{C}^n[R]$ in $\mathbf{C}^n$. We maintain the previous notation:

$$\theta_l = (dd^c \log \|z\|^2)^l$$

$$\varphi_l = (dd^c \|z\|^2)^l.$$

Suppose that $Z \subset U$ is an analytic set of dimension k; let $Z[r] = Z \cap \mathbf{C}^n[r]$ and for $r < R$, $Z[r, R] = Z[R] - Z[r]$. Then the $2k$-dimensional area $v[Z, r]$ is

$$v(Z, r) = \int_{Z[r]} \varphi_k.$$

(1.16) LEMMA. *The area $v(Z, r)$ satisfies*

$$\int_{Z[r,R]} \theta_k = \frac{v(Z, R)}{R^{2k}} - \frac{v(Z, r)}{r^{2k}}.$$

Proof. An easy computation shows that

$$d^c \log \|z\|^2 \wedge \theta_{k-1} = \frac{d^c \|z\|^2 \wedge \varphi_{k-1}}{\|z\|^{2k}} + d\|z\|^2 \wedge \lambda$$

where λ is some form. Thus by Stokes' theorem, noting that $\theta_k = d(d^c \log \|z\|^2 \wedge \theta_{k-1})$,

$$\int_{Z[r,R]} \theta_k = \int_{\partial Z[r,R]} d^c \log \|z\|^2 \wedge \theta_{k-1} = \int_{\partial Z[R]} \frac{d^c \|z\|^2 \wedge \varphi_{k-1}}{\|z\|^{2k}} - \int_{\partial Z[r]} \frac{d^c \|z\|^2 \wedge \varphi_{k-1}}{\|z\|^{2k}}$$

since the restriction of $d\|z\|^2$ to $\partial Z[r]$ is zero. But $\|z\|^{2k} = R^{2k}$ on $\partial Z[R]$, etc., so

$$\int_{Z[r,R]} \theta_k = \frac{1}{R^{2k}} \int_{\partial Z[R]} d^c \|z\|^2 \wedge \varphi_{k-1} - \frac{1}{r^{2k}} \int_{\partial Z[r]} d^c \|z\|^2 \wedge \varphi_{k-1}$$

$$= \frac{1}{R^{2k}} \int_{Z[R]} \varphi_k - \frac{1}{r^{2k}} \int_{Z[r]} \varphi_k = \frac{v(Z, R)}{R^{2k}} - \frac{v(Z, r)}{r^{2k}}.$$

Remark. This lemma remains true if we replace Z by any closed, positive current, cf. [15].

It follows from the lemma that the limit

$$\mathfrak{L}_0(Z) = \lim_{r \to 0+} \frac{v(Z, r)}{r^{2k}}$$

exists and is called the *Lelong number* of Z at the origin. Although not strictly necessary for our purposes, we shall prove the following result of Thie [20] and Draper [8].

(1.17) PROPOSITION. *$\mathcal{L}_0(Z)$ is an integer and, in fact, it is the multiplicity of Z at the origin.*

Proof. We first show, following roughly the argument of 1.10, that for any $\lambda \in A_c^0(U)$,

$$dd^c(Z \wedge \Theta_{k-1})\,(\lambda) = \mathrm{Mult}_0\,(Z)\lambda(0).$$

We again use the blow-up $\pi = \hat{\mathbf{C}}^n \to \mathbf{C}^n$ and preserve the notation of § 1(b). Let $\hat{Z}$ be the closure of $\pi^{-1}(Z - \{0\})$. Then if $\pi^{-1}(0) = E \cong \mathbf{P}^{n-1}$, the intersection $\hat{Z} \cdot E$ is the Zariski tangent cone and $\mathrm{Mult}_0\,(Z)$ is the degree of $\hat{Z} \cdot E$ in $E \cong \mathbf{P}^{n-1}$, which in turn equals $\int_{\hat{Z} \cdot E} \varrho^* \omega^{k-1}$.

Now by (1.11) and Theorem (1.1)

$$\int_Z \Theta_{k-1} dd^c \lambda = \int_{\hat{Z}} \log \|z\|^2 dd^c\,(\pi^* \lambda \varrho^* \omega^{k-1}) = \int_{\hat{Z} \cdot E} \pi^* \lambda \varrho^* \omega^{k-1} = \lambda(0) \int_{\hat{Z} \cdot E} \varrho^* \omega^{k-1}.$$

On the other hand, if $\lambda \equiv 1$ for small r,

$$dd^c(Z \wedge \Theta_{k-1})\,(\lambda) = \lim_{r \to 0+} \int_{Z - Z[r]} \Theta_{k-1} \wedge dd^c \lambda$$

and the right-hand integral is by Stokes' theorem (for small $r, d^c\lambda = 0$)

$$\begin{aligned} -\int_{Z-Z[r]} d\Theta_{k-1} \wedge d^c \lambda &= \int_{Z-Z[r]} d^c \Theta_{k-1} \wedge d\lambda \\ &= -\int_{Z-Z[r]} d(\lambda d^c \Theta_{k-1}) \\ &= \int_{\partial Z[r]} \lambda d^c \Theta_{k-1} = \frac{1}{r^{2k}} \int_{\partial Z[r]} d^c \|z\|^2 \varphi_{k-1} = \frac{1}{r^{2k}} \int_{Z[r]} \varphi_k = \frac{v(r)}{r^{2k}}. \end{aligned}$$

2. Special exhaustion functions on algebraic varieties

(a) Definition and some examples

Let M be a complex manifold of dimension m. We will say that a function $\tau: M \to [-\infty, +\infty)$ has a *logarithmic singularity* at $z_0 \in M$ if, in a suitable coordinate system $(z_1, \ldots, z_m)$ around z_0,

$$\tau(z) = \log \|z\| + r(z)$$

where $r(z)$ is a C^∞ function. An *exhaustion function* is given by

$$\tau: M \to [-\infty, +\infty)$$

which is C^∞ except for finitely many logarithmic singularities and is such that the half-spaces

$$M[r]=\{z\in M: e^{\tau(z)}\leqslant r\}$$

are compact for all $r\in[0, +\infty)$. The *critical values* of such an exhaustion function τ are, as usual, those r such that $d\tau(z)=0$ for some $z\in\partial M[r]=\{z: \tau(z)=r\}$. If r is not a critical value, then the level set $\partial M[r]$ is a real C^∞ hypersurface in M and we shall denote by $T_z^{(1,0)}(\partial M[r])$ the holomorphic tangent space to $\partial M[r]$ at z.

Definition. A special exhaustion function is given by an exhaustion function $\tau: M\to[-\infty, +\infty)$ which has only finitely many critical values and whose *Levi form* $dd^c\tau$ satisfies the conditions

$$\begin{cases} dd^c\tau\geqslant 0 \\ (dd^c\tau)^{m-1}\not\equiv 0 \text{ on } T_z^{(1,0)}(\partial M[r]) \\ (dd^c\tau)^m=0 \end{cases} \tag{2.1}$$

Examples. (i) Let M be an affine algebraic curve. Then $M=\bar{M}-\{z_1, \ldots, z_N\}$ where $\bar{M}$ is a compact Riemann surface. Given a fixed point $z_0\in M$, we may choose a harmonic function τ_α on $\bar{M}$ such that

$$\tau_\alpha\sim\log|z-z_0| \text{ near } z_0$$

$$\tau_\alpha\sim -\log|z-z_\alpha| \text{ near } z_\alpha$$

where z is a local holomorphic coordinate in each case. The sum $\tau=\sum_{\alpha=1}^N\tau_\alpha$ gives a special exhaustion function (=*harmonic exhaustion function*) for M.

(ii) On $\mathbf{C}^m$ with coordinates $(z_1, \ldots, z_m)$, we may take $\tau=\log\|z\|$ to obtain a special exhaustion function. We shall explain the geometric reasons for this, following to some extent the proof of Proposition (1.5).

Observe first that the level set $\partial M[r]$ is just the sphere $\|z\|=r$ in $\mathbf{C}^m$. There is the usual Hopf fibration

$$\pi: \partial M[r]\to\mathbf{P}^{m-1}$$

of $\partial M[r]$ over the projective space of lines through the origin in $\mathbf{C}^m$. The differential

$$\pi_*: T_z^{(1,0)}(\partial M[r])\to T_{\pi(z)}^{(1,0)}(\mathbf{P}^{m-1}) \tag{2.2}$$

is an isomorphism, and the Levi form is given by

$$2\, dd^c\log\|z\|=\pi^*(\omega) \tag{2.3}$$

where ω is the $(1, 1)$ form associated to the Fubini-Study metric on $\mathbf{P}^{m-1}$. It follows from (2.3) that $dd^c\log\|z\|\geqslant 0$ and $(dd^c\log\|z\|)^m=0$, while (2.2) gives that $(dd^c\log\|z\|)^{m-1}$ is positive on $T_z^{(1,0)}(\partial M[r])$. Consequently $\log\|z\|$ gives a special exhaustion function on $\mathbf{C}^m$.

(b) Construction of a special exhaustion function

These two examples are combined in the

(2.4) PROPOSITION. *Let A be a smooth affine algebraic variety. Then there exists a special exhaustion function τ on A.*

The proof uses resolution of singularities and proceeds in two steps.

Step 1. We shall first describe the exhaustion function on $\mathbf{C}^{n+1}$ given by example (ii) in a somewhat different manner.

Let $\mathbf{P}^n$ be complex projective space and $H \xrightarrow{\pi} \mathbf{P}^n$ the standard positive line bundle. There are distinguished holomorphic sections $\sigma_0, \ldots, \sigma_n$ of $H - \mathbf{P}^n$ such that the associated map

$$[\sigma_0, \ldots, \sigma_n]: \mathbf{P}^n \to \mathbf{P}^n$$

is just the identity. On the other hand, there is a tautological section ζ of the pull-back bundle

$$\pi^* H \to H$$

such that $\zeta = 0$ defines the zero section embedding of $\mathbf{P}^n$ in H. The induced map

$$[\pi^*\sigma_0, \ldots, \pi^*\sigma_n;\ \zeta]: H \to \mathbf{P}^{n+1}$$

is an embedding of H into $\mathbf{P}^{n+1}$ such that the zero section $\mathbf{P}^n$ of H goes into the hyperplane given in homogeneous coordinates $[\xi_0, \ldots, \xi_{n+1}]$ on $\mathbf{P}^{n+1}$ by $\xi_{n+1} = 0$. The image of H is the complement of the point $\xi = [0, \ldots, 0, 1]$ in $\mathbf{P}^{n+1}$, and the fibration $H \to \mathbf{P}^n$ is geometrically just the projection from ξ onto the hyperplane $\xi_{n+1} = 0$ in $\mathbf{P}^{n+1}$.

The metric in $\mathbf{C}^{n+1}$ induces a metric in $H \xrightarrow{\pi} \mathbf{P}^n$ whose curvature form $c(H)$ is the usual Kähler form ω on $\mathbf{P}^n$. This metric in turn induces a metric in $\pi^* H \to H$, and we consider the function

$$\tau_0 = -\log |\zeta|$$

on $H - \mathbf{P}^n$. The level sets $\{z \in H: \tau_0(z) = r\}$ are just the boundaries of tubular neighborhoods of the zero section $\mathbf{P}^n$ in H. Using the inclusion $H \hookrightarrow \mathbf{P}^{n+1}$, we see that τ_0 gives a special exhaustion function on $\mathbf{P}^{n+1} - \mathbf{P}^n = \mathbf{C}^{n+1}$. In fact, this is the same as the exhaustion function constructed in example (ii) above, only we are now focusing our attention around the hyperplane at infinity for $\mathbf{C}^{n+1}$. The distinguished point ξ is just the origin in $\mathbf{C}^{n+1}$.

Step 2. Let $\bar{A}$ be a smooth completion of A satisfying the following conditions: (i) $\bar{A}$ is a smooth, projective variety; (ii) $D_\infty = \bar{A} - A$ is a divisor with normal crossings on $\bar{A}$; and (iii) there is a projective embedding $\bar{A} \hookrightarrow \mathbf{P}^N$ such that $D_\infty = \bar{A} \cap \mathbf{P}^{N-1}$ is a hyperplane section of $\bar{A}$ (*not* counting multiplicities). Such an embedding exists by [12] and the assumption that A is affine.

Assume that $\dim_{\mathbf{C}} A = m$ and choose a linear subspace $\mathbf{P}^{N-m-1}$ of $\mathbf{P}^N$ which lies in the hyperplane $\mathbf{P}^{N-1}$ and does not meet $\bar{A}$. Selecting a generic $\mathbf{P}^m$ not meeting $\mathbf{P}^{N-m-1}$, we consider the projections

$$\begin{tikzcd} \bar{A} \arrow[r, hook] \arrow[d, "\pi"] & \mathbf{P}^N - \mathbf{P}^{N-m-1} \arrow[d, "\pi"] & \mathbf{P}^{N-1} - \mathbf{P}^{N-m-1} \arrow[l, hook'] \arrow[d, "\pi"] \\ \mathbf{P}^m \arrow[r, phantom, "="] & \mathbf{P}^m & \mathbf{P}^{m-1} \arrow[l, hook'] \end{tikzcd} \tag{2.5}$$

Then $\pi^{-1}(\mathbf{P}^{m-1}) \cap \bar{A} = D_\infty$, so that (2.5) induces a finite branched covering mapping

$$\pi: A \to \mathbf{C}^m \tag{2.6}$$

where $\mathbf{C}^m = \mathbf{P}^m - \mathbf{P}^{m-1}$. We let $\tau = \pi \circ \tau_0$ where τ_0 was constructed in Step 1. From the geometric discussion there together with example (ii) we see that the $A[r]$ are compact and the conditions (1.1) on the Levi form are satisfied. What we must show is that τ has only finitely many critical values, which is not immediately clear since the branch locus of (2.6) extends to infinity if $m > 1$.

We now localize around infinity. Let $L \to \bar{A}$ be the pullback $\pi^* H$ in (2.5), take the metric in L induced from that in H, and let $\zeta \in H^0(\bar{A}, O(L))$ be the section which defines D_∞ on $\bar{A}$. Then $\tau = -\log|\zeta|$ near D_∞ on $\bar{A}$.

Around a point on D_∞ we choose holomorphic coordinates $w_1, \ldots, w_m$ such that

$$\zeta = w_1^{\alpha_1} \cdots w_k^{\alpha_k}.$$

Then it follows that

$$\tau = -\sum_{\mu=1}^{k} \alpha_\mu \log |w_\mu| + \varrho(w) \tag{2.7}$$

where $\varrho(w)$ is a C^∞ function. From (2.7) we obtain

$$\partial\tau = -\tfrac{1}{2}\sum_{\mu=1}^{k} \alpha_\mu \frac{dw_\mu}{w_\mu} + \partial\varrho,$$

from which it follows that $d\tau \neq 0$ for $\|w\| < \varepsilon$. Using the compactness of D, we see that $d\tau \neq 0$ outside a compact set of A. Q.E.D.

(c) Some properties of the projection (2.5)

If A is a smooth affine variety of dimension m, then we have constructed an algebraic branched covering

$$A \xrightarrow{\pi} \mathbf{C}^m \tag{2.8}$$

such that $\tau = \log \|\pi(x)\|$ gives a special exhaustion function on A. We may assume that τ is unramified over the origin, so that

$$\pi^{-1}(0) = \{x_1, \ldots, x_d\} \quad (d = \text{degree of } A)$$

where $x_1, \ldots, x_d$ are the logarithmic singularities of τ. There are several properties of the covering (2.8) which we wish to record here.

Let $\Phi = \prod_{j=1}^m (\sqrt{-1}/2\pi)\,(dz_j \wedge d\bar{z}_j)$ be the Euclidean volume form on $\mathbf{C}^m$ and $\Phi_\pi = \pi^*\Phi$ the pull-back of Φ to A. Suppose that Ω is an everywhere positive C^∞ volume form on A, and write $\Omega = \zeta\Phi_\pi$ where $\zeta \geqslant 0$ on A and $\log \zeta \in L^1(\text{loc}, A)$.

(2.9) Lemma. *In the sense of currents, we have*

$$dd^c \log \zeta = \operatorname{Ric} \Omega - B$$

where $\operatorname{Ric}\Omega$ *is the Ricci form of* Ω *and* B *is the branch locus of the projection* (2.8).

Proof. Using local holomorphic coordinates $w_1, \ldots, w_m$ on A, we have the relations

$$\Phi_\pi = |j(w)|^2 \prod_{i=1}^m \frac{\sqrt{-1}}{2}\,(dw_i \wedge d\bar{w}_i)$$

$$\Omega = a(w) \prod_{i=1}^m \frac{\sqrt{-1}}{2}\,(dw_i \wedge d\bar{w}_i)$$

where $j(w) = 0$ is the local equation of B and $a(w) > 0$ is the coefficient of Ω. It follows that

$$\zeta = \frac{a(w)}{|j(w)|^2}$$

so that using the Poincaré equation (1.1) we obtain

$$dd^c \log \zeta = dd^c \log a - B$$

as an equation of currents. Q.E.D.

Before proving our next property of the situation (2.8), we need to have the following

(2.10) Lemma. *Suppose that* Z *is a* k*-dimensional analytic subset of* $\mathbf{C}^n$ *such that in* $\mathbf{P}^n$ *there is a* $\mathbf{P}^{n-k-1}$ *in the* $\mathbf{P}^{n-1}$ *at infinity with* $\bar{Z} \cap \mathbf{P}^{n-k-1} = \emptyset$. *Then* Z *is algebraic.*

Proof. Assume first that $k = n-1$, so that $Z \subset \mathbf{C}^n$ is a hypersurface and there is a point $\xi \in \mathbf{P}^n - \mathbf{C}^n$ with $\xi \cap \bar{Z} = \emptyset$. The projection $\mathbf{P}^n - \xi \xrightarrow{\pi} \mathbf{P}^{n-1}$ is the total space of the standard positive line bundle $H \xrightarrow{\pi} \mathbf{P}^{n-1}$ (cf. Step 1 in the proof of Proposition (2.4)). The restriction $\bar{Z} \xrightarrow{\pi} \mathbf{P}^{n-1}$ is proper and realizes $Z \xrightarrow{\pi} \mathbf{C}^{n-1}$ as a finite analytic covering with d sheets. Thus we may write

$$\pi^{-1}(z) = \{\sigma_1(z), \ldots, \sigma_d(z)\}$$

where $\sigma_r(z)$ are multi-valued holomorphic sections of $H \xrightarrow{\pi} \mathbf{C}^{n-1}$. Each homogeneous symmetric function of the $\sigma_r(z)$, such as

$$\sigma_1(z) \ldots \sigma_d(z),$$

may be considered as a single-valued holomorphic section $\zeta(z)$ of $H^b \xrightarrow{\pi} C^n$ for suitable b. Moreover, around the $\mathbf{P}^{n-1}$ at infinity in $\mathbf{C}^n$, the $\zeta(z)$ are locally given by *bounded* holomorphic functions in a punctured polycylinder. Applying the Riemann extension theorem, it follows that any such $\zeta(z)$ is a holomorphic section of $H^b \xrightarrow{\pi} \mathbf{P}^{n-1}$. The holomorphic sections of $H^b \xrightarrow{\pi} \mathbf{P}^{n-1}$ are, essentially by definition, given by holomorphic functions $F_\zeta(z)$ on $\mathbf{C}^n - \{0\}$ which satisfy

$$F_\zeta(\lambda z) = \lambda^b F_\zeta(z).$$

By Hartogs' theorem, F_ζ extends to give a holomorphic function of $\mathbf{C}^n$, which is then a polynomial of degree b by the homogeneity condition. Thus any such ζ is algebraic, and it follows from this that Z is algebraic. In fact, for each such ζ it follows that Z satisfies the polynomial equation

$$\xi_n^b = F_\zeta(\xi_0, \ldots, \xi_{n-1}).$$

In general, the situation

$$\begin{array}{ccccc} Z & \subset & \bar{Z} & \subset & \mathbf{P}^n - \mathbf{P}^{n-k-1} \\ \downarrow & & \downarrow & & \downarrow \pi \\ \mathbf{C}^k & \subset & \mathbf{P}^k & = & \mathbf{P}^k \end{array}$$

gives a vector bundle $E \to \mathbf{P}^k$ such that Z may be considered as a multi-valued section over $\mathbf{C}^k \subset \mathbf{P}^k$. Choosing coordinates such that π is given by

$$[\xi_0, \ldots, \xi_n] \to [\xi_0, \ldots, \xi_k]$$

gives an isomorphism

$$E \cong \underbrace{H \oplus \ldots \oplus H}_{n-k}.$$

Using this we may repeat the above argument to find a homogeneous polynomial $P(\xi_0, \ldots, \xi_n)$ such that $P(\xi) = 0$ on $\bar{Z}$ but $P(\xi) \neq 0$ at a given point $\xi \in \mathbf{P}^n - \bar{Z}$. This $P(\xi)$ will be a section of a symmetric power of $\pi^* E$.

One may also note that for a generic $\mathbf{P}^{n-1}$ there is an open set $U \subset \mathbf{P}^{n-1}$ at infinity so that for each $x \in U$, and $\pi_x: \mathbf{P}^n - \{x\} \to \mathbf{P}^{n-1}$, the set $\pi_x(Z) \subset \overline{\pi_x(Z)} \subset \mathbf{P}^{n-1} - \pi(\mathbf{P}^{n-k-1})$ satisfies the hypotheses of the theorem. Thus by induction on codimension Z, $\pi_x^{-1}(\pi_x(Z))$ is algebraic and $Z\subset \bigcap_{x\in U}\pi_x^{-1}(\pi_x(Z))$, an algebraic set. These two sets are actually equal, since if $y\in \mathbf{C}^n-Z$ the set of $x\in U$ such that $\pi_x(y)\in\pi_x(Z)$ has dimension equal to dimension $Z<$dimension U. Therefore, $Z=\bigcap_{x\in U}\pi_x^{-1}(\pi_x(Z))$ is algebraic.

COROLLARY (Chow's theorem). *Any analytic set in $\mathbf{P}^n$ is defined by polynomial equations, and is therefore algebraic.*

Remark: The above proof uses only the Riemann extension theorem and Hartogs' theorem, and is thus both elementary and reasonably simple.

Now we return to the finite algebraic projection $A\xrightarrow{\pi}\mathbf{C}^m$ given in (2.8). Let $Z\subset A$ be an analytic subset and $\pi(Z)$ its projection onto $\mathbf{C}^n$.

(2.11) LEMMA. *Z is an algebraic subset of A if, and only if, $\pi(Z)$ is an algebraic subset of $\mathbf{C}^m$.*

Proof. It will suffice to assume that $\pi(Z)$ is algebraic and then prove that Z is also. There is a linear $\mathbf{P}^{m-k-1}\subset\mathbf{P}^{m-1}=\mathbf{P}^m-\mathbf{C}^m$ such that $\overline{\pi(Z)}\cap\mathbf{P}^{m-k-1}=\emptyset$. Considering the situation

$$\begin{array}{ccc} A & \subset & \mathbf{P}^N-\mathbf{P}^{N-m-1}\\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi}\\ \mathbf{C}^m & \subset & \mathbf{P}^m \end{array}$$

it follows that $\overline{\pi^{-1}(\mathbf{P}^{m-k-1})}$ is a $\mathbf{P}^{N-k-1}$ in $\mathbf{P}^N-\mathbf{C}^N$ such that $\bar{Z}\cap\mathbf{P}^{N-k-1}=\emptyset$. By Lemma (2.11), Z is algebraic. Q.E.D.

3. Some integral formulas and applications

(a) Jensen's theorem

Let M be a complex manifold having an exhaustion function $\tau: M\to[-\infty,+\infty)$. We set $M[r]=\{x\in M: \tau(x)\leqslant\log r\}$ and assume that r_0 is such that $\tau(x)$ has all of its critical values in the interval $[-\infty,\log r_0)$. We denote the Levi form of τ by

$$\psi=dd^c\tau\geqslant 0.$$

Let D be a divisor on M and set $D[r]=D\cap M[r]$. If D does not pass through any of the logarithmic singularities of τ, we define

$$(3.1)\qquad \begin{cases} n(D,t)=\displaystyle\int_{D[t]}\psi^{m-1}\\ N(D,r)=\displaystyle\int_0^r n(D,t)\frac{dt}{t}\qquad (\textit{counting function}).\end{cases}$$

In case D passes through some of the logarithmic singularities of τ, the formulae in (3.1) must be modified by using Lelong numbers as discussed in § 1(d). We shall assume without further comment that this has been done whenever necessary.

(3.2) PROPOSITION (Jensen's theorem). *Let α be a meromorphic function on M with divisor D. Then we have*

$$N(D, r) = \int_{\partial M[r]} \log |\alpha|^2 \eta + \int_{M[r]} \log \frac{1}{|\alpha|^2} \psi^m + O(1) \quad (r \geq r_0)$$

where $\eta = d^c\tau \wedge \psi^{m-1} \geq 0$ on $\partial M[r]$ and where the term

$$O(1) = N(D, r_0) + \int_{M[r_0]} \log \frac{1}{|\alpha|^2} \psi^m - \int_{\partial M[r_0]} \log |\alpha|^2 \eta$$

depends on D but not on r.

Proof. We recall the Poincaré equation of currents (1.1)

$$dd^c \log |\alpha|^2 = D.$$

Let γ_t be the characteristic function of $M[t]$. Since the current $d^c \log|\alpha|^2$ is a Radon measure, $d^c \log|\alpha|^2 \wedge \gamma_t$ is defined. We claim that

$$d(d^c \log|\alpha|^2 \wedge \gamma_t) = D \wedge \gamma_t - \partial M[r] \wedge d^c \log|\alpha|^2. \tag{3.3}$$

By Equation (1.1) or by the usual Stokes' theorem this is clearly true around all $x \notin D \cap \partial M[r]$. As in (1.10) there are two ways to verify this on $D \cap \partial M[r]$. One way is to observe that these are flat currents of dimension $2m-2$ in the sense of Federer. Since the two sides differ on $D \cap \partial M[r]$, a set of real dimension $2m-3$, the two sides must be equal [13].

The other way is the method used in the proof of (1.10). In this case it will be better to blow up the origin by inserting a real sphere instead of a complex projective space. This leads to a real analytic set with boundary $\hat{\Gamma}$ and $\pi: \hat{\Gamma} \to M$ such that $\pi^*(d^c \log|\alpha|^2)$ is smooth and the usual Stokes' theorem gives (3.3) for the forms pulled up to $\hat{\Gamma}$ and this implies (3.3) (cf. (1.13) and the proof of Proposition (1.10)).

Now given Equation (3.3), apply these currents to the form ψ^{m-1}, replacing γ_t by $\gamma_t - \gamma_{r_0}$. Then since $d\psi^{m-1} = 0$,

$$\int_{D[t]} \psi^{m-1} = \int_{\partial M[t]} d^c \log |\alpha| \wedge \psi^{m-1} \tag{3.4}$$

(one must check also that the boundary term arising from the logarithmic singularities of τ in zero). In (3.4) we have assumed that D does not pass through any of the logarithmic singularities of τ and that $\log t$ is not a critical value of τ. As mentioned above, the first of these restrictions may be disposed of using Lelong numbers.

Now we integrate (3.4) with respect to dt/t from r_0 to r and apply Fubini's theorem to have

$$N(D, r) = \int_{M[r_0, r]} d\tau \wedge d^c \log |\alpha|^2 \wedge \psi^{m-1} + O(1). \tag{3.5}$$

Using the relation

$$d\tau \wedge d^c \log |\alpha|^2 \wedge \psi^{m-1} = d(\log |\alpha|^2 \eta) - \log |\alpha|^2 \psi^m$$

and applying Stokes' theorem to the R.H.S. of (3.5) gives Jensen's theorem. Q.E.D.

Suppose now that $\alpha \in O(M)$ is holomorphic and let

$$M_\alpha(r) = \max_{x \in M[r]} \log |\alpha(x)|^2$$

be the *maximum modulus* $\log |\alpha|$ on $M[r]$. From (3.2) we obtain the estimate

$$N(D, r) \leqslant M_\alpha(r) \left(\int_{\partial M[r]} \eta \right) + \int_{M[r]} \log \frac{1}{|\alpha|^2} \psi^m + O(1). \tag{3.6}$$

In general, this inequality does not seem to be very useful because, at least on the face of it, the zeroes of α will contribute positively to the term $\int_{M[r]} \log (1/|\alpha|^2) \psi^m$. However, if τ is a special exhaustion function as defined in § 2, then $\psi^m = 0$ and $\int_{\partial M[r]} \eta$ is a constant independent of r. Taking this constant to be $1/2$, (3.6) reduces to the *Nevanlinna inequality*

$$N(D, r) \leqslant M_\alpha(r) + O(1). \tag{3.7}$$

Although simple to derive, this inequality has the remarkable effect of bounding the size of the divisor $\alpha = 0$ in terms of the maximum modulus of α. To illustrate the strong global consequences which result from a special exhaustion function, we shall prove the

(3.8) PROPOSITION (Casorati-Weierstrass theorem). *Let M have a special exhaustion function and α be a non-constant meromorphic function on M. Then the image $\alpha(M)$ is dense in* $\mathbf{P}^1$.

Proof. If the proposition were false, then after a suitable linear fractional transformation we may assume that α is a bounded holomorphic function such that the divisor $\alpha = 0$ is non-empty and does not pass through any of the logarithmic singular points of τ. Thus, for some $c > 0$, we will have from (3.7) the inequality

$$c \log r \leqslant N(D, r) \leqslant O(1)$$

for arbitrarily large r. This is a contradiction. Q.E.D.

Remark. For a Riemann surface M, there is a notion of what it means for M to be *parabolic* (cf. Ahlfors-Sario, *Riemann Surfaces*, Princeton University Press (1960), pp. 26 and 204). It is then a theorem that this is equivalent to M having a special exhaustion function in the sense of § 2 (cf. M. Nakai, *On Evans Potential*, Proc. Japan Acad., vol. 38 (1962), pp. 624–629). This together with Proposition (3.8) perhaps suggests a generalization of the notion of parabolic to general complex manifolds.

(b) The Nevanlinna characteristic function

Let M be a complex manifold with a special exhaustion function $\tau: M-[-\infty, +\infty)$. Following R. Nevanlinna [16], we shall put Jensen's theorem in a more symmetric form. Let α be a meromorphic function on M and denote by D_a the divisor

$$\alpha(z)=a$$

for a point $a\in\mathbf{P}^1$. Then, using the notations

$$\log^+ t=\max(\log t, 0), \quad (t\geqslant 0), \qquad m(\alpha, r)=\int_{\partial M[r]}\log^+\frac{1}{|\alpha|^2}\eta, \tag{3.9}$$

Jensen's theorem (3.2) may be rewritten as

$$N(D_0, r)+m(\alpha, r)=N(D_\infty, r)+m(1/\alpha, r)+O(1). \tag{3.10}$$

The R.H.S. of (3.10) will be denoted by $T_0(\alpha, r)$ and called the *Nevanlinna characteristic function* of α. Using the inequalities

$$\begin{aligned}&\log^+(t_1t_2)\leqslant\log^+ t_1+\log^+ t_2\\ &\log^+(t_1+t_2)\leqslant\log^+ t_1+\log^+ t_2+\log 2,\end{aligned}$$

we obtain from (3.10) the relations

$$\begin{cases}T_0(\alpha_1\alpha_2, r)\leqslant T_0(\alpha_1, r)+T_0(\alpha_2, r)\\ T_0(\alpha_1\alpha_2, r)\leqslant T_0(\alpha_1, r)+T_0(\alpha_2, r)+O(1)\\ T_0(\alpha-a, r)=T_0(\alpha, r)+O(1)\\ T_0(1/\alpha, r)=T_0(\alpha, r)+O(1).\end{cases} \tag{3.11}$$

From (3.11) we immediately deduce

(3.12) PROPOSITION. *Let $\Lambda(r)$ be a positive, increasing function of r such that $\lim_{r\to\infty}\Lambda(r)=+\infty$. Then the set of all meromorphic functions α on M which $r\to\infty$ satisfy the growth condition*

$$T_0(\alpha, r)=O(\Lambda(r))$$

forms a subfield $\mathfrak{M}_\Lambda$ of the field $\mathfrak{M}$ of all meromorphic functions on M.

Following still the classical theory, we let $N(r, e^{i\theta}) = N(D_{e^{i\theta}}, r)$ and shall prove an identity due to H. Cartan when $M = \mathbf{C}$.

(3.13) PROPOSITION. *The Nevanlinna characteristic function satisfies*

$$\frac{1}{2\pi}\int_0^{2\pi} N(r, e^{i\theta})\, d\theta = T_0(\alpha, r) + O(1).$$

Proof. Jensen's theorem applied to the function $\alpha(z) = z - a$ on the complex plane gives

$$\frac{1}{2\pi}\int_0^{2\pi} \log|a - e^{i\theta}|\, d\theta = \log^+|a|, \tag{3.15}$$

even including the limiting case $a = \infty$. Replacing $\alpha(z)$ by $\alpha(z) - e^{i\theta}$ and using (3.2) we have

$$N(r, e^{i\theta}) = N(D_\infty, r) + \int_{\partial M[r]} \log|\alpha(z) - e^{i\theta}|^2 \eta(z) + O(1).$$

Integrating this equation with respect to $d\theta$ and using (3.14) yields

$$\frac{1}{2\pi}\int_0^{2\pi} N(r, e^{i\theta}) = \int_{\partial M[r]} \log^+|\alpha(z)|^2 \eta(z) + N(D_0, r) + O(1).$$

Comparing the R.H.S. of this relation with the R.H.S. of (3.10) gives the proposition.

Q.E.D.

It follows from (3.13) that $T_0(\alpha, r)$ is an increasing convex function of $\log r$, an assertion which may be viewed as a sort of "three-circles" theorem.

(c) Jensen's theorem for vector valued functions

We continue to let M be a complex manifold with exhaustion function $\tau: M \to [-\infty, +\infty)$ and Levi form $\psi = dd^c\tau$. Let $f: M \to \mathbf{C}^n$ be a holomorphic mapping such that $Z = f^{-1}(0)$ has pure codimension n. Using the notations

$$\left\{\begin{aligned} M[r] &= \{z \in M : \tau(z) \leqslant \log r\} \\ Z[r] &= Z \cap M[r] \\ n(Z, t) &= \int_{Z[t]} \psi^{m-n} \\ N(Z, r) &= \int_0^r n(Z, t)\frac{dt}{t} \end{aligned}\right. \tag{3.15}$$

we want to have a formula for the counting function $N(Z, r)$ in terms of f and ψ. Referring to (7.6), we let

$$\begin{cases} \omega_l = (dd^c \log \|f\|^2)^l \\ \Omega_l = \log \|f\|^2 \omega_l \\ \mu_l = d^c\Omega_l = d^c \log \|f\|^2 \wedge \omega_l. \end{cases} \tag{3.16}$$

(3.17) PROPOSITION. *Using the notations (3.15) and (3.16) we have*

$$N(Z, r) = \int_{\partial M[r]} \Omega_{n-1} \wedge \eta_{m-n} - \int_{M[r]} \Omega_{n-1} \wedge \psi^{m-n+1} + O(1)$$

$$\int_{M[r]} \omega_l \wedge \psi^{m-l} = \int_{\partial M[r]} \Omega_{l-1} \wedge \eta_{m-l} - \int_{M[r]} \Omega_{l-1} \wedge \psi^{m-l+1} + O(1) \quad (l \leqslant n-1)$$

where $\eta_k = d^c\tau \wedge \psi^k \geqslant 0$ *on* $\partial M[r]$.

Proof. This proposition follows in the same manner as 3.2 by integrating twice the equations of currents

$$\begin{cases} dd^c \omega_l = 0 \\ dd^c\Omega = \omega_{l+1} \qquad (l < n-1) \\ dd^c\Omega_{n-1} = Z. \end{cases} \tag{3.18}$$

The restriction of these equations to $M[r]$ is handled in the same way as in 3.2.

Remarks. For $f: M \to \mathbf{C}^n$ introduce the notations

$$M_f(r) = \max_{z \in M[r]} \log \|f(z)\|^2$$

$$V(r) = \int_{M[r]} \omega^{n-1} \wedge \psi^{m-n+1}$$

$$S(r) = \int_{M[r]} \frac{1}{\|f\|^2} \omega^{n-1} \wedge \psi^{m-n+1}.$$

Then Jensen's theorem gives the estimate

$$N(Z, r) \leqslant M_f(r)\, V(r) + S(r) + O(1). \tag{3.19}$$

The first term $M_f(r)V(r)$ on the R.H.S. is intrinsic and involves, so to speak, only the growth of the mapping f and not the particular value "0" where $Z = f^{-1}(0)$. On the other hand, the remainder term $S(r)$ is not intrinsic. One's initial hope might be that $M_f(r)V(r)$ is the more important term. If, e.g., we have the special case of a holomorphic mapping

$$\mathbf{C}^2 \xrightarrow{f = (f_1, f_2)} \mathbf{C}^2$$

where f is of *finite order* in the sense that $M_f(r) \leqslant c_2 r^\lambda$, then the relative unimportance of $S(r)$ would imply an estimate

$$N(Z, r) \leqslant c_2 r^{2\lambda} \tag{3.20}$$

for the number of common zeroes of f_1 and f_2 in the ball $\|z\| \leq r$. A recent example of Cornalba and Shiffman [7] shows that the *Bezout estimate* (3.20) may be *false* (cf. Stoll [19] for a Bezout estimate "on the average"). Thus in general there will be no Nevanlinna inequality (3.7) in higher codimension. Moreover, the Casorati-Weierstrass theorem (3.8) fails also in higher codimension, as illustrated by the well-known Fatou-Bieberbach example. Outside of the one result due to Chern-Stoll-Wu (cf. Proposition (5.20) page 186), the value distribution theory for higher codimensional subvarieties remains a mystery.

4. Conditions that a divisor be algebraic

Let A be an m-dimensional affine variety, and

$$\pi: A \to \mathbf{C}^m, \qquad \tau(z) = \log \|\pi(z)\|^2$$

the generic projection and special exhaustion function constructed in § 2(b). Let D be an effective analytic divisor on A, $\psi = dd^c\tau$ the Levi form of τ, and

$$N(D, r) = \int_0^r \left\{ \int_{D[t]} \psi^{m-1} \right\} \frac{dt}{t}$$

the counting function for D (cf. (3.1)).

(4.1) PROPOSITION. *D is an algebraic divisor if and only if,*

$$N(D, r) = O(\log r).$$

Proof. It is immediate from the definition (3.1) that $N(D, r)$ is $O(\log r)$ if, and only if,

$$\int_{D[r]} \psi^{m-1} = O(1) \tag{4.2}$$

for all r. On the other hand, by Lemma (2.10), D is an algebraic subset of A if and only if, $\pi(D)$ is an algebraic subset of $\mathbf{C}^m$. Using this together with (4.2) we are reduced to the following result of Stoll [17]:

(4.3) PROPOSITION. *Let D be an effective analytic divisor in $\mathbf{C}^m$ and $\psi = dd^c \log \|z\|^2$. Then D is algebraic if, and only if,*

$$\int_{D[r]} \psi^{m-1} = O(1).$$

We will give two proofs of this result. The first uses elementary properties of plurisubharmonic functions together with the Cousin II problem in $\mathbf{C}^n$. The second uses Nevanlinna theory and provides one of the few occasions where the remainder term in the First Main Theorem 5.14 can be dealt with. In giving this proof we shall use the F.M.T. (5.14) below and refer the reader to that section for its proof.

First Proof. Suppose that D is algebraic of degree d in $\mathbf{C}^m$. Assuming that the origin does not lie on D, we consider the projection

$$D \to \mathbf{P}^{m-1}$$

of D over the lines through the origin in $\mathbf{C}^m$. For each such line ξ the intersection $\xi \cdot D$ consists of $\leqslant d$ points, counted with multiplicities, on ξ. It follows that

$$\int_{D[r]} \psi^{m-1} \leqslant d\left\{\int_{\mathbf{P}^{m-1}} \omega^{m-1}\right\} = d, \qquad \lim_{r\to\infty}\left(\int_{D[r]} \psi^{m-1}\right) = d,$$

where $\omega = dd^c \log \|z\|^2$ is the usual (1, 1) form on $\mathbf{P}^{m-1}$.

Conversely, suppose that $\int_{D[r]} \psi^{m-1} \leqslant d$ for all r. Then, on the *average* (cf. (5.18) page 186), each line ξ through the origin meets D in at most d points. Our main step is to show in an a priori manner that this happens for *all* lines ξ.

For this we use the Cousin II problem on $\mathbf{C}^m$ to find an entire holomorphic function $\alpha(z)$ such that $(\alpha) = D$. Normalizing so that $\alpha(0) = 1$, Jensen's theorem (3.2) gives

$$N(D, r) = \int_{\|z\|=r} \log |\alpha(z)|\, \eta(z) \tag{4.4}$$

where $\eta(z) = d^c \log \|z\|^2 \wedge (dd^c \log \|z\|^2)^{m-1}$. For each point $z \neq 0$ in $\mathbf{C}^m$, we let ξ_z be the line determined by z and $\xi_z[r] = \xi_z \cap \mathbf{C}^m[r]$. Then Jensen's theorem applied to $\alpha(z)\,|\,\xi_z$ gives

$$N(D \cap \xi_z, r\|z\|) = \frac{1}{2\pi}\int_0^{2\pi} \log |\alpha(re^{i\theta} z)|\, d\theta. \tag{4.5}$$

From (4.5) we see that the counting function $N(D \cap \xi_z, r\|z\|)$ is a pluri subharmonic function of $z \in \mathbf{C}^m$ since it is the mean value of the pluri subharmonic functions $\log|a(re^{i\theta} z)|$.

We want to apply the sub-mean-value property of pluri subharmonic functions to $N(D \cap \xi_z, r\|z\|)$. For this let $B(z, \varrho)$ be the ball of radius ϱ around z in $\mathbf{C}^m$ and Φ be the Euclidean volume element normalized so that $\int_{\|z\|\leqslant 1} \Phi(z) = 1$. Then the sub-mean-value property gives the inequality

$$\begin{aligned} N(D \cap \xi_z, r\|z\|) &\leqslant \frac{1}{\varrho^{2m}} \int_{w \in B(z,\varrho)} N(D \cap \xi_w, r\|w\|)\, \Phi(w) \\ &\leqslant \frac{1}{\varrho^{2m}} \int_{w \in B(0,\|z\|+\varrho)} N(D \cap \xi_w, r\|w\|)\, \Phi(w) \\ &\leqslant \left(\frac{\|z\|+\varrho}{\varrho}\right)^{2m} \int_{\xi \in \mathbf{P}^{m-1}} N(D \cap \xi, r(\|z\|+\varrho))\, \psi^{m-1}(\xi) \\ &\leqslant \left(\frac{\|z\|+\varrho}{\varrho}\right)^{2m} d \log (r(\|z\|+\varrho)). \end{aligned}$$

Taking $\|z\|=1$, we obtain the estimate,

$$\frac{N(D \cap \xi, r)}{\log r} \leqslant d\left(\frac{1+\varrho}{\varrho}\right)^{2m} + O(1), \quad (\xi \in \mathbf{P}^{m-1}). \tag{4.6}$$

It follows from (4.6) that $D \cap \xi$ is a divisor of degree $\leqslant d((1+\varrho)/\varrho)^{2m}$ for every $\varrho > 0$. Letting $\varrho \to \infty$ we find that all intersections $D \cap \xi$ are divisors of degree $\leqslant d$ on the line ξ through the origin in $\mathbf{C}^m$.

Let d_0 be the smallest integer with the property that degree $(D \cap \xi) \leqslant d_0$ for all ξ. Let ξ_0 be some ξ at which the maximum is attained; then there is a neighborhood U of ξ_0 and $0 < R_0 < \infty$ such that $D \cap \xi \subset B(0, R_0)$ for $\xi \in U$. Thus D satisfies the hypothesis of Lemma (2.10) and is algebraic.

Second proof. As before, we must show that

$$\deg (D \cap \xi) \leqslant d < \infty \tag{4.7}$$

for all lines ξ through the origin in $\mathbf{C}^n$. Suppose we are able to prove the estimate

$$\int_{H \cap D} \psi^{n-2} \leqslant d < \infty \tag{4.8}$$

for all hyperplanes H passing through the origin in $\mathbf{C}^n$. Then we may repeat the proof of (4.8) with $H \cap D$ replacing D, and in this way work our way down to the desired estimate (4.7).

We consider the residual mapping

$$D \xrightarrow{f} \mathbf{P}^{n-1} \tag{4.9}$$

which sends each point $z \in D$ to the line $f(z)$ determined by z (here we assume that $\{0\} \notin D$). The function $\tau(z) = \log \|z\|^2$ gives an exhaustion function on D, and the Levi form

$$dd^c\tau = f^*(\omega) = \omega_f \tag{4.10}$$

where ω is the standard Kähler form on $\mathbf{P}^{n-1}$. We wish to apply the F.M.T. (5.14) for divisors to the mapping (4.9) and exhaustion function τ. Even though D may have singularities, this is possible because of Proposition (1.1).

Let $L \to \mathbf{P}^{n-1}$ be the standard line bundle whose sections are the linear functions on $\mathbf{C}^n$ and where $|L|$ is the complete linear system of hyperplanes in $\mathbf{P}^{n-1}$. We take in L the standard metric such that $c(L) = \omega$, and denote by H_σ the hyperplane in $\mathbf{C}^n$ determined by the section $\sigma \in H^0(\mathbf{P}^{n-1}, L)$. Using the notations

$$N(H_\sigma, r) = \int_0^r \left\{ \int_{H_\sigma[t]} \omega_f^{n-2} \right\} \frac{dt}{t};$$

$$T(r) = \int_0^r \left\{ \int_{D[t]} \omega_f^{n-1} \right\} \frac{dt}{t};$$

$$S(H_\sigma, r) = \int_{D[r]} \log \frac{1}{|\sigma|} \omega_f^{n-1};$$

the F.M.T. (5.14) together with (4.10) yields the estimate

$$N(H_\sigma, r) \leqslant T(r) + S(H_\sigma, r). \tag{4.11}$$

From (5.18) we have the averaging formula

$$T(r) = \int_{\substack{\sigma \in H^0(\mathbf{P}^{n-1}, L) \\ \sup|\sigma|=1}} N(H_\sigma, r)\, d\mu(\sigma),$$

and (4.8) will follow from an estimate

$$\lim_{r \to \infty} \frac{S(H_\sigma, r)}{T(r)} = 0. \tag{4.12}$$

Proof. From (4.11) and (4.12) we have

$$\lim_{r \to \infty} \frac{N(H_\sigma, r)}{T(r)} \leqslant 1.$$

Now $T(r) \leqslant d \log r$ since $\int_D \omega_f^{n-1} = d < \infty$, and if $\int_{H_\sigma[t]} \omega^{n-1} = c$, then $N(H_\sigma, r) \geqslant c \log r + O(1)$ for large r. Thus $c \leqslant d$ for all t, and consequently

$$\int_{H_\sigma} \omega_f^{n-2} \leqslant d.$$

To prove (4.12), we consider the singular volume form

$$\Omega_\sigma = \frac{1}{|\sigma|^{\frac{1}{2}}} \omega^{n-1}$$

on $\mathbf{P}^{n-1}$. We normalize $|\sigma|$ so that $\int_{\mathbf{P}^{n-1}} \Omega_\sigma = 2$, and then use the concavity of log to obtain

$$S(H_\sigma, r) \leqslant \log \left\{ \int_{D[r]} \Omega_\sigma + O(1) \right\}. \tag{4.13}$$

On the other hand, we have

$$dd^c(|\sigma|^{\frac{1}{2}}\omega^{n-2}) = \Omega_\sigma + \Theta$$

where Θ is a *bounded* volume form on $\mathbf{P}^{n-1}$. It follows that

$$S(H_\sigma, r) \leq \log\left\{\int_{D[r]} dd^c|\sigma|^{\frac{1}{2}}\omega_f^{n-2} + c\omega_f^{n-1}\right\} \quad (c>0). \tag{4.14}$$

Now we may apply Stokes' theorem twice to obtain

$$\int_0^R\left\{\int_0^r dd^c|\sigma|^{\frac{1}{2}}\omega_f^{n-2}\right\}\frac{dr}{r} = \int_{D[R]}|\sigma|^{\frac{1}{2}}\omega_f^{n-1} - \int_{\partial D[R]}|\sigma|^{\frac{1}{2}}\omega_f^{n-2}\wedge d^c\tau \tag{4.15}$$

(cf. the proof of Proposition (3.2)). Since $|\sigma|$ is bounded and $\int_{D[R]}\omega_f^{n-1} = \int_{\partial D[R]}\omega_f^{n-2}\wedge d^c\tau = O(1)$, we may combine (4.14) and (4.15) to have an estimate

$$\int_0^R e^{S(H_\sigma, r)}\frac{dr}{r} = O(\log R). \tag{4.16}$$

Since $S(H_\sigma, r)$ is a non-negative increasing function of r, (4.12) follows from (4.16). Q.E.D.

Remark. The above proof works for an arbitrary divisor $D \subset \mathbf{C}^n$ and yields an estimate

$$N(H_\sigma, r) \leq T(r) + o\{T(r)\} \qquad \|$$

(cf., the proof of Lemma (7.22) for an explanation of the symbol $\|$). This estimate bounds the growth of $H \cap D$ in terms of the growth of D for *all* hyperplanes H. There is no analogous inequality known in case codim $(D) > 1$.

5. The order function for holomorphic mappings

(a) Definition and basic properties

Let M be a complex manifold having a special exhaustion function $\tau: M \to [-\infty, +\infty)$ (cf., § 2). Suppose that V is a smooth, projective algebraic variety, and that $L \to V$ is a positive line bundle having a metric with positive curvature form ω (cf., § 0). Let $f: M \to V$ be a holomorphic mapping, set $\omega_f = f^*\omega$, and define order functions $T_1, \ldots, T_m$ by

$$T_q(r) = \int_0^r\left\{\int_{M[t]} w_f^q \wedge (dd^c\tau)^{m-q}\right\}\frac{dt}{t}. \tag{5.1}$$

The *total order function* $\underset{\sim}{T}(f, r)$ is defined by

$$\underset{\sim}{T}(f, r) = \sum_{q=0} T_q(r).$$

Here, $T_0(r) = \int_0^r \{\int_{m[t]} (dd^c\tau)^m\} dt/t =$ constant since τ is a special exhaustion function. Geometrically, if we let $\Gamma_f \subset M \times V$ be the graph of f, $\Gamma_f[t] = \Gamma_f \cap (M[t] \times V)$ that part of the graph lying above $M[t]$, and

$$v(t) = \int_{\Gamma_f(t)} (dd^c\tau + \omega)^m$$

the volume of $\Gamma_f[t]$ on $M \times V$ relative to the pseudo-Kähler metric $dd^c\tau + \omega$, then

$$\underset{\sim}{T}(r) = \int_0^r v(t)\frac{dt}{t}, \tag{5.2}$$

modulo some inessential constant factors.

(5.3) PROPOSITION. *If we choose another metric in $L \to V$ which leads to a new curvature form $\tilde{\omega}$ and order functions $\tilde{T}_q(r)$, then*

$$T_q(r) - \tilde{T}_q(r) = O\left(\frac{dT_{q-1}(r)}{dr} + 1\right).$$

Proof. Let $\theta \in A^{n-1,\, n-1}(M)$ be a $C^\infty\,(n-1, n-1)$ form on V. Then by Stokes' theorem

$$\int_0^r \left(\int_{M[t]} dd^c\theta\right)\frac{dt}{t} = \int_0^r \left(\int_{\partial M[t]} d^c\theta\right)\frac{dt}{t} = \int_{M[r]} d\tau \wedge d^c\theta,$$

which gives

$$\int_0^r \left(\int_{M[t]} dd^c\theta\right)\frac{dt}{t} = \int_{\partial M[r]} \theta \wedge d^c\tau - \int_{M[r]} \theta \wedge dd^c\tau. \tag{5.4}$$

From (0.3) we have $\tilde{\omega} = \omega + dd^c\varrho$ where $\varrho \in C^\infty(V)$. Plugging this into the definitions and using (5.4) we obtain

$$T_q(r) - \tilde{T}_q(r) = \sum_{k=0}^{q-1} a_k \int_{\partial M[r]} \varrho(dd^c\varrho)^{q-k-1} \wedge \omega^k \wedge d^c\tau + b_k \int_{M[r]} \varrho(dd^c\varrho)^{q-k-1} \wedge \omega^k \wedge dd^c\tau. \tag{5.5}$$

Now $\varrho(dd^c\varrho)^{q-k-1} \leqslant c_k \omega^{q-k-1}$, and using this together with Stokes' theorem in (5.5) gives the result. Q.E.D.

(5.6) COROLLARY. *The order functions associated to two different metrics in $L \to V$ satisfy*

$$T_1(r) = \tilde{T}_1(r) + O(1).$$

Remark. The above proposition and corollary suggest that $T_1(r)$ should perhaps be the most interesting term in the total order function $\underset{\sim}{T}(f, r)$. Thus, e.g., the order of growth of $T_q(r)$ for $q > 1$ will be well-defined only if

$$\lim_{r\to\infty} \frac{\dfrac{dT_{q-1}(r)}{dr}}{T_q(r)} = 0. \tag{5.7}$$

We shall now give two more indications that $T_1(r)$ is the most important term in $\underset{\sim}{T}(f, r)$ For the first we write $T_1(r) = T_1(f, r)$ in order to emphasize the dependence on f. Suppose given two holomorphic mappings $f: M \to V$ and $g: M \to W$ of M into smooth projective varieties V and W, so that we may consider the product mapping $f \times g: M \to V \times W$.

(5.8) PROPOSITION. *The mappings f, g and $f \times g$ satisfy*

$$T_1(f \times g, r) = T_1(f, r) + T_1(g, r) + O(1).$$

Moreover, this functorial property is not *necessarily true for the terms $T_q(r)$ $(q \geq 2)$.*

Proof. The equality follows immediately from the definition. The observation that, e.g., $T_2(r)$ does not necessarily have the functoriality property may be seen by letting $\dim_{\mathbf{C}} M = 2$. Then what we need to do is to be able to estimate $\int \omega_f \wedge \omega_g$ in terms of $\int \omega_f \wedge \omega_f$ and $\int \omega_g \wedge \omega_g$. In general this is not possible.

The second proposition will be proved at the end of § 5(c) below. To state it we assume that M is a smooth affine variety A with the special exhaustion function τ constructed in § 2(b).

(5.9) PROPOSITION. *The mapping f is rational if, and only if,*

$$T_1(r) = O(\log r).$$

This estimate is, in turn, satisfied if, and only if,

$$\underset{\sim}{T}(f, r) = O(\log r).$$

(b) The First Main Theorem (FMT)

We continue with the situation $f: M \to V$ of § 5(a). The F.M.T. for divisors, which is the global version of Jensen's theorem (3.2) for meromorphic functions, will be presented first. Let $D \in |L|$ be an arbitrary effective divisor given by the zeroes of a holomorphic section $\sigma \in H^0(V, L)$. Since σ and $\lambda\sigma$ $(\lambda \neq 0)$ define the same divisor, we shall assume that $|\sigma(x)| \leq 1$ for $x \in V$. Let $L_f \to M$ be the pull-back of $L \to V$ and σ_f the pull-back of σ. Assume that $\sigma_f \not\equiv 0$ and define the *proximity form*

$$m(D, r) = \int_{\partial M[r]} \log \frac{1}{|\sigma_f|^2}\, \eta \geq 0 \tag{5.10}$$

where $\eta = d^c\tau \wedge (dd^c\tau)^{m-1}$ is positive on $\partial M[r]$.

(5.11) PROPOSITION (F.M.T. for divisors). *Letting D_f be the divisor of $\sigma_f \in H^0(M, L_f)$, we have*

$$N(D_f, r) + m(D, r) = T_1(r) + O(1)$$

where O(1) depends on D but not on r.

Proof: This follows by integrating twice the equation of currents (1.5) applied to $L_f \to M$ and σ_f, in exactly the same way as Jensen's theorem (3.2) followed by integrating twice the equation (1.1). Q.E.D.

Remark. Combining (5.10) and (5.11) gives the estimate

$$N(D_f, r) < T_1(r) + O(1) \qquad (D \in |L|), \tag{5.12}$$

which is a variant of the basic Nevanlinna inequality (3.7). In both cases, the effect of the estimate is to bound the growth any particular divisor by the *average growth* of all the divisors in the same linear system (cf. (3.10), (3.13), and Proposition (5.18), page 186).

To give the general F.M.T., we assume that $\sigma_1, \ldots, \sigma_n \in H^0(V, L)$ are holomorphic sections such that the subvariety Z defined by $\sigma_1 = \ldots = \sigma_n = 0$ has pure codimension n on V. Assume that $f: M \to V$ is a holomorphic mapping such that $Z_f = f^{-1}(Z)$ has pure codimension n on M. We use the notations (1.15) and set

$$\begin{cases} \sigma_f = (f^{-1}(\sigma_1), \ldots, f^{-1}(\sigma_n)), & \Lambda_f = f^*\Lambda \\ \psi_l = (dd^c\tau)^l, & \eta_l = d^c\tau \wedge \psi^{l-1} \\ m(Z, r) = \int_{\partial M[r]} \Lambda_f \wedge \eta^{n-m} & (\textit{proximity form}) \\ S_n(Z, r) = \int_{M[r]} \Lambda_f \wedge \psi_{n-m+1} & (\textit{remainder}). \end{cases} \tag{5.13}$$

(5.14) PROPOSITION (F.M.T.—the general case). *Using the notations (5.13) and (1.15),*

$$N(Z_f, r) + m(Z, r) = T_n(r) + S_n(Z, r) + O(1),$$

where O(1) depends on Z but not on r.

Proof. This follows by integrating twice the equation of currents (1.18) cf. the proof of (3.17).

Remarks. As in the case of divisors, we may assume that $\|\sigma(x)\| \leq 1$ for all $x \in V$. Then by (1.22) the proximity form $m(Z, r) \geq 0$ and (5.14) gives the estimate

$$N(Z_f, r) < T_n(r) + S_n(Z, r) + O(1). \tag{5.15}$$

When $n=1$, (5.15) reduces to the Nevanlinna inequality (5.12) because $\psi^m=0$ and so the remainder $S_1(Z, r)\equiv 0$. However, for $n>1$ in general the remainder term will be positive and we are in the analogous situation to Proposition (3.17).

(c) Averaging and density theorems

In this section we assume that $L\to V$ is sufficiently *ample*, which means that the complete linear system $|L|$ should give an embedding of V in $\mathbf{P}^{N-1}$ where $\dim_{\mathbf{C}} H^0(V, L)=N$, and that the image of V should contain no proper linear subspaces. Choosing a metric in $H^0(V, L)$ induces the usual Fubini-Study form ω on $\mathbf{P}^{N-1}$ which is invariant under the unitary group, and we may assume that the metric and curvature form on $L\to V$ are induced from these on $\mathbf{P}^{N-1}$.

Let $G(n, N)$ be the Grassman manifold of all n-planes in $H^0(V, L)$, and denote by $C\{G(n, N)\}$ the *Grassman cone* of all decomposable vectors $\sigma=\sigma_1\wedge \dots \wedge\sigma_n\in \wedge^n H^0(V, L)$. For any such σ we denote by $Z(\sigma)$ the subvariety $\sigma=0$ on V and note that codim $(Z(\sigma))=n$ since $L\to V$ is ample. The proximity form $\Lambda=\Lambda(\sigma)$ given by (1.15) may be constructed, and $\Lambda(\sigma)$ is the restriction to V of the analogous form on $\mathbf{P}^{N-1}$ which is given by the same formula. In particular, if T: $H^0(V, L)\to H^0(V, L)$ is any unitary linear transformation, then by linear algebra T induces actions of $C\{G(n, N)\}$ and $\mathbf{P}^{N-1}$, and

$$\Lambda(T\sigma)=T^*\Lambda(\sigma). \tag{5.16}$$

We denote by $C_1\{G(n, N)\}$ the vectors in $C\{G(n, N)\}$ of length one and let $d\mu(\sigma)$ be the measure on $C_1\{G(n, N)$ which is invariant under the unitary group. Explicitly,

$$d\mu(\sigma)=c d^c \log \|\sigma\| \wedge (dd^c \log \|\sigma\|)^{n(N-n)}$$

where c is a constant to be determined.

(5.17) LEMMA. *For a suitable choice of constant c, the average*

$$\int_{\sigma\in C_1\{G(n,N)\}} \log \frac{1}{\|\sigma\|^2}\Lambda(\sigma)\, d\mu(\sigma)=\omega^{n-1}.$$

Proof. From our construction it follows that $\int_{\{\sigma\}} \log (1/\|\sigma\|^2)\ \Lambda(\sigma)\ d\mu(\sigma)$ is an $n-1, n-1)$ form on $\mathbf{P}^{N-1}$ which is invariant under the unitary group. It follows that $\int_{\{\sigma\}} \log (1/\|\sigma\|^2)\ \Lambda(\sigma)\ d\mu(\sigma)=c_1\omega^{n-1}$ since *any* invariant form is a multiple of ω^{n-1}. We may easily check that $c_1\neq \pm\infty$, and so we arrange that $c_1=1$ by a suitable choice of c. Q.E.D.

Let f: $M\to V$ be a holomorphic mapping such that codim $\{Z_f(\sigma)\}=n$ for almost all $\sigma\in C\{G(n, N)\}$.

(5.18) PROPOSITION. *We have the averaging formula*

$$\int_{\sigma \in C_1\{G(n,N)\}} N(Z_f(\sigma), r)\, d\mu(\sigma) = T_n(r) + O(1).$$

Proof. We shall give the formal computation. The convergence follows by justifying Fubini's theorem in the same way as in Stoll [18]. Referring to (5.14) it will suffice to prove that

$$\int_{\{\sigma\}} m(Z(\sigma), r)\, d\mu(\sigma) = \int_{\{\sigma\}} S_n(Z(\sigma), r)\, d\mu(\sigma) + O(1). \tag{5.19}$$

Interchanging the order of integration in (5.17) and using (5.17) we are left to verify that

$$\int_{\partial M[r]} \omega^{n-1} \eta_{n-m} = \int_{M[r]} \omega^{n-1} \psi_{n-m+1} + O(1),$$

which follows from $d\eta_{n-m} = \psi_{n-m+1}$ together with Stokes' theorem. Q.E.D.

Remark. The averaging formula (5.18) is a version of *Crofton's formula* from integral geometry, which says that the length of any piecewise smooth closed curve C in $\mathbf{R}^2$ is the average over the lines L in $\mathbf{R}^2$ of the number of points of intersection of L and C.

As an application of (5.18), we shall prove the following result which is a variant of those of Chern, Stoll and Wu (cf. Stoll [18]).

(5.20) PROPOSITION. *Let $f: M \to V$ be as above and assume that*

$$\varliminf_{r\to\infty} \frac{\dfrac{dT_{n-1}(1)}{dr} + O(1)}{T_n(r)} = 0.$$

Then the image $f(M)$ meets almost all $Z(\sigma)$ for $\sigma \in G(n, N)$.

Proof. Suppose that the set I of all $\sigma \in C_1\{G(n, N)\}$ such that $f(M)$ intersects $Z(\sigma)$ has measure $1-\varepsilon$ for some $\varepsilon > 0$. Combining (5.15) and (5.18) we have

$$\begin{aligned}
T_n(r) &= \int_{\{\sigma\}} N(Z_f(\sigma), r)\, d\mu(\sigma) && \text{(by (5.18))}\\
&= \int_{\sigma\in I} N(Z_f(\sigma), r)\, d\mu(\sigma) && \text{obviously}\\
&\leqslant \int_{\sigma\in I} \{T_n(r) + S_n(Z(\sigma), r) + O(1)\}\, d\mu(\sigma) && \text{(by (5.15))}\\
&\leqslant (1-\varepsilon)\, T_n(r) + \frac{dT_{n-1}(r)}{dr} + O(1),
\end{aligned}$$

where the last step follows because of

$$\begin{aligned}\int_{\sigma\in I} S_n(Z(\sigma),r)\,d\mu(\sigma) &\leqslant \int_{\{\sigma\}} S_n(Z(\sigma),r)\,d\mu(\sigma)\\ &= \int_{M[r]} \omega_f^{n-1}\wedge\psi_{n-m+1} \qquad \text{(by (5.17))}\\ &= \frac{dT_{n-1}(r)}{dr} \qquad \text{(by definition).}\end{aligned}$$

Combining the above inequalities gives

$$1 \leqslant (1-\varepsilon) + \frac{\dfrac{dT_{n-1}(r)}{dr} + O(1)}{T_n(r)}. \tag{5.21}$$

Taking lim-inf in (5.21) gives the proposition. Q.E.D.

(5.22) Corollary. *If M has a special exhaustion function, then the image $f(M)$ meets almost all divisors $D\in|L|$.*

Remarks. (i) This corollary is obviously the same type of assertion as the Casorati-Weierstrass theorem (3.8).

It is interesting to observe that the condition

$$\lim_{r\to\infty} \frac{\dfrac{dT_{n-1}(r)}{dr} + O(1)}{T_n(r)} = 0$$

which allows the density theorem to hold is the same as the condition (5.7) that the order function $T_n(r)$ be intrinsic.

(ii) Suppose now that our map

$$f\colon M\to V$$

satisfies the estimate

$$\frac{dT_{q-1}(r)}{dr} = o(T_n(r)) \qquad (q\leqslant n). \tag{5.23}$$

Then certainly the image $f(M)$ meets almost all $Z(\sigma)$ for $\sigma\in G(n, N)$.

Question. Assuming the estimate (5.23), do we then have the *Nevanlinna inequality*

$$N(Z_f(\sigma), r) \leqslant T_n(r) + o(N(Z_f(\sigma), r)$$

valid for any $Z(\sigma)$?

The motivation for this question is that the presence of an estimate bounding the growth of every $Z_f(\sigma)$ in terms of the average growth seems geometrically to be about the same as saying that the image $f(M)$ meets almost all $Z(\sigma)$. In order to prove (5.24), it would seem necessary to estimate the remainder term in the F.M.T. (5.14), and (with perhaps the exception of our proof of Stoll's theorem in § 4) nobody has been able to successfully do this, even in the case of divisors.

Proof of Proposition (5.9). Replacing the positive line bundle L by

$$L^k = \underbrace{L \otimes \ldots \otimes L}_{k\text{-times}}$$

changes $T_1(r)$ into $kT_1(r)$, and therefore does not alter the conditions of the proposition. Choosing k sufficiently large, we may assume that $L \to V$ is ample so that the complete linear system $|L|$ induces a projective embedding of V. Then it is clear that $f: A \to V$ is rational if, and only if, the divisors

$$D_f = f^{-1}(D)$$

are algebraic and of uniformly bounded degree for all $D \in |L|$.

Suppose first that f is rational. Then, referring to § 4, we see that for any $D \in |L|$

$$N(D_f, r) \leqslant d \log r + O(1) \tag{5.24}$$

where d is the degree of $\pi(D_f)$ in $\mathbf{C}^m$. Here the $O(1)$ depends on D, but from the discussion of Lelong numbers in § 1(d) it follows that, for fixed r, the estimate (5.24) holds for *all* $D \in |L|$. Integration of (5.24) with respect to the invariant measure $d\mu(D)$ on $|L|$ and an application of (5.18) gives

$$T_1(r) \leqslant d \log r + O(1)$$

where, as is easily checked, the $O(1)$ term is now independent of r. This proves one half of our proposition.

To prove the other, and more substantial, half we assume that $T_1(r) = d \log r + O(1)$. From the Nevanlinna inequality (5.12) it follows that

$$N(D_f, r) \leqslant d \log r + O(1)$$

for any $D \in |L|$. Applying Proposition (4.12) we find that all divisors D_f are algebraic and of degree $\leqslant d$ on A.

It remains to prove that:

$$T_1(r) = O(\log r) \Rightarrow \underset{\sim}{T}(f, r) = O(\log r).$$

Under the assumption $T_1(r) = O(\log r)$ we have just proved that f is rational. Choose a rational projective embedding $g: A \to \mathbf{P}^N$. Replacing f by the product $h = f \times g: A \to V \times \mathbf{P}^N$, we obviously have that

$$\underset{\sim}{T}(f, r) \leqslant \underset{\sim}{T}(h, r).$$

On the other hand, h has the advantage of being an algebraic embedding of A into a complete projective variety, and we may obviously assume that the image $h(A)$ is in general position with respect to a given family of algebraic subvarieties of the image variety. In conclusion, it will suffice to prove that $\underset{\sim}{T}(f, r)=O(\log r)$ under the assumption that $L \to V$ is ample and that

$$\operatorname{codim}\,[f^{-1}(Z(\sigma)]=n$$

for all subvarieties $Z(\sigma)$ corresponding to $0 \neq \sigma \in \Lambda^n H^0(V, L)$.

Now then all $Z_f(\sigma)=f^{-1}[Z(\sigma)]$ are algebraic subvarieties of dimension $m-n$ on A, and the degrees of $\pi[Z_f(\sigma)]$ in $\mathbf{C}^m$ are all bounded by some number d. It follows that

$$N(Z_f(\sigma), r) \leqslant d \log r + O(1),$$

and our result follows by averaging this inequality over all $Z(\sigma)$ and using (5.18).

(d) Comparison between the order function and Nevanlinna characteristic function.

Let M be a complex manifold with special exhaustion function $\tau: M \to [-\infty, +\infty)$ and $\alpha(z)$ a meromorphic function on M. In § 3(b) we defined the Nevanlinna characteristic function (cf. (3.10))

$$T_0(\alpha, r) = N(D_\infty, r) + \int_{\partial M[r]} \log^+ |\alpha|^2 \eta \tag{5.25}$$

where $\eta = d^c\tau \wedge (dd^c\tau)^{m-1} \geqslant 0$ on $\partial M[r]$. This characteristic function has the very nice algebraic properties given by (3.11). Moreover, in case M is an affine algebraic variety A, it follows from (3.10) and Proposition (4.1) that α is a rational function for the algebraic structure on A if, and only if,

$$T_0(\alpha, r) = O(\log r).$$

At this time we want to introduce another order function $T_1(\alpha, r)$ which, in case α may be interpreted as a holomorphic mapping $\alpha: M \to \mathbf{P}^1$, is just the order function $T_1(r)$ for the standard positive line bundle over $\mathbf{P}^1$ introduced in (5.1). Locally on M we may write $\alpha = \beta/\gamma$ where β and γ are relatively prime holomorphic functions. From the relation

$$\log(1+|\alpha|^2) = \log(|\gamma|^2+|\beta|^2) - \log|\gamma|^2$$

it follows that the locally L^1 differential form of type (1, 1)

$$\omega_\alpha = dd^c \log|(\gamma|^2+|\beta|^2)$$

is well-defined, and we have the equation of currents

(5.26) $$dd^c \log (1+|\alpha|^2)=\omega_\alpha - D_\infty.$$

Using the by now familiar notations

$$T_1(\alpha, r)=\int_0^r \left\{\int_{M[t]} \omega_\alpha \wedge (dd^c\tau)^{m-1}\right\}\frac{dt}{t}, \qquad m_1(\alpha, r)=\int_{\partial M[r]} \log (1+|\alpha|^2)\,\eta \geqslant 0$$

we may integrate (5.26) twice as in the proofs of (5.11) and (3.2) to have the formula

(5.27) $$N(D_\infty, r)+m_1(\alpha, r)=T_1(\alpha, r)+O(1).$$

Classically, (5.27) is called the spherical F.M.T. in Ahlfors-Shimizu form. Using the relations

$$\log^+|\alpha|^2 \leqslant \log (1+|\alpha|^2) \leqslant \log^+|\alpha|^2+\log 2,$$

we may compare (5.25) and (5.27) to obtain

(5.28) $$T_0(\alpha, r)=T_1(\alpha, r)+O(1).$$

Consequently, for studying orders of growth, the functions $T_0(\alpha, r)$ and $T_1(\alpha, r)$ are interchangeable.

It is hoped that (5.28) will tie together the discussion in § 3(b) with that in § 5(b).

6. Volume forms and the second main theorem (SMT)

(a) Singular volume forms on projective varieties

Let V be a smooth projective variety of dimension n and $L \to V$ a holomorphic line bundle. Our aim is to construct volume forms on V which are singular along certain divisors and which have positive Ricci forms; we will follow the proof in [6]. We shall consider divisors D of the following type:

(6.1) $$\begin{gathered} D \in |L| \text{ is a divisor with normal crossings;} \\ D=D_1+\ldots+D_k, \text{ where each } D_i \text{ is nonsingular;} \end{gathered}$$

that is, D has simple normal crossings (§ 0).

Let L_i be the line bundle $[D_i]$; there is a section σ_i of L_i such that $(\sigma_i)=D_i$. Then $L=L_1 \otimes \ldots \otimes L_k$ and the section $\sigma=\sigma_1 \otimes \ldots \otimes \sigma_k$ has divisor $(\sigma)=D$.

(6.2) PROPOSITION. *Suppose that $c(L)+c(K_V)>0$ and that $D \in |L|$ satisfies (6.1); then there is a volume form Ω on V and there exist metrics on the L_i such that the singular volume form*

$$\Psi = \frac{\Omega}{\prod_{i=1}^{k} (\log |\sigma_i|^2)^2 |\sigma_i|^2} \tag{6.3}$$

satisfies

$$\operatorname{Ric} \Psi > 0, \quad (\operatorname{Ric} \Psi)^n \geqslant \Psi, \quad \int_{V-D} (\operatorname{Ric} \Psi) < \infty. \tag{6.4}$$

Proof. We know that there is a metric on L with curvature form ω such that $\omega + \operatorname{Ric} \Omega > 0$. Choose metrics on $L_1, \ldots, L_{k-1}$ arbitrarily and set

$$|\zeta_k| = |\zeta_1 \otimes \ldots \otimes \zeta_k| / |\zeta_1| \ldots |\zeta_{k-1}|$$

for any nonvanishing sections ζ_i of L_i. Multiplying the metrics by a constant, we can require that $|\sigma_i| < \delta$ for any fixed $\delta > 0$.

Using (0.1), (0.2) and (0.5) we have

$$\operatorname{Ric} \Psi = \omega + \operatorname{Ric} \Omega - \sum_{i=1}^{k} dd^c \log (\log |\sigma_i|^2)^2. \tag{6.5}$$

Now
$$- dd^c \log (\log |\sigma_i|^2)^2 = \frac{-2\, dd^c \log |\sigma_i|^2}{\log |\sigma|^2} + \frac{4 d \log |\sigma_i|^2 \wedge d^c \log |\sigma_i|^2}{(\log |\sigma_i|^2)^2}.$$

The first term is a continuous form on V, so perhaps choosing a smaller δ we have, setting $\omega_0 = \omega + \operatorname{Ric} \Omega$, for some $c_1 > 0$

$$\operatorname{Ric} \Psi \geqslant c_1 \omega_0 + 4 \sum_{i=1}^{k} \frac{d \log |\sigma_i|^2 \wedge d^c \log |\sigma_i|^2}{(\log |\sigma_i|^2)^2} \geqslant 0. \tag{6.6}$$

The latter form is $\geqslant 0$ because $d\lambda \wedge d^c \lambda = (2\pi)^{-1} i \partial \pi \wedge \overline{\partial d} \geqslant 0$ for any real λ.

Around any point $x \in V$, one can choose coordinates $(z_1, \ldots, z_n)$ in a neighborhood U of x such that $x = (0, \ldots, 0)$ and $D_i = (z_i)$ in U, this being because D has normal crossings. Thus $\log |\sigma_i|^2 = \log b_i + \log |z_i|^2$ where $b > 0$ is a C^∞ function. Hence

$$d \log |\sigma_i|^2 \wedge d^c \log |\sigma_i|^2 = \frac{\sqrt{-1}}{2\pi} \frac{dz_i \wedge d\bar{z}_i}{|z_i|^2} + \varrho. \tag{6.7}$$

The form
$$\varrho_i = \frac{\sqrt{-1}}{2\pi} \left(\frac{\partial b \wedge \bar{\partial} b}{b^2} + \frac{\partial b \wedge d\bar{z}_i}{b \bar{z}_i} + \frac{dz_i \wedge \bar{\partial} b}{z_i b} \right)$$

has the property that $|z_i|^2 \varrho_i$ is a smooth form whose coefficients vanish on D_i.

Thus we see that, noting $\omega_0 \geqslant c_2 \sqrt{-1} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$ for some $c_2 > 0$

$$(\operatorname{Ric} \Psi)^n \geqslant c_3 (\sqrt{-1})^n \frac{dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z} + c_4 \Lambda}{\prod_{i=1}^{k} (\log |z_i|^2)^2 |z_i|^2} \tag{6.8}$$

where the coefficients of Λ are 0 at $(0, \ldots, 0)$ and $c_3, c_4 > 0$. There is then a $c_5 > 0$ such that $(\operatorname{Ric}\Psi)^n > c_5\Psi$ in some neighborhood $U' \subset U$ of x where the coefficients of Λ are small. Since V is compact, we cover V by a finite number of such U' and get $(\operatorname{Ric}\Psi)^n > c_6\Psi$ for $c_6 > 0$. Now we can redefine Ψ by replacing Ω by $c_6\Omega$. This does not affect $\operatorname{Ric}\Psi$, thus we have

$$(\operatorname{Ric}\Psi)^n > \Psi.$$

Finally, we must see that $\int_{V-D}(\operatorname{Ric}\Psi)^n < \infty$. By compactness it suffices to show convergence on a neighborhood of each x. Choose a compact neighborhood $U' \subset U$ of x, and we see by our previous calculations that locally

$$(\operatorname{Ric}\Psi)^n = \frac{\Phi}{\prod_{i=1}^{k}(\log|z_i|^2)^2|z_i|^2}$$

where Φ is a smooth form on U'. Thus $\int_{U'-V}(\operatorname{Ric}\Psi)^n < \infty$ since the function $\prod_{i=1}^{k}(\log|z_i|^2)^2|z_i|^2$ is locally L^1 in $\mathbf{C}^n$ because of $\int_0^\varepsilon(\log t)^{-2}t^{-1}dt < \infty$. Q.E.D.

We can modify the preceding proposition somewhat to include the case that $c(L) + c(K_V) = 0$ if we assume that $c(L) > 0$.

(6.9) PROPOSITION. *Suppose that $c(L) + c(K_V) = 0$ and that $D \in |L|$ satisfies (6.1); then there is a volume form Ω on V and there exist metrics on the L_i such that the singular volume form:*

$$\Psi_\varepsilon = \frac{\Omega}{\prod_{i=1}^{k}(\log|\sigma_i|^2)^2|\sigma_i|^{2+2\varepsilon}} \tag{6.10}$$

satisfies

$$\operatorname{Ric}\Psi_\varepsilon > 0, \qquad (\operatorname{Ric}\Psi_\varepsilon)^n \geq |\sigma|^{2\varepsilon}\Psi_\varepsilon. \tag{6.11}$$

Proof. We choose metrics on the L_i so that $\omega = -dd^c\log|\sigma|^2 = -\sum_{i=1}^{k}dd^c\log|\sigma_i|^2 > 0$. Then for any volume form Ω',

$$\omega + \operatorname{Ric}\Omega' = dd^c\varrho$$

for some C^∞ real-valued function ϱ; let $\Omega = e^{-\varrho}\Omega'$; then the form $\omega + \operatorname{Ric}\Omega = 0$.

Proceeding with the same computation as in (6.5) we have

$$\operatorname{Ric}\Psi_\varepsilon = \varepsilon\omega - \sum_{i=1}^{k}dd^c\log(\log|\sigma_i|^2)^2 \tag{6.12}$$

and since $\varepsilon\omega > 0$, as in (6.6)

$$\operatorname{Ric}\Psi_\varepsilon \geq c_1\varepsilon\omega + \sum_{k=1}^{k}\frac{d\log|\sigma_i|^2 \wedge d^c\log|\sigma_i|^2}{(\log|\sigma_i|^2)^2} \geq 0. \tag{6.13}$$

We continue as before and get, in the same notation:

$$(6.14)\qquad (\operatorname{Ric}\Psi_\varepsilon)^n \geqslant \prod_{i=1}^{k}|z_i|^{2\varepsilon}\frac{c_3\sqrt{-1}^n dz_1\wedge d\bar z_k\wedge\dots\wedge dz_n\wedge d\bar z_n + c_4\Lambda}{\prod_{i=1}^{k}(\log|z_i|^2)^2|z_i|^{2+2\varepsilon}} \geqslant c_6|\sigma|^{2\varepsilon}\Psi_\varepsilon.$$

Replacing Ω by $c_6\Omega$ we have $(\operatorname{Ric}\Psi_\varepsilon)^n \geqslant |\sigma|^{2\varepsilon}\Psi_\varepsilon$.

The rest follows exactly the same as in the proof of (6.2). Q.E.D.

(6.15) *Example.* We can apply (6.9) to the case when $c(K_V^*)>0$, especially the case where $V=\mathbf{P}^n$ and $D=\sum_{l=0}^{n}H_l$ is the union of $(n+1)$ hyperplanes in general position.

Remark. In case $n=1$, V is a compact Riemann surface of genus g and $D=\{x_1,\dots,x_N\}$ consists of N distinct points. The condition $c(L)+c(K_V)>0$ in (6.2) is

$$2g-2+N>0,$$

and in this case Proposition (6.2) amounts to finding a metric of Gaussian curvature $K(x)\leqslant -1$ on $V-\{x_1,\dots,x_N\}$. If $g>1$, we may take $N=0$; if $g=1$ we may take $N=1$; and for $V=\mathbf{P}^1$ we must have $N\geqslant 3$. In all cases the metric given by (6.3) is complete.

Proposition (6.10) applies only to $\mathbf{P}^1$, and it says that we may find a metric on $\mathbf{P}^1-\{0,1\}\simeq\mathbf{C}^*$ whose Gaussian curvature is everywhere negative and satisfies $K(z)\leqslant -|z|^{2\varepsilon}$ near $z=0$ and similarly near $z=\infty$. It follows from results of R. Greene and H. Wu that this estimate is sharp.

Our last proposition on volume forms deals with the opposite extreme to Proposition (6.10). Namely, recall that a smooth projective variety V_n is said to be of *general type* if

$$\limsup_{k\to\infty}\frac{\dim H^0(V,K_V^k)}{k^n}>0.$$

For example, this condition is satisfied whenever the canonical bundle is positive. From [14], we see that, if V is of general type and $L\to V$ is an ample line bundle, then

$$H^0(V,K_V^k\otimes L^*)\neq 0$$

for some sufficiently large k.

(6.16) PROPOSITION. *If Ω is a C^∞ volume form on the complex manifold V, $L\to V$ is a positive holomorphic line bundle, and $0\not\equiv\sigma\in H^0(V,K_V^k\otimes L^*)$, then the volume form $\Psi=|\sigma|^{2/k}\Omega$ satisfies the condition*

$$\operatorname{Ric}\Psi>0$$

on all of V. (The metric on K_V is that induced by Ω.)

Proof: Referring to (0.1), (0.2) and (0.5) we obtain

$$\begin{aligned}\operatorname{Ric}\Psi &= k^{-1}dd^c\log|\sigma|^2+\operatorname{Ric}\Omega\\ &= k^{-1}[-kc(K_V)+c(L)]+c(K_V)\\ &= \frac{1}{k}c(L)>0.\end{aligned}$$

Q.E.D.

(b) The Second Main Theorem

We keep the notations from § 6(a), and shall consider a holomorphic mapping

$$f\colon A_m\to V_n \qquad (m\geqslant n)$$

from a smooth affine variety A into V where we assume that f has maximal rank n. Equivalently, the image $f(A)$ should contain an open set on V. We shall also use the generic projection

$$\pi\colon A\to\mathbf{C}^m$$

discussed in § 2(b). The notations concerning π which we adopt throughout are:

$$\begin{aligned}&\tau=\log\|\pi(x)\|^2 \quad (x\in A), && \eta=d^c\tau\wedge\psi_{m-1},\\ &\psi=dd^c\tau, && \varphi=dd^c\|\pi(x)\|^2,\\ &\psi_k=\underbrace{\psi\wedge\ldots\wedge\psi}_{k}, && \Phi=\varphi_m.\end{aligned}$$

Before stating and proving our S.M.T., we need a local lemma about singular volume forms. For this we let $U\subset\mathbf{C}^m$ be an open set, $\Phi(w)=\prod_{j=1}^m(\sqrt{-1}/2\pi)(dw_j\wedge d\bar{w}_j)$ the Euclidean volume form on $\mathbf{C}^m$, and

$$\Psi=\frac{|\gamma|^2\Phi(w)}{(\log|\delta|^2)^2|\delta|^{2\lambda}} \tag{6.17}$$

a singular volume form where $\gamma=\alpha\,e^a$ and $\delta=\beta e^b$ with $\alpha,\beta\in\mathcal{O}(U)$ and $a,b\in C^\infty(U)$. Clearly, $\operatorname{Ric}\Psi\in L^1_{(1,1)}(\mathrm{loc},U)$.

(6.18) LEMMA. *Writing $\Psi=\zeta\Phi(w)$, the function $\log\zeta$ is locally L^1 on U and satisfies the equation of currents*

$$dd^c\log\zeta=\operatorname{Ric}\Psi+D_\alpha-\lambda D_\beta$$

where $D_\alpha=(\alpha)$ and $D_\beta=(\beta)$.

Proof. Using (0.5) and (1.1), we must show that

$$\left\{\begin{array}{l}dd^c\log(\log|\delta|^2)^2\\ \text{in the sense of}\\ \text{currents}\end{array}\right\}=\left\{\begin{array}{l}dd^c\log(\log|\delta|^2)^2\\ \text{in the sense of}\\ \text{differential forms}\end{array}\right\}$$

What this amounts to is proving that

$$\int_U \partial\bar{\partial} \log (\log |\delta|^2)^2 \wedge \sigma = \int_U \log (\log |\delta|^2)^2 \wedge \partial\bar{\partial}\sigma$$

for all $\sigma \in A_c^{n-1,n-1}(U)$. This equation is, in turn, easy to verify by a direct computation. Q.E.D.

Returning now to our holomorphic mapping $f: A \to V$, we consider a divisor D on V which satisfies (6.1) and let $\Psi_f = f^*\Psi$ be the pull-back of the volume form constructed in Proposition (6.2). Since f has maximal rank, Ψ_f is not identically zero. Thus we may choose linear coordinates $z_1, \ldots, z_m$ on $\mathbf{C}^m$ such that

$$\Psi_f \wedge \pi^* \left\{ \prod_{j=1}^{m-n} \frac{\sqrt{-1}}{2\pi} (dz_j \wedge d\bar{z}_j) \right\} = \xi \Phi \tag{6.19}$$

where $\xi \geqslant 0$ is not identically zero. Roughly speaking, the local behavior of ξ is as follows:

(i) $\xi = +\infty$ along the divisor $D_f = f^{-1}(D)$;

(ii) $\xi = +\infty$ along the branch locus B of $A \xrightarrow{\pi} \mathbf{C}^m$;

(iii) $\xi = 0$ along the ramification divisor R of f;

(iv) $\xi = 0$, along the divisor T given by $\Psi_f \wedge \pi^* \left\{ \prod_{j=1}^{m-n} \frac{\sqrt{-1}}{2\pi} (dz_j \wedge d\bar{z}_j) \right\} = 0$ but $\Psi \neq 0$

(v) otherwise, ξ is finite and non-zero.

(6.20) LEMMA. *Setting $S = R + T$, the function $\log \xi$ is locally L^1 on A and satisfies the equation of currents*

$$dd^c \log \xi = S - B - D_f + \operatorname{Ric} \Psi_f. \tag{6.21}$$

Proof. This follows from Lemma (2.9), (6.3), (6.17), and Lemma (6.18). Q.E.D.

Our S.M.T. will be the twice integrated version of (6.21), in the same way that the F.M.T. (5.11) was the twice integrated form of the equation of currents (1.5). Taking into account the discussion of Lelong numbers in § 1(c) and following (3.1), we assume that none of the divisors S, B, D_f passes through $\pi^{-1}(0)$ and introduce the notations

$$\left\{ \begin{aligned} T^{\neq}(r) &= \int_0^r \left\{ \int_{A[t]} \operatorname{Ric} \Psi_f \wedge \varphi_{m-1} \right\} \frac{dt}{t^{2m-1}}; \\ N(E, r) &= \int_0^r \left\{ \int_{E[t]} \Psi_{m-1} \right\} \frac{dt}{t}; \qquad (E = \text{divisor on } A) \\ \mu(r) &= \int_{\partial A[r]} \log \xi \eta. \end{aligned} \right. \tag{6.22}$$

(6.23) PROPOSITION (S.M.T.). *For $r \geqslant r_0$, we have the equation*

$$T^{\neq}(r) + N(S, r) = N(B, r) + N(D_f, r) + \mu(r).$$

Proof. Following the procedure used in the proof of Proposition (3.2), we may integrate (6.21) once to have for all but finitely many t

$$(6.24)\qquad \int_{A[t]} \operatorname{Ric} \Psi_f \wedge \psi_{m-1} + \int_{S[t]} \psi_{m-1} = \int_{B[t]} \psi_{m-1} + \int_{D_f[t]} \psi_{m-1} + \int_{\partial A[t]} d^c \log \xi \wedge \psi_{m-1}.$$

Now $\operatorname{Ric} \Psi_f \in L^1_{(1,1)}$ (loc, A) and $d \operatorname{Ric} \Psi_f = 0$ in the sense of currents. Since $\pi: A \to \mathbf{C}^m$ is a finite and therefore proper mapping, we may integrate $\operatorname{Ric} \Psi_f$ over the fibers to obtain $\pi_* \operatorname{Ric} \Psi_f \in L^1_{(1,1)}$ (loc, $\mathbf{C}^m$) which is still closed. Thus $\pi_* \operatorname{Ric} \Psi_f = d\varrho$ for ϱ a locally L^1 differential form on $\mathbf{C}^m$, and

$$\begin{aligned}
\int_{A[t]} \operatorname{Ric} \Psi_f \wedge \psi_{m-1} &= \int_{\mathbf{C}^m[t]} \pi_* \operatorname{Ric} \Psi_f \wedge (dd^c \log \|z\|^2)^{m-1} \\
&= \int_{\partial \mathbf{C}^m[t]} \varrho \wedge (dd^c \log \|z\|^2)^{m-1} && \text{(Stokes')} \\
&= \frac{1}{t^{2m-2}} \int_{\partial \mathbf{C}^m[t]} \varrho \wedge (dd^c \|z\|^2)^{m-1} && \text{(by (1.24))} \\
&= \frac{1}{t^{2m-2}} \int_{\mathbf{C}^m[t]} \pi_* \operatorname{Ric} \Psi_f \wedge (dd^c \|z\|^2)^{m-1} && \text{(Stokes')} \\
&= \frac{1}{t^{2m-2}} \int_{A[t]} \operatorname{Ric} \Psi_f \wedge \varphi_{m-1}.
\end{aligned}$$

Using this relation and integrating (6.24) with respect to dt/t from 0 to r gives

$$(6.25)\qquad T^{\#}(r) + N(S, r) = N(B, r) + N(D_f, r) + \int_{A[r]} d\tau \wedge d^c \xi \wedge (dd^c \tau)^{m-1}.$$

Now
$$\begin{aligned}
d\tau \wedge d\xi \wedge (dd^c \tau)^{m-1} &= -d^c\tau \wedge d\xi \wedge (dd^c \tau)^{m-1} \\
&= d\{d^c \tau \wedge (dd^c \tau)^{m-1}\} \\
&= d(\xi\eta)
\end{aligned}$$

since $(dd^c \tau)^m = 0$. Using this and applying Stokes' theorem to the last term on the R.H.S. of (6.25), we obtain our formula. Q.E.D.

7. The defect relations

(a) Nevanlinna defects and statement of the main result

Let A be a smooth affine algebraic variety and V a smooth projective variety having a positive line bundle $L \to V$ with curvature form ω. We want to study a holomorphic mapping

$$f: A \to V$$

with particular attention to the position of the image $f(A)$ relative to the divisors $D \in |L|$. For this we set $\omega_f = f^*\omega$ and let τ be the special exhaustion function of A constructed in Proposition (2.4). Define the *order function for the line bundle* $L \to V$ by the formula

$$T(L, r) = \left\{ \int_{A[t]} \omega_f \wedge (dd^c \tau)^{m-1} \right\} \frac{dt}{t}. \tag{7.1}$$

In Section 5, this order function was denoted by $T_1(r)$, but here we want to emphasize the dependence on L. Referring to (5.3), (5.8) and (5.9) we find that $T(L, r)$ has the following properties:

$$\begin{cases} T(L, r) \text{ is well defined up to an } O(1) \text{ term;} \\ T(L_1 \otimes L_2, r) = T(L_1, r) + T(L_2, r); \text{ and} \\ f \text{ is rational} \Leftrightarrow T(L, r) = O(\log r) \end{cases} \tag{7.2}$$

For any divisor $D \in |L|$ we have the First Main Theorem (5.11) and subsequent Nevanlinna inequality (5.12), repeated here for easy reference:

$$\begin{gathered} N(D_f, r) + m(D, r) = T(L, r) + O(1) \\ N(D_f, r) \leq T(L, r) + O(1). \end{gathered} \tag{7.3}$$

We refer once more to § 3(c) where the $O(1)$ term, which depends on D but not on r, is discussed. Using the inequality in (7.3) we may define the *defect* for the divisor D by

$$\delta(D) = 1 - \overline{\lim_{r \to \infty}} \frac{N(D_f, r)}{T(L, r)}, \tag{7.4}$$

which has the basic properties

$$0 \leq \delta(D) \leq 1; \text{ and } \delta(D) = 1 \text{ if } f(A) \text{ does not intersect } D. \tag{7.5}$$

In general, divisors $D \in |L|$ with $\delta(D) > 0$ are said to be *deficient*; this means that the divisor $D_f = f^{-1}(D)$ is smaller than on the average. From (5.10) we obtain the relation

$$\int_{D \in |L|} \delta(D)\, d\mu(D) = 0, \tag{7.6}$$

which may be interpreted as stating that, in the measure-theoretic sense, almost all divisors D_f have the same asymptotic growth given by the order function $T(L, r)$. Roughly speaking, the basic problem in the value distribution of divisors on algebraic varieties is the following:

(*) Show that there is a constant $c = c(V, L)$ with the property that if $D_1, \ldots, D_k \in |L|$ are divisors such that each D_i is smooth and $D = D_1 + \ldots + D_k$ has normal crossings and if the image $f(A)$ satisfies a mild general position requirement, then

$$\sum_{j=1}^{k} \delta(D_j) \leqslant c. \tag{7.7}$$

In particular, the *defect relation* (7.7) would imply that, if $L \to V$ is ample, then the deficient divisors lie on a countable family of subvarieties in $|L|$. Thus, if $\dim_{\mathbf{C}}|L| = N$ and if $\Delta = \{D \in |L| : \delta(D) > 0\}$ is the set of deficient divisors, then the $2N-1$ Hausdorff measure $\mathcal{H}_{2N-1}(\Delta)$ should be zero. Thus far, even this weak statement is not known.

Geometrically, the simplest situation to understand is when the image $f(A)$ contains an open set on V. In this case our main result is the following defect relation (D.R.):

(7.8) THEOREM. *Assume that the image $f(A)$ contains an open subset of V and $D_1, \ldots, D_k \in |L|$ are divisors such that each D_i is smooth and $D = D_1 + \ldots + D_k$ has normal crossings. Then*

$$\sum_{j=1}^{k} \delta(D_j) \leqslant \frac{c(K_V^*)}{c(L)} + \varkappa \tag{D.R.}$$

where $\varkappa$ is a constant which is zero if either $A = \mathbf{C}^m$ or f is transcendental.

Remark. Before embarking on a formal proof of (D.R.), let us give the heuristic reasoning behind it. For this purpose we let $L_1 \to V$ be a positive line bundle satisfying

$$c(L_1) + c(K_V) > 0,$$

and let $D \in |L_1|$ be a divisor with normal crossings. (In the proof of (7.8), we will take $L_1 = L^k$.) Then we may construct the volume form Ψ given by (6.3) which has singularities along D. Writing out the F.M.T. (5.11) and S.M.T. (6.23) together, we obtain the inequalities

$$\begin{cases} N(D_f, r) \leqslant T(L_1, r) + O(1) \\ T^{\#}(r) \leqslant N(B, r) + N(D_f, r) + \mu(r). \end{cases} \tag{7.9}$$

The first equation in (7.9) gives an *upper* bound on the counting function $N(D_f, r)$, and the second equation will turn out to give a *lower* bound on $N(D_f, r)$. Playing these off against each other will lead to (7.8).

More precisely, using the curvature condition $(\operatorname{Ric} \Psi)^n \geqslant \Psi$, we will obtain an approximate inequality

$$\mu(r) \leqslant \log \frac{d^2 T^{\#}(r)}{dr^2}. \tag{7.10}$$

From (6.5) we will also have approximately

$$T^{\#}(r) = T(L_1, r) + T(K_V, r). \tag{7.11}$$

By (7.10) it seems plausible that

$$\varliminf_{r\to\infty} \frac{\mu(r)}{T^{\#}(r)} = 0,$$

so that using (7.11) we may rewrite the second equation in (7.9) as

$$1 \leqslant \varkappa_1 + \frac{N(D_f, r)}{[T(L_1, r) + T(K_V, r)]} + o(1) \tag{7.12}$$

where $\varkappa_1 = \overline{\lim}_{r\to\infty} [N(B, r)/T^{\#}(r)]$ is a term not involving D and which is zero if $A = \mathbf{C}^m$. Neglecting $\varkappa_1$, the inequality (7.12) illustrates clearly how the S.M.T. acts as a lower bound on $N(D_f, r)$. When this is made precise, we will obtain (7.8).

(b) A preliminary defect relation

In this section we let $f\colon A \to V$ be a holomorphic mapping such that $f(A)$ contains an open subset of V, $L_1 \to V$ a positive line bundle satisfying

$$c(L_1) + c(K_V) > 0,$$

and $D \in |L_1|$ a divisor with simple normal crossings. Then the discussion in § 6(b) applies, and in particular the S.M.T. (6.23) may be used to study the divisor D_f on A. Referring to Lemma (6.20), we let $N_1(r) = \int_0^r \{\int_{R[t]} \psi_{m-1}\}\, dt/t$ be the counting function for the ramification locus of $f\colon A \to V$ and rewrite (6.23) as the inequality

$$T^{\#}(r) + N_1(r) \leqslant N(B, r) + N(D_f, r) + \mu(r). \tag{7.13}$$

(7.14) LEMMA. *There is a constant $c > 0$ such that, for $r \geqslant 1$,*

$$T^{\#}(r) \geqslant c \log r.$$

Proof. Referring to the proof of (6.23), we have for $r \geqslant 1$

$$T^{\#}(r) = \int_0^r \left\{ \int_{\mathbf{C}^m[t]} \pi_*(\operatorname{Ric} \Psi_f) \wedge \psi_{m-1} \right\} \frac{dt}{t} \geqslant c_1 \log r + O(1)$$

where $$c_1 = \int_{\mathbf{C}^m[1]} \left\{ \pi_*(\operatorname{Ric} \Psi_f) \wedge \psi_{m-1} \right\} = \int_{\mathbf{C}^m[1]} \{\pi_* \operatorname{Ric} \Psi_f \wedge \varphi_{m-1}\}$$

is a positive constant by the first condition in (6.4). Q.E.D.

(7.15) LEMMA. *We have that*

$$\overline{\lim_{r\to\infty}} \frac{N(B, r)}{T^{\#}(r)} = \varkappa_1 < \infty.$$

Proof. Since the branch locus B of $\pi: A \to \mathbf{C}^m$ is an algebraic divisor, it follows from (4.7) that, for some constant $d > 0$,

$$N(B, r) \leqslant d \log r$$

for large r. Using (7.14) we obtain

$$\frac{N(B, r)}{T^{\#}(r)} \leqslant \left(\frac{d}{c}\right)$$

for all large r. Q.E.D.

Our preliminary defect relation is the following.

(7.16) PROPOSITION. *Using the above notations*

$$1 + \varliminf_{r\to\infty} \frac{N_1(r)}{T^{\#}(r)} \leqslant \varkappa_1 + \varlimsup_{r\to\infty} \frac{N(D_f, r)}{T^{\#}(r)}.$$

Proof. We want to use the curvature condition

$$(\mathrm{Ric}\, \Psi_f)^n \geqslant \Psi_f \tag{7.17}$$

to obtain a lower bound on $T^{\#}(r)$. For this we adopt the following notations:

(i) $z = (a_1, \ldots, z_m)$ are coordinates in $\mathbf{C}^m$;

(ii) $I = \{1, \ldots, m\}$ and $A \subset I$ runs through all subsets containing n distinct elements;

(iii) $A_0 = \{1, \ldots, n\}$; and

(iv) $\Phi_B = \prod_{j\in B} \{\frac{1}{2}\pi^{-1}\sqrt{-1}\,(dz_j \wedge d\bar{z}_j)\}$ for any subset $B \subset I$.

Setting $\Psi = \sum_A \xi_A \Phi_A$, the definition (6.19) gives $\xi = \xi_{A_0}$. We define the auxiliary order function

$$T^{\#\#}(r) = \int_0^r \left\{ \int_{A[t]} n\xi^{1/n}\Phi \right\} \frac{dt}{2m-1}. \tag{7.18}$$

(7.19) LEMMA. *We have the estimate*

$$T^{\#\#}(r) \leqslant T^{\#}(r).$$

Proof. Writing $(\mathrm{Ric}\, \Psi_f)^n = \sum_A \zeta_A \Phi$ $(\zeta_A \geqslant 0)$, the curvature condition (7.17) gives $\zeta_A \geqslant \xi_A$ for all A, and in particular

$$n\xi^{1/n} \leqslant n\zeta_{A_0}^{1/n}. \tag{7.20}$$

We now write $$\mathrm{Ric}\, \Psi_f \wedge \varphi_{m-1} = \sum_A \left\{ \sum_{j\in A} \mathrm{Ric}\, \Psi_f \wedge \varphi_{A-\{j\}} \right\} \wedge \varphi_{I-A}.$$

Using the inequality trace $(H) \geqslant n(\det H)^{1/n}$ for a positive Hermitian matrix, we have that

(7.21) $$n\zeta_{A_0}^{1/n}\Phi \leqslant \sum_{j\in A_0} \operatorname{Ric}\Psi_f \wedge \varphi_{A_0-\{j\}} \wedge \varphi_{I-A_0} \leqslant \operatorname{Ric}\Psi_f \wedge \varphi_{m-1}.$$

The lemma now follows by combining (7.20), (7.21) and integrating. Q.E.D.

(7.22) LEMMA. *Setting* $d/ds = r^{2m-1}(d/dr)$, *we have*

$$\mu(r) \leqslant n \log \frac{d^2T^{\neq\neq}(r)}{ds^2} - n(4m-2)\log r.$$

Proof. Using the definition (6.17) and concavity of the logarithm function, we obtain

$$\mu(r) = \int_{\partial A[r]} n \log \xi^{1/n}\eta \leqslant n \log\left\{\int_{\partial A[r]} \xi^{1/n}\eta\right\} = n \log\left\{\frac{1}{r^{2m-1}}\frac{d}{dr}\int_{A[r]} \xi^{1/n}\,\Phi\right\}$$
$$= n \log\left\{\frac{1}{r^{2m-1}}\frac{d}{dr}\left[r^{2m-1}\frac{dT^{\neq\neq}(r)}{dr}\right]\right\} = 2n \log\left(\frac{1}{r^{2m-1}}\right) + n \log\left[\frac{d^2T^{\neq\neq}(r)}{ds^2}\right]. \qquad Q.E.D.$$

Now we must eliminate the derivatives in front of $T^{\neq\neq}(r)$. For this we use the following real-variables lemma from [16], page 253:

(7.23) LEMMA. *Suppose that* $f(r)$, $g(r)$, $\alpha(r)$ *are positive increasing functions of* r *where* $g'(r)$ *is continuous and* $f'(r)$ *is piecewise continuous. Suppose moreover that* $\int^\infty (dr/\alpha(r)) < \infty$. *Then*

$$f'(r) \leqslant g'(r)\,\alpha(f(r))$$

except for a union of intervals $I \subset \mathbf{R}^+$ *such that*

$$\int_I dg \leqslant \int^\infty \frac{dr}{\alpha(r)}.$$

We use the notation

$$a(r) \leqslant b(r) \qquad \|_g$$

to mean that the stated inequality holds except on an open set $I \subset \mathbf{R}^+$ such that $\int_I dg < \infty$. Taking $f(r) = T^{\neq\neq}(r)$, $g(r) = r^\mu/\mu$, $\alpha(r) = r^\nu$ with μ and $\nu > 1$, we obtain from (7.23) that

(7.24) $$\frac{dT^{\neq\neq}(r)}{dr} \leqslant r^{\mu-1}(T^{\neq\neq})^\nu \qquad \|_g.$$

Keeping the same α and g and taking $f(r) = r^{2m-1}(dT^{\neq\neq}(r))/dr = \int_{A[r]} n\xi^{1/n}\Phi$, we find

(7.25) $$\frac{d}{dr}\left(r^{2m-1}\frac{dT^{\neq\neq}(r)}{dr}\right) \leqslant r^{\mu-1} r^{(2m-1)\nu}\left(\frac{dT^{\neq\neq}(r)}{dr}\right)^\nu \qquad \|_g.$$

Combining (7.24) and (7.25) we obtain

$$\frac{d^2 T^{\#\#}(r)}{ds^2} \leqslant r^{4m-2+\varepsilon}(T^{\#\#}(r))^{2+\delta} \qquad \|_g \tag{7.26}$$

where $\varepsilon, \delta > 0$ may be made as small as we wish by choosing μ and ν close to 1. Combining (7.26) with (7.22) and (7.19), we have

$$\mu(r) \leqslant n\varepsilon \log r + (2+\delta) \log T^{\#}(r). \qquad \|_g. \tag{7.27}$$

Now we are almost done, because (7.13) and (7.27) together give the estimate

$$1 + \frac{N_1(r)}{T^{\#}(r)} \leqslant \frac{N(B,r)}{T^{\#}(r)} + \frac{N(D_f,r)}{T^{\#}(r)} + \frac{n\varepsilon \log r}{T^{\#}(r)} + (2+\delta)\frac{\log T^{\#}(r)}{T^{\#}(r)} \qquad \|_g. \tag{7.28}$$

Passing to the limit in (7.28) using (7.15) and (7.14) yields the inequality

$$1 + \varliminf_{r\to\infty} \frac{N_1}{T^{\#}} \leqslant \varkappa_1 + \varlimsup_{r\to\infty} \frac{N(D_f)}{T^{\#}} + \frac{n\varepsilon}{c}.$$

Letting $\varepsilon \to 0$ we obtain our proposition. Q.E.D.

(c) Proof of the main defect relation

We use the notations and assumptions from Theorem (7.8). Because of (7.6), almost all divisors $D^{\#} \in |L|$ will have defect zero. Adding a finite number of such $D^{\#}$ to the L.H.S. of (7.8) will increase k without affecting the sum $\sum_j (D_j)$. Thus we may assume that

$$c(L^k) + c(K_V) = kc(L) + c(K_V) > 0. \tag{7.29}$$

We want to use Proposition (7.16) with L^k playing here the role of L_1 in that result. In order to do this, it is necessary to be able to compare $T^{\#}(r)$ given by (6.22) with $kT(L, r) + T(K_V, r)$.

(7.30) LEMMA. *We have the inequalities*

$$0 \leqslant [kT(L,r) + T(K_V,r)] - T^{\#}(r) \leqslant 2 \log [kT(L,r) + c].$$

Proof: Let $\sigma \in H^0(V, L^k)$ define D. Then from (6.5) we have, using $kT(L,r) = T(L^k, r)$, that

$$kT(L,r) + T(K_V,r) - T^{\#}(r) = \int_0^r \left(\int_{A[t]} dd^c \log (\log |\sigma_f|^2)^2 \wedge \psi_{m-1} \right) \frac{dt}{t}. \tag{7.31}$$

By the same argument as in the proof of the S.M.T. (6.23) (cf. the proof of (6.18)), the R.H.S. of (7.31) is equal to

$$\int_{\partial A[r]} \log (\log |\sigma_f|^2)^2 \eta.$$

Making $\|\sigma\|_V$ sufficiently small, this term is non-negative, which gives the left hand inequality in (7.30). To obtain the other one, we use concavity of the logarithm together with (5.10) and (5.11) to have

$$\int_{\partial A[r]} \log (\log |\sigma_f|^2)^2 \eta \leqslant 2 \log \left(\int_{\partial A[r]} \log \frac{1}{|\sigma_f|^2} \eta \right)$$
$$= 2 \log [m(D, r)] \leqslant 2 \log (T(L, r) + c). \qquad \text{Q.E.D.}$$

Referring to (7.29), we let $l > 0$ be any real number such that

$$k - l \geqslant \frac{c(K_V^*)}{c(L)}. \tag{7.32}$$

Then, using the definition (7.4) we have

$$\sum_{j=1}^{k} \delta(D_j) = \sum_{j=1}^{k} \left[1 - \overline{\lim_{r\to\infty}} \frac{N(D_j, r)}{T(L, r)} \right] \leqslant k - \overline{\lim_{r\to\infty}} \frac{N(D, r)}{T(L, r)}$$
$$\leqslant k - l \overline{\lim_{r\to\infty}} \frac{N(D, r)}{lT(L, r)} \leqslant k - l \overline{\lim_{r\to\infty}} \frac{N(D, r)}{kT(L, r) + T(K_V, r)}$$
$$= k - l \overline{\lim_{r\to\infty}} \frac{N(D, r)}{T^{\neq}(r)} \leqslant k - l[1 - \varkappa_1]. \qquad \text{(by (7.30))}$$

Combining, we obtain the inequality

$$\sum_{j=1}^{k} \delta(D_j) \leqslant (k - l) + \varkappa$$

where $\varkappa = l\varkappa_1$. Since l is subject only to (7.32), we have proved our theorem except for the assertion that $\varkappa = 0$ if either $A = \mathbf{C}^m$ or f is transcendental.

Referring to Lemma (7.15), it is obvious that $\varkappa = \varkappa_1 = 0$ if $A = \mathbf{C}^m$, since in this case the branch divisor $B = 0$. If now $\varkappa = \varkappa_1 l > 0$, then by the proof of (7.15)

$$\overline{\lim_{r\to\infty}} \frac{\log r}{T^{\neq}(r)} = c > 0.$$

Using (7.30) this converts into

$$\overline{\lim_{r\to\infty}} \frac{\log r}{T(L, r)} = c_1 > 0. \tag{7.33}$$

Setting $v(L, r) = \int_{A[r]} \omega_f \wedge \psi_{m-1}$, by definition

$$T(L,r)=\int_0^r v(L,t)\frac{dt}{t}.$$

Since $v(L, r)$ is an increasing function of r, the order function $(T(L, r)=O(\log r)$ if, and only if, there is an estimate

$$v(L, r)\leqslant c_3 \tag{7.34}$$

for all r. If (7.34) does not hold, then given $\varepsilon>0$ we will have

$$v(L, r)>\frac{1}{\varepsilon}$$

for $r\geqslant r_0(\varepsilon)$. It follows that

$$T(L,r)>\frac{1}{\varepsilon}(\log r)-\frac{1}{\varepsilon}(\log r_0(\varepsilon))$$

for $r\geqslant r_0(\varepsilon)$. Thus

$$\frac{\log r}{T(L,r)}<\varepsilon\frac{\log r}{\log r-\log r_0},$$

from which it follows that

$$\overline{\lim_{r\to\infty}}\frac{\log r}{T(L,r)}\leqslant\varepsilon.$$

Comparing this with (7.33), we arrive at the statement:

$$\varkappa\neq 0\Rightarrow T(L, r)=O(\log r).$$

Using Proposition (5.9), it follows that f is rational. Q.E.D.

8. Some applications

(a) Holomorphic mappings into algebraic varieties of general type

Let V be a smooth projective variety. We recall that V is of general type if

$$\overline{\lim_{k\to\infty}}\frac{\dim H^0(V,K_V^k)}{k^n}>0,$$

where $K_V\to V$ is the canonical bundle of V. If K_V is positive, then V is of general type, but the converse is not quite true. Indeed, the concept of being of general type is birationally invariant, whereas the positivity of K_V is not. Special cases of the following result were given in [11] and [14].

(8.1) PROPOSITION. *Let A be an algebraic variety. Then any holomorphic mapping $f: A\to V$ whose image contains an open set is necessarily rational.*

Proof. Obviously it will suffice to assume that A is smooth and affine. Let $\pi: A \to \mathbf{C}^m$ be the generic projection constructed in § 2, and consider the volume form Ψ given in Proposition (6.16). Since f is of maximal rank, $f^*\Psi = \Psi_f$ is not identically zero on A, and we may choose coordinates on $\mathbf{C}^m$ such that

$$\Psi_f \wedge \pi^* \left\{ \prod_{j=m-n}^{m} \frac{\sqrt{-1}}{2\pi} (dz_j \wedge d\bar{z}_j) \right\} = \xi \Phi$$

where $\xi \geqslant 0$ is not identically zero. Using (6.11), the same proof as that of Lemma (6.20) gives the equation of currents on A

$$dd^c \log \xi = S + \frac{1}{k}(D_f) - B + \operatorname{Ric} \Psi_f. \tag{8.2}$$

The proof of the S.M.T. (6.23) may now be repeated to give, using the notations (6.22),

$$T^{*}(r) + N(S, r) + \frac{1}{k} N(D_f, r) = N(B, r) + \mu(r). \tag{8.3}$$

Using that $\operatorname{Ric} \Psi$ is C^∞ and positive definite on V, we set

$$T^{\#\#}(r) = \int_0^r \left\{ \int_{A[t]} \xi^{1/n} \Phi \right\} \frac{dt}{t^{2m-1}}$$

and, as in the proof of (7.19), have an estimate

$$cT^{\#\#}(r) \leqslant T^{\#}(r) \quad (c > 0). \tag{8.4}$$

Utilizing now the facts that $N(B, r) \leqslant c_2 \log r$ (cf. (4.1)) and

$$\mu(r) \leqslant n \log \frac{d^2 T^{\#\#}(r)}{ds^2} + c_3 \log r \qquad \text{(cf. (7.22))},$$

we obtain from (8.4) and (8.3) the inequality

$$T^{\#\#}(r) + \varepsilon N(D_f, r) \leqslant c_4 \log r + \log \frac{d^2 T^{\#\#}(r)}{ds^2}. \tag{8.5}$$

Proceeding in the same way as just below Lemma (7.23), (8.5) leads to

$$1 + \varepsilon \varliminf_{r\to\infty} \frac{N(D_f, r)}{T^{\#\#}(r)} \leqslant c_4 \varlimsup_{r\to\infty} \frac{\log r}{T^{\#\#}(r)}. \tag{8.6}$$

The R.H.S. of (8.6) gives that

$$T^{\#\#}(r) \leqslant c_5 \log r,$$

and using this the L.H.S. yields the estimate

$$N(D_f, r) \leqslant c_6 \log r.$$

By Proposition (4.1), all the divisors D_f are algebraic and of bounded degree. This implies that f is rational. Q.E.D.

(8.7) COROLLARY (Kodaira). *Let V_m be an algebraic variety of general type. Then any holomorphic mapping $f: \mathbf{C}^m \to V_n$ has everywhere rank less than $n = \dim_{\mathbf{C}} V$.*

(b) Generalizations of the Picard theorems

In one complex variable, the big Picard theorem implies the following global version: "Let A be an affine algebraic curve. Then any non-degenerate holomorphic mapping $f: A \to \mathbf{P}^1 - \{0, 1, \infty\}$ is rational. If $A = \mathbf{C}$, then no such mapping exists."

To give our generalization of this result, we assume that V is a smooth projective variety, $L \to V$ is a positive line bundle with

$$c(L) + c(K_V) > 0,$$

and that $D \in |L|$ is a divisor with simple normal crossings.

(8.8) PROPOSITION. *Let $f: A \to V - D$ be a holomorphic mapping from an algebraic variety A into V such that the image $f(A)$ contains an open set on V. Then f is rational, and if $A = \mathbf{C}^m$ no such mapping exists.*

Proof. Referring to Proposition (7.16), the counting function $N(D_f, r) \equiv 0$ since $f(A)$ misses D. Thus $\varkappa_1 > 0$ and so f is rational. Q.E.D.

Remark. This big Picard theorem will be proved in local form on the domain space A in the Appendix below. This alternate proof will only use Proposition (6.2), the Ahlfors lemma (cf. Proposition (2.7) in [11]), and elementary properties of currents and plurisubharmonic functions.

(c) Holomorphic mappings of finite order

Let V be a smooth, projective variety, $L \to V$ a positive line bundle with order function $T(L, r)$, and

$$f: A \to V$$

a holomorphic mapping of an affine variety A into V.

Definition. The holomorphic mapping f is of *finite order* if $T(L, r) = O(r^\lambda)$ for some $\lambda > 0$.

Remarks. From (7.2) we see that the maps of finite order have the following functorial properties:

(i) The definition is intrinsic (i.e., it is independent of the positive line bundle L and choice of metric in L); and

(ii) Given two maps $f: A \to V$ and $g: A \to W$, the product $f \times g: A \to V \times W$ is of finite order if, and only if, both of f and g are of finite order.

One importance of finite order maps is that these form the class of transcendental maps which turns up most naturally in the study of the analytic Grothendieck ring of an affine algebraic variety. Moreover, classically the finite order functions on $\mathbf{C}$ include most of those transcendental functions which appear in analysis and number theory.

In value distribution theory the maps of finite order have the very pleasant property that the exceptional intervals which appeared in the proof of Proposition (7.16) are no longer necessary.

(8.9) PROPOSITION. *Keeping the notations and assumptions of Theorem* (7.8), *we assume that f is of finite order. Then the F.M.T. and S.M.T. yield the following inequalities, valid for* all *large r,*

$$\begin{aligned}N(D_f, r) &\leq kT(L, r) + O(1)\\ T^{\#}(r) &\leq N(D_f, r) + O(\log r)\\ T^{\#}(r) &= kT(L, r) + T(K_V, r) + O(\log r)\end{aligned}$$

Remark. These inequalities again clearly illustrate just how the F.M.T. and S.M.T. act as upper and lower bounds respectively on the counting function $N(D_f, r)$.

Proof. The first inequality is just a restatement of (5.12), and the third one follows from (7.30) and the finite order assumption

$$T(L, r) = O(r^{\lambda}). \tag{8.10}$$

For the remaining inequality, we will utilize the S.M.T. (6.23)

$$T^{\#}(r) + N(S, r) = N(B, r) + N(D_f, r) + \mu(r). \tag{8.11}$$

Using (8.11) and (4.1), our proposition follows from

(8.12) LEMMA. *The term $\mu(r)$ in* (8.11) *satisfies, for* all *large r,*

$$\mu(r) = O(\log r).$$

Proof. Referring to (7.22) and (7.19) we obtain

$$
(8.13) \qquad \begin{cases} \mu(r) \leqslant n \log \dfrac{d^2 T^{\#\#}(r)}{ds^2} + O(\log r) \\ T^{\#\#}(r) \leqslant T^{\#}(r). \end{cases}
$$

By the argument following Lemma (7.23), given any $\mu_0 > 0$, the inequalities (8.13) lead to the estimate

$$
(8.14) \qquad \mu(r) \leqslant c \log T^{\#}(r) + O(\log r) \qquad \|_g
$$

where the exceptional intervals I satisfy $\int_I dr^{\mu_0} < \infty$.

Using (8.10) we will be done if we can show that (8.14) holds for *all* large r.

Choose $\mu_0 > \lambda$ where λ appears in the estimate (8.10). Setting

$$
n(D_f, r) = \int_{D_f[r]} \psi_{m-1}
$$

the usual integration by parts formula for $N(D_f, r)$ and $n(D_f, r)$ ([16], page 217) gives $n(D_f, r) = O(r^\lambda)$.

It follows that
$$
\int_I \frac{n(D_f, r)\, dr}{r} \leqslant d \int_I d(r^{\mu_0}) < \infty.
$$

Let $r_1 < r < r_2$ be a component of the exceptional set I. Then by (8.11) and (4.1),

$$
\begin{aligned}
\mu(r) &= T^{\#}(r) + N(S, r) - N(B, r) - N(D_f, r) \leqslant T^{\#}(r_2) + N(S, r_2) - N(D_f, r_1) + O(\log r) \\
&\leqslant \mu(r_2) + N(D_f, r_2) - N(D_f, r_1) + O(\log r) = O(\log r_2) + \int_{r_1}^{r_2} \frac{n(D_f, r)\, dr}{r} = O(\log r_2) + O(1).
\end{aligned}
$$

Furthermore, $\log r_2 = \log r + \int_r^{r_2} (dr/r) < \log r + O(1)$, and it follows that

$$
\mu(r) = O(\log r) + O(1). \qquad \text{Q.E.D.}
$$

(d) Sharpness of results

In the case of a holomorphic mapping

$$
A \xrightarrow{f} V
$$

where $\dim_{\mathbf{C}} A = 1 = \dim_{\mathbf{C}} V$, the defect relation (7.8) and its applications, such as Proposition (8.8), are well-known to be sharp. In the case where $\dim_{\mathbf{C}} V > 1$, the conditions on the divisor D in which we are interested are

$$
(8.15) \qquad \begin{gathered} c(D) + c(K_V) > 0, \text{ and} \\ D \text{ has simple normal crossings.} \end{gathered}
$$

The question arises as to whether the conditions (8.15) are sharp. There is some evidence that this is so, but it is by no means proved.

To present this evidence, let $V = \mathbf{P}^2$ and $D = L_1 + \dots + L_k$ be a sum of lines. We ask whether a holomorphic mapping

$$f: \mathbf{C}^2 \to \mathbf{P}^2 - D$$

is necessarily degenerate. If $k \leqslant 3$, then $c(D) + c(K_{\mathbf{P}^2}) \leqslant 0$, and there are non-degenerate rational mappings if $k \leqslant 2$ and non-degenerate transcendental mappings if $k \leqslant 3$. For example, if $k = 3$ and D has normal crossings, then

$$\mathbf{P}^2 - D \cong \mathbf{C}^* \times \mathbf{C}^*.$$

Suppose now that $k = 4$ but $D = L_1 + L_2 + L_3 + L_4$ does not have normal crossings; for example, we may assume that L_1, L_2, L_3 all pass through a point. Taking L_4 to be the line at infinity, it follows that

$$\mathbf{P}^2 - D \cong \mathbf{P}^1 - \{0, 1, \infty\} \times \mathbf{C}^*$$

Then any map $\mathbf{C}^2 \xrightarrow{f} \mathbf{P}^2 - D$ is degenerate, but taking $A = \mathbf{P}^1 - \{0, 1, \infty\} \times \mathbf{C}$, the mapping

$$A \xrightarrow{f} \mathbf{P}^2 - D, \qquad f(z, w) = (z, e^w)$$

is transcendental and so the big Picard theorem (8.8) fails.

In general, suppose that M is a (possibly non-compact) complex manifold of dimension n having a C^∞ volume form Ω. Let $\varrho = (\varrho_1, \dots, \varrho_n)$ and $P(\varrho)$ be the polycylinder

$$P(\varrho) = \{z = (z_1, \dots, z_n) \in \mathbf{C}^n \colon |z_j| \leqslant \varrho_j\}.$$

We denote by $\Phi = \prod_{j=1}^n \{\frac{1}{2}\pi^{-1}\sqrt{-1}\,(dz_j \wedge d\bar{z}_j)\}$ the standard volume form on $\mathbf{C}^n$, and say that M has the *Schottky-Landau property* if for any normalized holomorphic mapping

$$f: P(\varrho) \to M, \qquad (f^*\Omega)\,(0) \geqslant \Phi(0),$$

it follows that the product of the radii

$$\varrho_1 \cdots \varrho_n \leqslant c < \infty.$$

(8.16) PROPOSITION. *If* $\operatorname{Ric}\Omega > 0$ *and* $(\operatorname{Ric}\Omega)^n \geqslant \Omega$, *then* M *satisfies the Schottky-Landau property.*

Proof. This follows from the Ahlfors lemma; cf. Proposition (2.7) in [11]. Equivalently, (8.16) may be proved using the S.M.T. as was done in § 6(a) of [6]. Q.E.D.

Suppose now that V is a projective variety, D is a divisor on V, and $M = V - D$. Then M satisfies the Schottky-Landau property if the conditions 8.15 are met. Conversely, in case $V = \mathbf{P}^n$ and D is a sum of hyperplanes, then if $V - D$ satisfies the Schottky-Landau property, the conditions 8.15 are met for some divisor $\tilde{D} \leqslant D$ (we verified the case $n = 2$ above).

In general, the converse question is quite interesting, even in the case where D is empty. To state the question which arises here, we first remark that (8.16) is too strong in order that a smooth, projective variety V satisfy the Schottky-Landau property. Indeed, it follows from (6.16) that V satisfies this property if it is of general type (cf. [14] for details).

(8.17) *Question*. If V satisfies the Schottky-Landau property, then is V of general type?

Remark. For V a curve, this question is obviously O.K. For V a surface, it can be verified with the possible exception of $K3$ surfaces. One does this by checking the classification of surfaces, where only the elliptic case is nontrivial.

In general, the problem in verifying (8.17) is the absence of a uniformization theorem for $\dim_{\mathbf{C}} V > 1$, so that there is no obvious way of constructing holomorphic mappings to V.

9. Two further variations on curvature and the second main theorem

(a) An analogue of R. Nevanlinna's "lemma on the logarithmic derivative"

All of the results in §§ 7 and 8 were based on having available a volume form Ψ on $V - D$ satisfying the three conditions in (6.4). The middle inequality there may be thought of as being "negative curvature bounded away from zero" (cf. the discussion following (0.6)), and the point we wish to make here is that it is sometimes possible to relax this condition to "the curvature is negative, but may tend to zero as we approach D". When this method applies, it seems likely to yield somewhat more delicate estimates than the previous case.

Let V be a smooth, projective variety whose anti-canonical bundle $K_V^* \to V$ is ample. We consider a meromorphic n-form Λ on V which does not have zeroes and whose polar divisor D has simple normal crossings.

Example. Let $V = \mathbf{P}^n$ with affine coordinates $(w_1, \ldots, w_n)$ and homogeneous coordinates $[\xi_0, \ldots, \xi_n]$. Then the rational n-form

$$\Lambda = \frac{\sum_{\alpha=0}^{n} (-1)^\alpha \xi_\alpha d\xi_0 \wedge \ldots \wedge d\hat{\xi}_\alpha \ldots d\xi_n)}{\xi_0 \ldots \xi_n} = \frac{dw_1 \wedge \ldots \wedge dw_n}{w_1 \ldots w_n} \tag{9.1}$$

satisfies our requirements.

Suppose that $f: \mathbf{C}^n \to V$ is a transcendental, non-degenerate, equidimensional, holomorphic mapping. Then $f^*\Lambda = \Lambda_f$ is a non-identically zero meromorphic n-form on $\mathbf{C}^n$, and we set

$$\Lambda_f = \zeta dz_1 \wedge \ldots \wedge dz_n, \qquad \nu_f(r) = \int_{\partial \mathbf{C}^n[r]} \log^+ |\zeta| \eta \tag{9.2}$$

Denote by $T_1(r) = T(K_V^*, r)$ the order function (7.1) for the anti-canonical bundle.

(9.3) Proposition. *We have the estimate*

$$\varliminf_{r\to\infty} \frac{\nu_f(r)}{T_1(r)} = 0.$$

Remark. To see better what this proposition amounts to, consider the classical case of an entire transcendental meromorphic function $w = f(z)$. Taking Λ to be given by (9.1) in the case $n = 1$, we have from (9.2) that

$$\nu_f(r) = \int_{|z|=r} \log^+ \left| \frac{f'(z)}{f(z)} \right| d\theta \qquad (z = re^{i\theta}).$$

Denoting the order function of f simply by $T(r)$, Proposition (9.3) becomes

$$\varliminf_{r\to\infty} \frac{\int_{|z|=r} \log^+ \left| \frac{f'(z)}{f(z)} \right| d\theta}{T(r)} = 0. \tag{9.4}$$

This result is a weak form of R. Nevanlinna's "lemma on the logarithmic derivative", given on [16], pages 241–247. We recall that Nevanlinna proved the stronger estimate, valid for any $f(z)$ which need not be transcendental,

$$\int_{|z|=r} \log^+ \left| \frac{f'}{f} \right| d\theta = O(\log r + \log T(r)) \qquad \|_\sigma \tag{9.5}$$

and it it possible that our method might be refined to give (9.5) (cf. [16], page 259). At any event, (9.4) is sufficient to deduce R. Nevanlinna's defect relation from his rather elementary Second Main Theorem given on page 240 of [16].

Proof. Let $\sigma \in H^0(V, K_V^*)$ be a holomorphic section which defines the polar divisor D of Λ, and take a C_∞ volume form Ω on V such that $\operatorname{Ric}\Omega = c_1(K_V) = dd^c \log|\sigma|^2$. As usual we may assume that $\|\sigma\|_V < \delta_0$ for any given $\delta_0 > 0$. We consider the singular volume form

$$\Psi_\varepsilon = \frac{\Omega}{(\log|\sigma|^2)^2 \|\sigma\|^{2+2\varepsilon}}$$

given by (6.10). Writing $f^*\Psi_\varepsilon = \xi_\varepsilon \Phi$, it follows directly that

$$\log^+|\zeta| \leqslant \log^+|\xi_\varepsilon| + \varepsilon \log \frac{1}{|\sigma|^2} + \log(\log|\sigma|^2)^2. \tag{9.6}$$

Setting $\mu_\varepsilon(r) = (1/n) \int_{\partial \mathbf{C}^n[r]} \log^+|\xi_\varepsilon| \eta$ and recalling (5.10), it follows by integrating (9.6) and using concavity of the logarithm that

$$2\nu(r) \leqslant n\mu_\varepsilon(r) + \varepsilon m(D, r) + 2\log[m(D, r)] + O(1).$$

Using (5.11) we obtain $$\varliminf_{r\to\infty} \frac{\nu(r)}{T_1(r)} \leqslant \binom{n}{2} \varliminf_{r\to\infty} \frac{\mu_\varepsilon(r)}{T_1(r)} + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, our proposition will follow from the estimate

$$\varliminf_{r\to\infty} \frac{\mu_\varepsilon(r)}{T_1(r)} \leqslant \varepsilon. \tag{9.7}$$

We will prove (9.7) by deriving a S.M.T. for the volume form Ψ_ε. Referring to (6.9), the function $\log \xi_\varepsilon$ is locally L^1 and we have the equation of currents

$$dd^c \log \xi_\varepsilon = R - (1+\varepsilon) D_f + \operatorname{Ric} \Psi_\varepsilon \tag{9.8}$$

where R is the ramification divisor of f. Integrating (9.8) twice as in the proof of (6.23) leads to the relation

$$\int_0^r \left\{ \int_{\mathbf{C}^m[t]} \operatorname{Ric} f^*\Psi_\varepsilon \wedge \varphi_{n-1} \right\} \frac{dt}{t^{2m-1}} + N_1(r) = (1+\varepsilon) N(D_f, r) + \int_{\partial \mathbf{C}^m[r]} \log \xi_\varepsilon \eta. \tag{9.9}$$

From (9.9) and the F.M.T. we deduce the inequality

$$\int_0^r \left\{ \int_{\mathbf{C}^n[t]} \operatorname{Ric} f^*\Psi_\varepsilon \wedge \varphi_{n-1} \right\} \frac{dt}{t^{2n-1}} \leqslant cT_1(r) + n\mu_\varepsilon(r). \tag{9.10}$$

Using Proposition (6.10) and the same reasoning as in the proof of (7.19), the estimate (9.10) leads to

$$\int_0^r \left\{ \int_{\mathbf{C}^n[t]} \xi_\varepsilon^{1/n} |\sigma|^{2\varepsilon/n} \Phi \right\} \frac{dt}{t^{2n-1}} \leqslant c_1 T_1(r) + \mu_\varepsilon(r). \tag{9.11}$$

Because of $\alpha \geqslant e^{\log^+\alpha} - 1$, $(\alpha \geqslant 0)$ and $\log|\sigma| \leqslant 0$,

$$\exp\left(\frac{1}{n}\log^+ \xi_\varepsilon + \frac{\varepsilon}{n}\log|\sigma|^2\right) \leqslant \xi_\varepsilon^{1/n}|\sigma|^{2\varepsilon/n} + 1.$$

Plugging this into (9.11) and interating the integral, we arrive at the inequality

$$(9.12)\qquad \int_0^r\left\{\int_0^t\left(\int_{\partial C^n[s]}\exp\left(\frac{1}{n}\log^+\xi_\varepsilon+\frac{\varepsilon}{n}\log|\sigma|^2\right)\eta\right)s^{2n-1}ds\right\}\frac{dt}{t^{2n-1}}\leqslant c_1T_1(r)+c_2r^2+\mu_\varepsilon(r).$$

To eliminate the integrals in (9.12), we refer to the calculus Lemma (7.23), and taking $g(r)=r$ and $\alpha(r)= r^{1+\lambda}(\lambda>0)$, we have

$$(9.13)\qquad f'(r)\leqslant[f(r)]^{1+\lambda}\qquad \|$$

Applying (9.13) when $f(r)$ is the L.H.S. of (9.12), we obtain

$$(9.14)\qquad \int_0^r\left(\int_{\partial C^n[t]}\exp\left(\frac{1}{n}\log^+\xi_\varepsilon+\frac{\varepsilon}{n}\log|\sigma|^2\right)\eta\right)t^{2n-1}dt\leqslant r^{2n-1}(c_1T_1(r)+c_2r^2+\mu_\varepsilon(r))\qquad \|$$

Utilizing (9.13) once more where $f(r)$ is again the L.H.S. of (9.14) yields the estimate

$$(9.15)\qquad \int_{\partial C^n[r]}\exp\left(\frac{1}{n}\log^+\xi_\varepsilon+\frac{\varepsilon}{n}\log|\sigma|^2\right)\eta\leqslant r^{\delta_1}(c_1T_1(r)+c_2r^2+\mu_\varepsilon(r))^{1+\delta_2}$$

where δ_1 and δ_2 may be made as small as we wish. Now using concavity of the logarithm on the L.H.S. of (9.15) together with $\log^+(\alpha+\beta)\leqslant\log^+\alpha+\log^+\beta+\log 2$ gives

$$(9.16)\qquad \mu_\varepsilon(r)\leqslant\frac{\varepsilon}{n}m(D,r)+c_3\log T_1(r)+c_4\log r+c_5\log\mu_\varepsilon(r).$$

Dividing by $T_1(r)$ and using that f is transcendental, so that $\varliminf_{r\to\infty}\log r/T_1(r)=0$, we obtain from (9.16) that

$$\varlimsup_{r\to\infty}\frac{\mu_\varepsilon(r)}{T_1(r)}\leqslant\frac{\varepsilon}{n}.\qquad \text{Q.E.D.}$$

Remark. The above proof gives the estimate

$$(9.17)\qquad \mu_\varepsilon(r)\leqslant\left(\frac{\varepsilon}{n}\right)T_1(r)+O(\log T_1(r))+O(\log r)\qquad \|_\varepsilon$$

where the exceptional intervals depend on ε. Evidently, this is not as strong as (9.5).

(b) Holomorphic mappings into negatively curved algebraic varieties

Thus far our applications of the S.M.T. have been restricted to holomorphic mappings $f: A\to V$ where the image $f(A)$ contains an open set on V. This method is also applicable to other situations, and as an illustration we shall prove a variant of the recent theorem of Kwack.

We begin with two definitions:

Definition. A complex manifold V is *negatively curved* if there exists an Hermitian metric ds_V^2 on V all of whose *holomorphic sectional curvatures* K satisfy $K \leqslant -c < 0$.

The following lemma is standard (cf. [11] for further discussion and references).

(9.18) LEMMA. *Suppose that V is negatively curved, that S is a complex manifold, and $S \xrightarrow{f} V$ is a holomorphic immersion. Then the holomorphic sectional curvatures of the induced metric $f^*ds_V^2$) are less than or equal those of ds_V^2. In particular, S is negatively curved.*

We denote by ω_V the (1, 1) form associated to ds_V^2.

Definition. Suppose that V is a quasi-projective negatively curved complex manifold. An ample line bundle $L \to V$ is said to be *bounded* if there exist a metric in L and sections $\sigma_0, \ldots, \sigma_N \in H^0(V, L)$ such that (i) the curvature $c(L)$ of the metric satisfies

$$0 < c(L) < A\omega_V \qquad (A = \text{constant});$$

(ii) the sections $\sigma_0, \ldots, \sigma_N$ have bounded length and $[\sigma_0, \ldots, \sigma_N]: V \hookrightarrow \mathbf{P}^N$ induces an algebraic embedding of V.

Example 1. If V is projective (thus compact) and negatively curved, then any ample line bundle is bounded. (*Remark.* It does not seem to be known whether a negatively curved compact manifold is necessarily projective.)

Example 2. Suppose that X is a *bounded symmetric domain* and Γ is an *arithmetic subgroup* of the automorphism group of X. In general, Γ may not act freely on X, but a subgroup of finite index will act without fixed points, and we lose no essential generality in assuming that this is true for X. It is well known that $V = X/\Gamma$ is negatively curved, since in fact the *Bergman metric* on X has negative holomorphic sectional curvatures $\leqslant -c < 0$ and is Γ-invariant. It is a basic theorem of Baily-Borel [2] that V is quasiprojective.

(9.19) LEMMA. *There exists an ample, bounded line bundle on V.*

Proof. Let $K \to X$ be the canonical line bundle with unique (up to a constant) metric invariant under the automorphism group of X. For this metric the curvature form

$$c(K) = \omega_X$$

is the (1, 1) form associated to ds_X^2. Thus, it will suffice to show that for sufficiently large μ, there are Γ-invariant holomorphic sections $\sigma_0, \ldots, \sigma_N$ of $K^\mu \to X$ which have bounded length and which induce a projective embedding of X/Γ. Such sections σ are generally termed *automorphic forms of weight* μ for Γ, and among these automorphic forms are the *cusp forms* which, so to speak, "vanish at infinity" on X/Γ [2]. These cusp forms have bounded length, and by the results in [2] will induce a projective embedding of X/Γ for large μ. Q.E.D.

(9.20) PROPOSITION. *Suppose that V is a quasi-projective, negatively curved complex manifold having a bounded ample line bundle $L\to V$. Then any holomorphic mapping $f: A\to V$ from an algebraic variety A into V is rational.*

Remark. This big Picard theorem will be proved in local form in the appendix below (the following proof may also be localized).

(9.21) COROLLARY (Kwack). *In case V is negatively curved and projective, any holomorphic mapping $A\xrightarrow{f} V$ is rational.*

(9.22) COROLLARY [5]. *In case $V=X/\Gamma$ is the quotient of a bounded, symmetric domain by an arithmetic group, any holomorphic mapping $A\xrightarrow{f} X/\Gamma$ is rational.*

Proof: Obviously we may assume that A is affine. Let σ be a section of L having bounded length and divisor D. We must show that the divisors

$$D_f=f^{-1}(D)$$

are algebraic and of uniformly bounded degree on A. Simple considerations of the algebraic curves lying in A show that for this it will suffice to prove that

$$\deg (D_f)\leqslant c<\infty$$

in case A is itself an affine curve.

Thus let $A\subset \mathbf{C}^N$ be an affine algebraic curve with harmonic exhaustion function

$$\tau(x)=\log|\pi(x)|$$

where $A\xrightarrow{\pi}\mathbf{C}$ is a generic projection (cf. § 2). We want to prove an estimate

$$N(D_f, r)=O(\log r) \tag{9.23}$$

for the counting function associated to D_f and with a uniform "O". We may assume that $|\sigma(z)|\leqslant 1$ for all $z\in V$ and set

$$\left\{\begin{aligned} &\varphi=dd^c|\pi(x)|^2,\\ &\theta_\varepsilon=|\sigma_f|^{2\varepsilon}w_f=\xi_\varepsilon\varphi\\ &T(r)=\int_0^r\left(\int_{A[t]}\omega_f\right)\frac{dt}{t} \qquad \text{(order function)}\end{aligned}\right. \tag{9.24}$$

where $\omega_f=f^*\omega_V$ and $\sigma_f=f^*(\sigma)\in H^0(A, f^*L)$. It follows from Lemma (9.18) and the definition of $L\to V$ being bounded that

$$\mathrm{Ric}\,\theta_\varepsilon = -\varepsilon f^* c(L) + \mathrm{Ric}\,\omega_f \geq c_1\omega_f \qquad (c_1 > 0) \tag{9.25}$$

provided we choose ε sufficiently small. Combining (9.24) and (9.25) we have the equation of currents on A (cf. § 6(b))

$$dd^c \log \xi_\varepsilon = R + \varepsilon D_f - B + \mathrm{Ric}\,\theta_\varepsilon \geq R + \varepsilon D_f - B + c_1\omega_f \tag{9.26}$$

where R is the ramification divisor of f and B is the branch locus of $A \xrightarrow{\pi} \mathbf{C}$. Setting

$$\omega_f = \xi \circ \varphi \geq \xi_\varepsilon \varphi \quad (\text{since } |\sigma_f| \leq 1) \qquad \mu(r) = \int_{\partial A[r]} \log \xi\, d^c\tau,$$

we may integrate (9.26) twice to obtain the estimate

$$c_1 T(r) + N(R, r) + \varepsilon N(D_f, r) \leq N(B, r) + \mu(r). \tag{9.27}$$

Now $N(B, r) \leq d \log r$ where d is the number of branch points of $A \xrightarrow{\pi} \mathbf{C}$, and by (7.22)

$$\mu(r) \leq \log \frac{d^2 T(r)}{ds^2} \qquad \left(\frac{d}{ds} = r\frac{d}{dr}\right).$$

Using these two inequalities in (9.27) gives

$$T(r) + \left(\frac{\varepsilon}{c_1}\right) N(D_f, r) \leq c_2 \log r + c_3 \log \frac{d^2 T(r)}{dr^2}. \tag{9.28}$$

Dividing by $T(r)$ and taking limits in (9.28) leads, as in § 7 (b), to

$$1 + \frac{\varepsilon}{c_1} \varliminf_{r\to\infty} \frac{N(D_f, r)}{T(r)} \leq c_3 \varlimsup_{r\to\infty} \frac{\log r}{T(r)}, \tag{9.29}$$

From the R.H.S. of (9.29) we obtain

$$T(r) \leq c_4 \log r,$$

and then using this the L.H.S. gives

$$N(D_f, r) \leq c_5 \log r.$$

This is the desired estimate (9.23). Q.E.D.

Remark. The original version of Kwack's theorem goes as follows: Suppose that $\bar{V}$ is a compact analytic space which contains the complex manifold V as the complement of a subvariety S. Suppose that V is negatively curved and let $d_V(p, q)$ be the distance from p to q using the ds_V^2 on V. We assume the following condition:

$$\begin{gathered}\text{“If } \{p_n\}, \{q_n\} \in V \text{ and } p_n \to p, q_n \to q \text{ where} \\ p, q \in \bar{V} \text{ and } d_V(p_n, q_n) \to 0, \text{ then } p = q.\text{”}\end{gathered} \tag{9.30}$$

Then any holomorphic map $f\colon D^* \to V$ from the punctured disc $D^* = \{0 < |t| < 1\}$ into V extends across $t = 0$ to give $\bar{f}\colon D \to \bar{V}$ where $D = \{|t| < 1\}$.

Our Nevanlinna-theoretic proof applies to give an analogue of the above result. To state this, we assume that $\bar{V}$ is projective (it may be singular) and has a Kähler metric $\varphi_{\bar{V}}$. Let ω_V be the (1, 1) form associated to a negatively curved ds_V^2 on V, and let $K_V(\xi)$ be the holomorphic sectional curvature for the (1, 0) vector $\xi \in T'(V)$. Assume the following condition:

$$cK_V(\xi)\omega_V(\xi) \leq -\varphi_{\bar{V}}(\xi) \qquad (\xi \in T'(V),\ c > 0). \tag{9.31}$$

(In particular, this is satisfied if

$$\varphi_{\bar{V}}(\xi) \leq c\omega_V(\xi) \qquad (\xi \in T'(V)), \tag{9.32}$$

which is a sort of analogue to (9.30.)

(9.33) PROPOSITION. *Under condition* (9.31), *any holomorphic mapping* $A \xrightarrow{f} V$ *is necessarily rational.*

We do not know whether (9.31) is automatically satisfied in case ds_V^2 is complete.

Appendix

Proof of the big Picard theorems in local form

Let M be a connected complex manifold, $S \subset M$ an analytic subset, and

$$f\colon M - S \to W \tag{A.1}$$

a holomorphic mapping into a quasi-projective variety W. We say that f extends to a *meromorphic mapping* $M \xrightarrow{f} W$ if the pull-back $f^*(\varphi)$ of every rational function φ on W extends meromorphically across S.

(A.2) PROPOSITION. *Suppose that* (i) $W = V - D$ *where* V *is a smooth, projective variety and* D *is a divisor with simple normal crossings satisfying* $c(K_V^*) + c(D) > 0$, *and* (ii) *that the image* $f(M - S)$ *contains an open subset of* W. *Then any holomorphic mapping* f *is meromorphic.*

Remarks. (i) Since an affine algebraic variety

$$A = \bar{A} - S$$

where $\bar{A}$ is a smooth, projective variety and $S \subset \bar{A}$ is a divisor, and since a meromorphic function ζ on A extends meromorphically across $S \Leftrightarrow \zeta$ is rational for the algebraic structure on A, it follows the Proposition (A.2) implies Proposition (8.8.) (ii) Our proof of (A.2) will apply equally well to the situation of § 9(b) to yield the following result.

(A.3) PROPOSITION. *Let V be a quasi-projective, negatively curved complex manifold having a bounded ample line bundle $L \to V$ (cf. § 9(b)). Then any holomorphic mapping (A.1) extends meromorphically across S.*

In particular this will give Borel's theorem (9.22) in its original form.

Proof. We will assume for simplicity that $\dim_{\mathbf{C}} M = \dim_{\mathbf{C}} V$, so that the Jacobian determinant of f is not identically zero. Since every meromorphic function defined outside an analytic set of codimension at least two automatically extends (theorem of E. Levi, we may assume that S is a smooth hypersurface in M. Localizing around a point $x \in S$, we may finally assume that

$$M = \{(z_1, \ldots, z_n): |z_i| < 1\}; \; S = \{z_1 = 0\}; \quad \text{so that } M - S \text{ is a } \textit{punctured polycylinder}.$$

We let $P \xrightarrow{\pi} M - S$ be the universal covering of $M - S$ by the usual polycylinder P, and denote by Θ the volume form on $M - S$ induced by the *Poincaré metric* on P (cf. [11]). Explicitly, there is the formula

$$\Theta = \frac{c\, dz_1 d\bar{z}_1}{|z_1|^2 (\log |z_1|^2)^2} \left\{ \prod_{j=2}^{n} \frac{dz_j d\bar{z}_j}{(1 - |z_j|^2)^2} \right\} \tag{A.4}$$

(cf. Lemma 3.4 on page 447 of [11]).

Let $L \to V$ be an ample line bundle. For each divisor $E \in |L|$, we set $E_f = f^{-1}(E)$ considered as a divisor on $M - S$. It will suffice to show that every such E_f extends to a divisor $E_{\bar{f}}$ on $M(E_{\bar{f}} = \text{closure of } E_f \text{ in } M)$. This is because a meromorphic function ψ on $M - S$ extends to a meromorphic function on $M \Leftrightarrow$ each level set $\psi = a$ extends as a divisor to M, as is easily seen from the ordinary *Riemann extension theorem.*

Consider the volume form Ψ on $V - D$ given by (6.3). Choose a metric in $L \to V$ and let $\sigma \in H^0(V, L)$ be the section whose divisor is $E \in |L|$. Define the new volume form

$$\Psi_\varepsilon = |\sigma|^{2\varepsilon} \Psi \quad (\varepsilon > 0). \tag{A.5}$$

From the relation
$$\operatorname{Ric} \Psi_\varepsilon = \varepsilon c(L) + \operatorname{Ric} \Psi$$

on $M - D$, we see that, after choosing ε sufficiently small and adjusting constants, we may assume

$$\{\operatorname{Ric} \Psi_\varepsilon > 0, \qquad (\operatorname{Ric} \Psi_\varepsilon)^n \geqslant \Psi_\varepsilon. \tag{A.6}$$

It follows from the *Ahlfors' lemma* (Proposition 2.7 in [11]) that

$$f^* \Psi_\varepsilon \leqslant \Theta. \tag{A.7}$$

Now let Φ be the Euclidean volume form on $M \subset \mathbf{C}^n$ and set

$$f^*\Psi_\varepsilon = \zeta_\varepsilon \Phi. \tag{A.8}$$

On $M-S$ we have the equation of currents (cf. (6.16))

$$R+\varepsilon E_f + f^*(\operatorname{Ric}\Psi_\varepsilon) = dd^c \log \zeta_\varepsilon. \tag{A.9}$$

This gives the basic inequality

$$E_f \leqslant \left(\frac{1}{\varepsilon}\right) dd^c \log \zeta_\varepsilon \tag{A.10}$$

between the positive currents E_f and $dd^c \log \zeta_\varepsilon$ on $M-S$. Taking into account the Ahlfors' lemma (A.7) and explicit formula for Θ given by (A.4), we have

$$0 \leqslant \zeta_\varepsilon \leqslant \frac{c}{|z_1|^2 (\log |z_1|^2)^2} \left\{ \prod_{j=2}^{n} \frac{1}{(1-|z_j|^2)^2} \right\}. \tag{A.11}$$

It follows from (A.11) that, given $x \in S$, there is a neighborhood U of x in M and $\delta > 0$ such that in U the function

$$\mu_{\delta,\varepsilon} = \log \zeta_\varepsilon + (1+\delta) \log |z_1|^2 \tag{A.12}$$

is everywhere plurisubharmonic, including on $U \cap S$ where it is $-\infty$. Using [4] we may solve the equation

$$\int_U |u|^2 e^{-N \log \mu_{\delta,\varepsilon}} \Phi < \infty$$

for holomorphic functions $u \in \mathcal{O}(U)$, $u \not\equiv 0$, and N sufficiently large. Taking into account (A.5), (A.9), and (A.12) we see that

$$R \cup E_f \cup S \subset \{u = 0\}.$$

It follows that $\bar{E}_f \cap U$ is an analytic divisor in U. Q.E.D.

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