New Algorithms for Minimum Area k-gons

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Abstract

Given a set P of n points in the plane, we wish to find a set $Q \subset P$ of k points for which the convex hull $conv(Q)$ has the minimum area. We solve this, and the related problem of finding a minimum area convex k-gon, in time $O(n^2 \log n)$ and space $O(n \log n)$ for fixed k, almost matching known bounds for the minimum area triangle problem. Our algorithm is based on finding a certain number of *nearest vertical neighbors* to each line segment determined by two input points. We use a classical result of Ramsey theory to prove that these nearest neighbors suffice to determine the minimum convex k-gon.

1 Introduction

One of the initial results of Ramsey theory [12] was the discovery that for every k there is some $n = f(k) = 2^{O(k)}$, so that if n points are given in general position, a subset of k points can be found forming the vertices of a convex k-gon. This does not work for empty convex polygons: arbitrarily large point sets are known that do not contain an empty 7-gon [13, 14].

This naturally raises the question whether such a k-gon can be computed efficiently; several papers study this problem [3, 6, 14]. Because of Ramsey theory, finding a convex k-gon takes time depending on k but not on n; therefore it can be solved in constant time for fixed k. The best known algorithm for empty convex k-gons takes time $O(T(n))$ where $T(n)$ is the number of empty triangles in the set, which varies between $O(n^2)$ and $O(n^3)$.

Here we study the related *geometric optimization* problems of finding a k-gon minimizing or maximizing a certain objective function. A celebrated result in this area is that a minimum area triangle can be found in time $O(n^2)$ by using geometric duality to transform the problem into one of searching a line arrangement [7, 8]. Algorithms are also known for optimizing other functions including minimum perimeter [1, 5, 9] and maximum perimeter and area [2, 4].

For some time it remained open whether the minimum area triangle result could be generalized to finding minimum area k-gons. There are actually four reasonable ways of generalizing this: one could search for (1) a minimum area k-gon, (2) a minimum area *convex* k-gon, (3) a minimum area *empty* convex k-gon, or (4) a minimum area polygon that is the convex hull of k points. All of these problems can be solved trivially in $O(kn^k)$ time, but this is not very satisfactory. Problem 1 remains open (except for $k = 4$, for which it can be solved in $O(n^2)$ time using the same methods as in the minimum triangle problem), but in a recent breakthrough by Eppstein et al. [9], $O(kn^3)$ time algorithms were developed for problems 2, 3, and 4. However these algorithms are a factor of $O(n)$ away from the time for the triangle problem.

In this paper we solve the minimum area convex k-gon problem in time $O(n^2 \log n + 2^{6k} n^2)$, which for fixed k is an improvement by almost a factor of n over the previous algorithm and is only a factor of $O(\log n)$ away from the minimum triangle algorithm. We use Ramsey theory to prove that the minimum k-gon can be found in a small set of points, the *nearest vertical neighbors* of a segment determined by two of the input points. The k-gon can then be found by applying the algorithm of Eppstein et al. to these small

sets. This is similar to the approach of Aggarwal et al. [1], who find small sets for minimum perimeter problems using high-order Voronoi diagrams.

We also solve the related problem of finding a set of k points with minimum area convex hull, in time $O(n^2 \log n + k^3 n^2)$. We again use nearest vertical neighbors, but fewer of them, and the proof no longer needs Ramsey theory. It is a curious fact that, for the algorithms presented here, the minimum k-point set problem is easier than the minimum k-gon, whereas in the algorithms of Eppstein et al. the difficulty of the problems was reversed (the minimum k-point set used an extra $O(k)$ factor in space).

Unfortunately our methods do not suffice to solve the third problem treated by Eppstein et al., finding a minimum area *empty* convex k-gon, except for the special cases $k = 4$ and $k = 5$. This happens because of the lack of an appropriate Ramsey theorem, due to the counterexamples described by Horton [13].

2 Nearest vertical neighbors

We begin with the problem of *nearest vertical neighbors* for points and line segments; we use this as a subroutine in our minimum k-gon algorithm.

Given a point x and a non-vertical line l, the *vertical distance* $d(x, l)$ is simply the length of a vertical line segment connecting x and l. The *nearest vertical neighbor* to l from a point set P is the point $x \in P$ minimizing the vertical distance to l.

The connection between this concept and minimum area polygons is as follows. If a triangle is formed by connecting point x to the endpoints of a line segment s, where s is contained in line l, the area of the triangle is $c \cdot d(x, l)$, where c is half the length of the horizontal projection of s. Therefore the point in P forming the minimum area triangle with s is the nearest vertical neighbor of l. This observation was used to develop $O(n^2)$ algorithms for the minimum triangle problem [7, 8].

We can tighten this characterization as follows. Let xyz be the minimum area triangle, and assume that the horizontal projection of y is between those of x and z. Then as before y is the nearest neighbor of line xz, but the vertical segment connecting y and line xz actually touches segment xz. In other words, y is within the *slab* defined by vertical lines through x and z. In general we say x is a neighbor of segment s if it appears vertically above or below s, as in this case y appears above or below segment xz. Then the triangle problem can be solved by finding, for each segment xz, the

slab of the points; for each such structure there are two smaller structures corresponding to slabs containing half as many points.

Then for each query segment s, we can find a set of $O(\log n)$ slabs that contain exactly those points above and below s. We can solve the k nearest vertical neighbors problem by performing ray-shooting queries within each slab, and selecting the best k points found.

We now describe the data structures which allow us to perform these ray-shooting operations and therefore find the nearest vertical neighbors. Rather than build a data structure that allows line segments to be tested in arbitrary order, we test the segments in left-to-right order of the points dual to the lines containing them. This allows us to perform our algorithm as a *plane sweep* of the dual line arrangement; i.e. we sweep a vertical line from left to right across the dual plane, and perform line segment neighbor finding queries and data structure updates as the vertical line crosses appropriate features in the arrangement. Such a plane sweep could be transformed into a static data structure using persistence techniques; however we do not need these techniques for our algorithms.

At any point in the algorithm, the sweep line will cross all of the n dual lines. Our data structure for a single slab (corresponding to a single dual line arrangement) simply consists of an array of n elements, listing those lines in the vertical order of their points of intersection with the sweep line. Then a vertical ray shooting query along the sweep line could be performed by a binary search in the array, to locate the starting point among the dual lines. Successive queries would then take $O(1)$ time by simply moving up to the next element in the array. The order of the dual lines changes exactly when the sweep line crosses an intersection between two dual lines. The change consists simply of swapping two adjacent elements in the array. The rays at which we wish to shoot correspond to segments xy, which are also found as the intersection of two lines dual to x and y. Each successive intersection can be found in $O(\log n)$ time, by keeping a priority queue of the $n - 1$ possible intersections between adjacent elements in the array.

Now let us consider putting several slabs together again. Starting each ray shooting query by binary searching in each slab separately would take $O(\log^2 n)$ time per segment. We can reduce this time, by using another data structure to relate locations in different slabs to each other. Recall that for each slab, corresponding to an arrangement of some m dual lines, there are two smaller slabs with $m/2$ dual lines each. The sweep line in the large slab is divided into $m + 1$ regions by the m lines crossing it. Each region in the

large slab corresponds to part of a region in each of the smaller slabs. We keep another array, of $m + 1$ elements, listing the correspondence between regions of the large and small slabs. This correspondence only changes when an intersection occurs in the large slab; other intersections will rearrange the correspondence of lines to regions but will not change the numbering of the regions. For each intersection of two lines, we only need to update the correspondence for the region between the lines. Thus again each update takes $O(1)$ time per slab.

Now we can use these arrays to locate the point dual to each segment in the $O(\log n)$ slabs we wish to search. The point is an intersection in the outer slab containing all n points, and its location will already be known when the sweep line crosses that intersection. Then, while we have a location within a slab that contains points not above or below the queried segment, we need to move to locations in the two child slabs. This can be done simply by looking in the appropriate arrays, in constant time per move. In this way it takes $O(\log n)$ time to find the initial locations for ray shooting in each of the $O(\log n)$ appropriate slabs.

Once we have found the initial locations for vertical ray shooting, we can find each successive vertical neighbor of the segment simply by moving from element to adjacent element in the appropriate array. But we must somehow combine the neighbors found in different slabs. To do this, we use a final array, which tells us for each line the position of that line as it crosses the sweep line in the root slab of all n points. This array is updated as before by swapping two elements per intersection encountered. Using this array, we can compare neighbors from different slabs, by examining their positions in the sweep line. Each vertical neighbor for the line segment must be found by selecting among $O(\log n)$ candidates, one from each slab, each of which can be thought of as an integer having $O(\log n)$ bits, representing the position in the sweep line. This selection can be performed in constant time per operation, using the *atomic heap* data structure of Fredman and Willard [11].

Theorem 1. *Given n points, we can enumerate all point sets found as the k nearest vertical neighbors of each segment formed by a pair of points, in total time $O(kn^2 + n^2 \log n)$ and space $O(n \log n)$.* □

Proof: Each intersection of two dual lines, causing a search from the corresponding segment as well as updates to the data structures, can be selected in time $O(\log n)$ from a priority queue of possible intersections. Each search

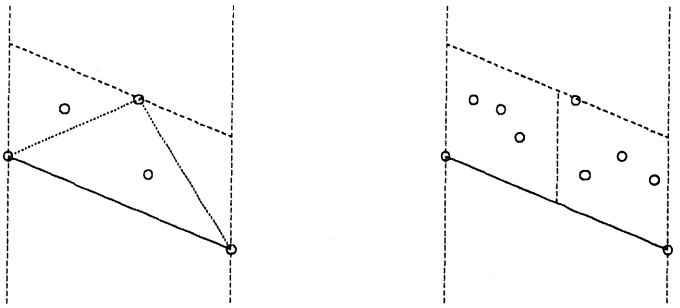

Figure 2. Nearest neighbors to xy: (a) five point set with triangle to furthest point; (b) seven nearest neighbors have five-point subset in one parallelogram.

for k nearest neighbors takes time $O(\log n)$ to find the initial positions for the ray shooting, and $O(1)$ time per neighbor found. Each update takes constant time per slab, and involves changes in $O(\log n)$ slabs (only those slabs containing both dual lines that intersect to cause the update). Therefore the total time for searches and updates is $O(n^2(\log n + k))$. Each dual line is involved in $O(\log n)$ slabs, and uses constant space per slab, so the total space is $O(n \log n)$. □

3 Minimum k-point sets

We now describe how to use the data structure of the previous section to find k-point sets with minimum area convex hulls. This problem is easier than that of finding minimum area k-gons, and serves as a warm-up to the k-gon question.

Lemma 1. *Let Q be the minimum area k-point set of some n point set P, and let x and y be the leftmost and rightmost points of Q respectively. Then each point in Q is one of the $2k-4$ nearest neighbors above or below line segment xy.*

Proof: Let z be the point with largest distance above xy in Q. Then the area of $conv(Q)$ is at least that of triangle xyz. Figure 2(a) shows a set of five points, with triangle xyz outlined.

Suppose z is not one of the $2k-4$ nearest neighbors. Then at least $2k-1$ points (including x, y, and z) are contained in the parallelogram with two vertical sides through x and y, one side equal to segment xy, and the

This is an improvement over the previous $O(kn^3)$ algorithm [9] when $k < (\log_2 n)/6$. The total space complexity is $O(n \log n + f(k)^2)$, improving the previous $O(n^2)$ bound.

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