



## New Analytical Approach to Nonlinear Behavior Study of Asymmetrically LCBs on Nonlinear Elastic Foundation under Steady Axial and Thermal Loading

### Abstract

In this paper, nonlinear behavior analysis of an asymmetrically laminated composite beam (LCB) on nonlinear foundation under axial and in-plane thermal loading is considered. To solve the obtained governing equation, a novel method based on Laplace transform is used. The resulted approximate analytical solution allows us the parametric study of different parameters which influence the nonlinear behavior of the system. The numerical results illustrate that proposed technique yields a very rapid convergence of the solution as well as low computational effort. The accuracy of the proposed method is verified by those available in literatures.

### Keywords

Nonlinear analytical analysis; Laplace Transform; Asymmetrically Laminated Composite Beam; Thermal loading; Nonlinear elastic foundation

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## 1 INTRODUCTION

Beam is one of the important mechanical elements and has numerous applications in different fields of engineering and industries such as civil, marine and aerospace structures or vehicles. Among these, laminated composite beams with high stiffness and strength to weight ratio are increasingly used in many engineering structures.

In most applications, they are subjected to non-linear vibrations which lead to material fatigue and structural damage due to increment of the oscillation amplitude. Therefore, it is necessary and very important to study dynamic nonlinear behavior and natural responses of these structures at large amplitudes. Furthermore, it is desirable to provide an accurate analysis towards the understanding of the non-linear vibration characteristics of these structures.

Generally, it is often difficult to find an analytical solution for a given nonlinear problem unless some simplifying assumptions are considered. Therefore, the application of different numerical techniques seems to be obligatory. It should be noted that it is hard to have a complete understanding of a nonlinear problem out of numerical results. Furthermore, numerical difficulties appear if a nonlinear problem has singularities or multiple solutions.

17 However, closed form solutions are more interesting to research community even if they are  
18 approximate solutions since they have various advantages such as ease of parametric studies and  
19 considering of physics of the problem. Among approximate analytical solutions for nonlinear  
20 problems, one may refer to the homotopy perturbation method (HPM) [8], the variational iter-  
21 ation method (VIM) [9], the modified Lindstedt-Poincare method (MLPM)[11], the harmonic  
22 balance method (HBM) [4], the energy balance method (EBM)[20], the parameter-expansion  
23 method (PEM) [19] and He's variational method (HVM) [12].

24 Most studies for nonlinear vibration and buckling analysis of beams are concerned with  
25 isotropic and symmetrically LCBs, [1, 3, 6, 15, 16]. Due to the bending-stretching coupling  
26 in asymmetrically laminated beams, their nonlinear vibrations analyses are significantly dif-  
27 ferent from that of isotropic beams and symmetrically LCBs. A few studies can be found in  
28 the literature for nonlinear analysis of asymmetrical LCBs [2, 7, 14]. For example, Patel et  
29 al.[14] used a three-nodded shear flexible beam element in order to investigate nonlinear free  
30 flexural vibrations and post-buckling of orthotropic laminated beams resting on a class of two  
31 parameter elastic foundation. Gunda et al. [7] employed the Rayleigh-Ritz method to study  
32 large amplitude vibration analysis of LCB with symmetric and asymmetric layup orientations.  
33 Baghani et al. [2] employed the variational iteration method for large amplitude free vibrations  
34 and post-buckling analysis of asymmetrically LCBs on nonlinear elastic foundation.

35 In this paper, geometrically nonlinear vibration and post-buckling analysis of asymmet-  
36 rically LCB on nonlinear foundation under axial and in-plane thermal loading is considered.  
37 First, Galerkin method is used and the governing nonlinear partial differential equation is re-  
38 duced to a single nonlinear ordinary differential equation. Afterwards, a novel method based  
39 on Laplace transform [13] that is called Laplace iteration method (LIM) is applied to obtain  
40 analytical solution for the nonlinear governing equation. Finally, an approximate analytical  
41 expression will be obtained which allows us to study effect of different parameters on nonlin-  
42 ear behavior of the system. In this paper for the first time, the effect of thermal loading in  
43 addition to the other effects is taking into account. The proposed technique yields very rapid  
44 convergence of the solution as well as low computational effort.

## 45 2 SYSTEM DYNAMICS

46 Consider a straight LCB of length  $l$ , width  $b$ , total thickness  $h$  and mass per unit length  $m$   
47 which rests on an elastic nonlinear foundation subjected to an axial force of magnitude  $\bar{P}$  and  
48 a thermal load i.e. temperature varies linearly from  $T_b$  at bottom side to  $T_t$  at top side of the  
49 beam as shown in Figure 1. A Cartesian coordinate is located while its origin is at left end  
50 and its  $\tilde{x}$  direction crosses through the neutral axis of the beam.

51 If  $\tilde{w}$  and  $\tilde{u}$  are the transverse and longitudinal displacements of the beam along the  $\tilde{z}$  and  
52  $\tilde{x}$  directions, respectively,  $\varepsilon_0$  shows the beam's neutral axis strain,  $\kappa$  points up the flexural or  
53 bending strain of the beam which is known as the curvature and  $\varepsilon_{th}$  represents the thermal  
54 strain. Employing the Von Karman large deformation assumption, the strain-displacement  
55 relation with considering thermal effect can be shown as [10]

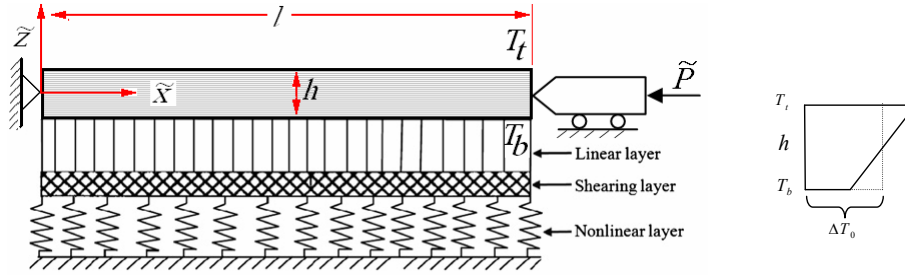


Figure 1 Schematic of the straight LCB on a nonlinear foundation and subjected to an axial force and thermal loading

$$\varepsilon = \varepsilon_m + \varepsilon_{th} \quad (1)$$

56 where

$$\varepsilon_m = \varepsilon_0 + \tilde{z}\kappa, \quad \varepsilon_0 = \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{1}{2} \left( \frac{\partial \tilde{w}}{\partial \tilde{x}} \right)^2, \quad \kappa = -\frac{\partial^2 \tilde{w}}{\partial \tilde{x}^2}, \quad \varepsilon_{th} = \alpha_{th} \Delta T, \quad \Delta T = \Delta T_0 + \tilde{z} \Delta T_1 \quad (2)$$

57 And  $\tilde{z}$  measures the distance of beam's material element from midline,  $\alpha_{th}$  is coefficient of  
58 thermal expansion. Moreover,  $\Delta T_0$  is temperature variation at midline of the beam and  $\Delta T_1$   
59 stands for temperature difference between top and bottom sides and they can be presented as:

$$\Delta T_0 = \frac{T_t + T_b}{2}, \quad \Delta T_1 = \frac{T_t - T_b}{h} \quad (3)$$

60 The force and moment resultants per unit length based on the classical laminate beam  
61 theory can be written as [10, 18]:

$$\begin{Bmatrix} N_{\tilde{x}} \\ M_{\tilde{x}} \end{Bmatrix} = \begin{bmatrix} A_{11} & B_{11} \\ B_{11} & D_{11} \end{bmatrix} \begin{Bmatrix} \varepsilon_0 \\ \kappa \end{Bmatrix} - \begin{bmatrix} A_{11th} & B_{11th} \\ B_{11th} & D_{11th} \end{bmatrix} \begin{Bmatrix} \Delta T_0 \\ \Delta T_1 \end{Bmatrix} \quad (4)$$

62 where its stiffness coefficients are given as follows [10, 18]:

$$\begin{aligned} A_{11} &= \sum_{k=1}^n \bar{Q}_{11}^{(k)} (h_k - h_{k-1}), & A_{11th} &= \sum_{k=1}^n \bar{Q}_{11}^{(k)} \alpha_{th}^{(k)} (h_k - h_{k-1}) \\ B_{11} &= \frac{1}{2} \sum_{k=1}^n \bar{Q}_{11}^{(k)} (h_k^2 - h_{k-1}^2), & B_{11th} &= \frac{1}{2} \sum_{k=1}^n \bar{Q}_{11}^{(k)} \alpha_{th}^{(k)} (h_k^2 - h_{k-1}^2) \\ D_{11} &= \frac{1}{3} \sum_{k=1}^n \bar{Q}_{11}^{(k)} (h_k^3 - h_{k-1}^3), & D_{11th} &= \frac{1}{3} \sum_{k=1}^n \bar{Q}_{11}^{(k)} \alpha_{th}^{(k)} (h_k^3 - h_{k-1}^3) \end{aligned} \quad (5)$$

63 Each layer  $k$  is referred to by the  $\tilde{z}$  coordinates of its lower face ( $h_{k-1}$ ) and upper face ( $h_k$ )  
64 and  $\bar{Q}_{11}^{(k)}$  is the elements of the stiffness matrix in the  $\tilde{x}$  direction,  $n$  is the number of laminas  
65 and  $\alpha_{th}^{(k)}$  is coefficient of thermal expansion of the  $k$ th layer.

66 Finally, using the Extended Hamilton's principle [17, 18], the governing equation of trans-  
67 verse vibration of an LCB including thermal effect and axial stretching on a nonlinear elastic  
68 foundation can be obtained as

$$m \frac{\partial^2 \tilde{w}}{\partial \tilde{t}^2} + b \left( D_{11} - \frac{B_{11}^2}{A_{11}} \right) \frac{\partial^4 \tilde{w}}{\partial \tilde{x}^4} + \beta \frac{\partial^2 \tilde{w}}{\partial \tilde{x}^2} = F_{\tilde{w}} \quad (6)$$

69 where

$$\beta = \left[ \tilde{P} + b(A_{11th}\Delta T_0 + B_{11th}\Delta T_1) - \frac{bA_{11}}{2l} \int_0^l \left( \frac{\partial \tilde{w}}{\partial \tilde{x}} \right)^2 d\tilde{x} - \frac{bB_{11}}{2l} \left( \frac{\partial \tilde{w}}{\partial \tilde{x}} \Big|_{(l,\tilde{t})} - \frac{\partial \tilde{w}}{\partial \tilde{x}} \Big|_{(0,\tilde{t})} \right) \right] \quad (7)$$

$$F_{\tilde{w}} = -\tilde{k}_L \tilde{w} - \tilde{k}_{NL} \tilde{w}^3 + \tilde{k}_{Sh} \frac{\partial^2 \tilde{w}}{\partial \tilde{x}^2}$$

70  $\tilde{k}_L$  and  $\tilde{k}_{NL}$  are linear and nonlinear elastic foundation coefficients,  $\tilde{k}_{Sh}$  is the shear stiffness  
71 of the elastic foundation.

72 By defining non-dimensional variables

$$x = \frac{\tilde{x}}{l}, \quad w = \frac{\tilde{w}}{r}, \quad t = \tilde{t} \sqrt{\frac{b}{ml^4} \gamma}, \quad r = \sqrt{\frac{I}{A}} \quad (8)$$

73 it can be written in a simple form as

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} + K_L w + K_{NL} w^3 - K_{Sh} \frac{\partial^2 w}{\partial x^2} + \left[ P + F_{0th} + F_{1th} - B \int_0^1 \left( \frac{\partial w}{\partial x} \right)^2 dx - \Lambda \left( \frac{\partial w}{\partial x} \Big|_{(1,t)} - \frac{\partial w}{\partial x} \Big|_{(0,t)} \right) \right] \frac{\partial^2 w}{\partial x^2} = 0 \quad (9)$$

74 where  $r$  is the radius of gyration of the beam's cross-section, and

$$\begin{aligned} K_L &= \frac{\tilde{k}_L l^4}{b\gamma} & K_{NL} &= \frac{\tilde{k}_{NL} r^2 l^4}{b\gamma} & K_{Sh} &= \frac{\tilde{k}_{Sh} l^2}{b\gamma} \\ P &= \frac{\tilde{P} l^2}{b\gamma} & F_{0th} &= \frac{l^2 \Delta T_0 A_{11th}}{\gamma} & F_{1th} &= \frac{l^2 \Delta T_1 B_{11th}}{\gamma} \\ B &= \frac{A_{11} r^2}{2\gamma} & \Lambda &= \frac{B_{11} r}{\gamma} & \gamma &= \left( D_{11} - \frac{B_{11}^2}{A_{11}} \right) \end{aligned} \quad (10)$$

75 To achieve the aims of the paper, the solution of Eq. (9) is assumed to be

$$w(x, t) = \varphi(x) \eta(t) \quad (11)$$

76 where  $\varphi(x)$  is the first normal mode of the beam [17] that is defined for simply supported and  
77 fixed-fixed boundary conditions in Table 1 and  $\eta(t)$  is an unknown time dependent function.

Table 1 The first normal modes for beam with various boundary conditions

Boundary Condition	$\varphi(x)$
Simply Supported	$\varphi(x) = \sin(\pi x)$
Fixed-Fixed	$\varphi(x) = (\sinh(qx) - \sin(qx)) - \frac{\sinh(q) - \sin(q)}{\cosh(q) - \cos(q)} (\cosh(qx) - \cos(qx)), q = 4.730041$

78 Applying the Galerkin method [17], Eq. (9) yields

$$\frac{d^2 \eta(t)}{dt^2} + [\alpha_1 + (P + F_{0th} + F_{1th}) \alpha_P + \alpha_L + \alpha_{Sh}] \eta(t) + \alpha_2 \eta^2(t) + (\alpha_{NL} + \alpha_3) \eta^3(t) = 0 \quad (12)$$

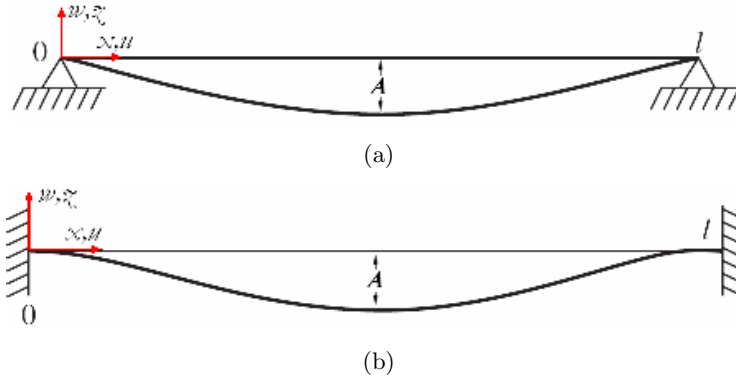


Figure 2 The first normal functions of the beam with a) Simply supported, b) Fixed-Fixed boundary conditions

79 where

80 Now, it can be assumed that the beam is subjected to an initial displacement according  
 81 to its first modal shape and zero initial velocity. So, the initial conditions of Eq. (12) can be  
 82 presented as

$$\alpha_1 = \frac{\int_0^1 \varphi^{(iv)} \varphi dx}{\int_0^1 \varphi^2 dx}, \quad \alpha_2 = -\Lambda (\varphi'(1) - \varphi'(0)) \alpha_P, \quad \alpha_3 = -B \alpha_P \int_0^1 \varphi'^2 dx \quad (13)$$

$$\alpha_P = \frac{\int_0^1 \varphi'' \varphi dx}{\int_0^1 \varphi^2 dx}, \quad \alpha_{Sh} = -K_{Sh} \alpha_P, \quad \alpha_L = K_L, \quad \alpha_{NL} = K_{NL} \frac{\int_0^1 \varphi^4 dx}{\int_0^1 \varphi^2 dx}$$

$$\eta(0) = A, \quad \frac{d\eta(0)}{dt} = 0 \quad (14)$$

83 where according to the Fig. 2,  $A$  denotes the non-dimensional maximum amplitude of oscillation  
 84 at the beam's center.

85 Based on the Eq. (12) the nonlinear post-buckling load of the considered LCB can be  
 86 written as

$$P_{NB} = -\frac{[\alpha_1 + \alpha_P + \alpha_L + \alpha_{Sh}] + \alpha_2 A + (\alpha_{NL} + \alpha_3) A^2}{\alpha_P} - (F_{0th} + F_{1th}) \quad (15)$$

87 Neglecting the  $A$  in Eq. (15), the linear buckling load will be derived as

$$P_{LB} = -\frac{\alpha_1 + \alpha_P + \alpha_L + \alpha_{Sh}}{\alpha_P} - (F_{0th} + F_{1th}) \quad (16)$$

88 The next step is to find the natural frequency of the system. Since the governing equation  
 89 Eq. (12) is nonlinear, the free vibration of the system has a nonlinear natural frequency  
 90 which is introduced by  $\omega_{NL}$ . Indeed, the nonlinear free vibration response of the system  $\eta(t)$   
 91 and its nonlinear natural frequency  $\omega_{NL}$  depend on the system parameters, the boundary  
 92 condition and the initial conditions. Eq. (12) is strongly nonlinear and nobody can find an  
 93 exact analytical closed form solution for  $\eta(t)$  and  $\omega_{NL}$ . Although numerical methods can be  
 94 implemented to get over this problem but, they cannot offer any suitable way for parametric

95 study. Therefore, it will be valuable if a powerful analytical approximate method exists that  
96 presents an accurate approximation of  $\eta(t)$  and  $\omega_{NL}$  while providing the ability to parametric  
97 study of the problem.

### 98 3 DESCRIPTION OF THE PROPOSED METHOD

99 Using the Laplace Transformation method an analytical approximated technique is proposed  
100 to present an accurate solution for nonlinear differential equations. To clarify the basic ideas  
101 of proposed method consider the following second order differential equation,

$$\ddot{u}(t) + N \{ u(t) \} = 0 \quad (17)$$

102 with artificial zero initial conditions and  $N$  is the nonlinear operator. Adding and subtracting  
103 the term  $\omega^2 u(t)$ , the Eq. (17) can be written in the form

$$\ddot{u}(t) + \omega^2 u(t) = L \{ u(t) \} = f(u(t)) \quad (18)$$

104 where  $L$  is the linear operator and

$$f(u(t)) = \omega^2 u(t) - N \{ u(t) \} \quad (19)$$

105 Taking Laplace transform of both sides of the Eq. (18) in the usual way and using the  
106 homogenous initial conditions gives

$$(s^2 + \omega^2) U(s) = \mathfrak{J} \{ f(u(t)) \} \quad (20)$$

107 where  $s$  and  $\mathfrak{J}$  are the Laplace variable and operator, correspondingly. Therefore it is obvious  
108 that

$$U(s) = \mathfrak{J} \{ f(u(t)) \} G(s) \quad (21)$$

109 where

$$G(s) = \frac{1}{s^2 + \omega^2} \quad (22)$$

110 Now, implementing the Laplace inverse transform of Eq. (21) and using the Convolution  
111 theorem offer

$$u(t) = \int_0^t f(u(\tau)) g(t - \tau) d\tau \quad (23)$$

112 where

$$g(t) = \mathfrak{J}^{-1} \{ G(s) \} = \frac{1}{\omega} \sin(\omega t) \quad (24)$$

113 Substituting Eq. (19) and (24) into (23) gives

$$u(t) = \int_0^t (\omega^2 u(\tau) - N \{ u(\tau) \}) \frac{1}{\omega} \sin(\omega(t - \tau)) d\tau \quad (25)$$

114 Now, the actual initial conditions must be imposed. Finally the following iteration formu-  
115 lation can be used [5]

$$u_{n+1} = u_0 + \frac{1}{\omega} \int_0^t (\omega^2 u_n(\tau) - N \{ u_n(\tau) \}) \sin(\omega(t - \tau)) d\tau \quad (26)$$

116 Knowing the initial approximation  $u_0$ , the next approximations  $u_n, n > 0$  can be determined  
117 from previous iterations. Consequently, the exact solution may be obtained by using:

$$u = \lim_{n \rightarrow \infty} u_n \quad (27)$$

118 In this method, the problems are initially approximated with possible unknowns and it can  
119 be applied in non-linear problems without linearization or small parameters. The approximate  
120 solutions obtained by the proposed method rapidly converge to the exact solution.

#### 121 4 IMPLEMENTATION OF THE PROPOSED METHOD

122 Eq. (12) can be rewritten in the standard form Eq. (18)

$$\frac{d^2 \eta(t)}{dt^2} + \omega^2 \eta(t) = f(\eta(t)) \quad (28)$$

123 where

$$\begin{aligned} f(\eta(t)) &= \omega^2 \eta(t) - N \{ \eta(t) \}, & \lambda_1 &= \alpha_1 + (P + F_{0th} - F_{1th}) \alpha_P + \alpha_L + \alpha_{Sh} \\ N \{ \eta(t) \} &= \lambda_1 \eta(t) + \lambda_2 \eta^2(t) + \lambda_3 \eta^3(t), & \lambda_2 &= \alpha_2, & \lambda_3 &= (\alpha_{NL} + \alpha_3) \end{aligned} \quad (29)$$

124 Applying the proposed method, the following iterative formula is assembled

$$\eta_{n+1}(t) = \eta_0(t) + \frac{1}{\omega} \int_0^t f(\eta_n(\tau)) \sin(\omega(t - \tau)) d\tau \quad (30)$$

125 Eq. (28) will be homogeneous, if  $f(\eta(t))$  is considered to be zero. So, its homogeneous  
126 solution

$$\eta_0(t) = A \cos(\omega t) \quad (31)$$

127 is considered as the zero approximation for using in iterative Eq.(30).

128 Expanding  $f(\eta_0(\tau))$ , we have:

$$f(\eta_0(\tau)) = \left(-\lambda_1 A + \omega^2 A - \frac{3}{4}\lambda_3 A^3\right) \cos(\omega t) - \frac{1}{4}\lambda_3 A^3 \cos(3\omega t) - \frac{1}{2}\lambda_2 A^2 (1 + \cos(2\omega t)) \quad (32)$$

129 Considering the relation:

$$\frac{1}{\omega} \int_0^t (\cos(m\omega\tau)) \sin(\omega(t-\tau)) d\tau = \begin{cases} \frac{\cos(\omega t) - \cos(m\omega t)}{\omega^2(m^2-1)} & m \neq 1 \\ \frac{t \sin(\omega t)}{2\omega} & m = 1 \end{cases} \quad (33)$$

130 To avoid secular terms in the next iterations, the coefficient of the  $\cos(\omega t)$  in  $f(\eta_0(\tau))$   
 131 should be vanished. So the first approximation of the frequency is obtained as:

$$\omega = \sqrt{\lambda_1 + \frac{3}{4}\lambda_3 A^2} \quad (34)$$

132 Substituting Eq. (31) into (30) and neglecting the secular terms that are the coefficient of  
 133  $\cos(\omega t)$  in forcing function  $f(\eta)$  give

$$\eta_1(t) = \frac{1}{96\omega^2} \left\{ (32\lambda_2 A^2 + 96\omega^2 A - 3\lambda_3 A^3) \cos(\omega t) + 16\lambda_2 A^2 \cos(2\omega t) + 3\lambda_3 A^3 \cos(3\omega t) - 48\lambda_2 A^2 \right\} \quad (35)$$

134 This is the first approximation of  $\eta(t)$ . Substituting Eq. (35) in Eq. (30) and implementing  
 135 the procedure for second time yields the second approximation of  $\eta(t)$  as

$$\eta_2(t) = \frac{1}{1981808640} \frac{1}{A^3} \begin{pmatrix} I_0 + I_1 \cos(\omega t) + I_2 \cos(2\omega t) + I_3 \cos(3\omega t) \\ + I_4 \cos(4\omega t) + I_5 \cos(5\omega t) + I_6 \cos(6\omega t) \\ + I_7 \cos(7\omega t) + I_8 \cos(8\omega t) + I_9 \cos(9\omega t) \end{pmatrix} \quad (36)$$

136 where  $I_i$  are given in Appendix.

137 In this step, to avoid the secular terms the coefficient of  $\cos(\omega t)$  in forcing function must  
 138 be zero. So,

$$\omega^8 + \beta_6 \omega^6 + \beta_4 \omega^4 + \beta_2 \omega^2 + \beta_0 = 0 \quad (37)$$

139 where

$$\begin{aligned} \beta_6 &= \left(-\frac{5^2}{2^5}\lambda_3 A^2 + \frac{1}{3}\lambda_2 A - \lambda_1\right) \\ \beta_4 &= \left(\frac{3}{2^6}\lambda_3^2 A^4 - \frac{3}{2^2}\lambda_2 \lambda_3 A^3 + \left(\frac{1}{2^5}\lambda_1 \lambda_3 + \frac{5}{2^3}\lambda_2^2\right) A^2 - \frac{1}{3}\lambda_1 \lambda_2 A\right) \\ \beta_2 &= \left(-\frac{3^2}{2^{12}}\lambda_3^3 A^6 + \frac{1}{2^5}\lambda_2 \lambda_3^2 A^5 + \frac{5}{2 \cdot 3^2}\lambda_2^3 A^3 - \frac{79}{2^5 \cdot 3}\lambda_2^2 \lambda_3 A^2\right) \\ \beta_0 &= \left(\frac{3}{2^{16}}\lambda_3^4 A^8 - \frac{3}{2^{12}}\lambda_2 \lambda_3^3 A^7 + \frac{3 \cdot 5}{2^9}\lambda_2^2 \lambda_3^2 A^6 - \frac{5}{2^3 \cdot 3}\right) \end{aligned} \quad (38)$$

140 Solution of Eq. (37) gives estimation  $\omega$  for the actual natural frequency of the system.



5 NUMERICAL RESULTS

To illustrate the robustness of the proposed LIM method and to compare with other methods, some cases are studied. First, an isotopic beam in two cases of simply supported and fixed-fixed boundary conditions is taken. In these cases, the effects of thermal loading and elastic foundation are ignored. The amounts of the nonlinear to the linear frequency ratio  $\omega_{NL}/\omega_L$  are derived for four non-dimensional amplitudes  $A$ . Table 2 shows the results of three references as well as the numerical results that are computed by the fourth Runge-Kutta method in both cases. The two last columns in each case show the results based on LIM and by one step and two step iteration. As it is mentioned, the proposed method offers the results with excellent accordance with the numerical results even by one step iteration.

Table 2 Comparison of nonlinear to linear frequency ratio,  $\omega_{NL}/\omega_L$

A	Simply Supported						Clamped-Clamped					
	Ref. [1]	Ref. [16]	Ref. [15]	Numerical results[2]	Present 1 step	Present 2 step	Ref. [1]	Ref. [16]	Ref. [15]	Numerical results [2]	Present 1 step	Present 2 step
1	1.0891	1.0897	1.0897	1.0891	1.0892	1.0892	1.0221	1.0628	1.0572	1.0553	1.0566	1.0566
2	1.3177	1.3229	1.3228	1.3175	1.3179	1.3178	1.0856	1.2140	1.2125	1.2042	1.2058	1.2057
3	1.6256	1.6394	1.6393	1.6255	1.6263	1.6257	1.1831	1.3904	1.4344	1.4158	1.4179	1.4176
4	-	-	1.9999	1.9758	1.9774	1.9761	1.3064	1.5635	1.6171	1.6658	1.6687	1.6679

In the second step, it is assumed that the composite beam is made by AS4/3501 Graphite-Epoxy. Its mechanical properties [21] are  $E_{11} = 138 GPa$ ,  $E_{22} = 8.9 GPa$ ,  $\nu_{12} = 0.3$ ,  $\alpha_{th}^{(1)} = -0.5 \times 10^{-6}/^{\circ}C$  and  $\alpha_{th}^{(2)} = 28.5 \times 10^{-6}/^{\circ}C$ . To study the effect of cross-ply lay-up configuration on the nonlinear vibration of the considered LCB, three cases of configuration are used. Figure 3 illustrates this effect on the nonlinear to the linear frequency ratio  $\omega_{NL}/\omega_L$  and Figure 4 shows the influence on the ratio of the nonlinear post buckling load to buckling load  $P_{NB}/P_{LB}$  for both cases of boundary conditions, respectively. It can be seen that the nonlinear behavior of the LCB with  $[0/90/90/0]$ ,  $[0/90/0/90]$  and  $[90/0/0/90]$  lay-up configuration increases from lower values to higher values, respectively. So, the LCB behavioral response can be controlled by its lay-up configuration, passively.

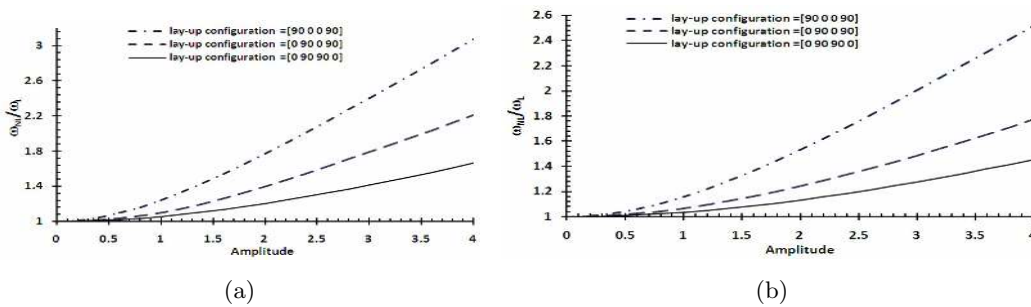


Figure 3 The effect of cross-ply lay-up configuration on the nonlinear to the linear frequency ratio (Left) Simply supported, (Right) Fixed-Fixed boundary conditions

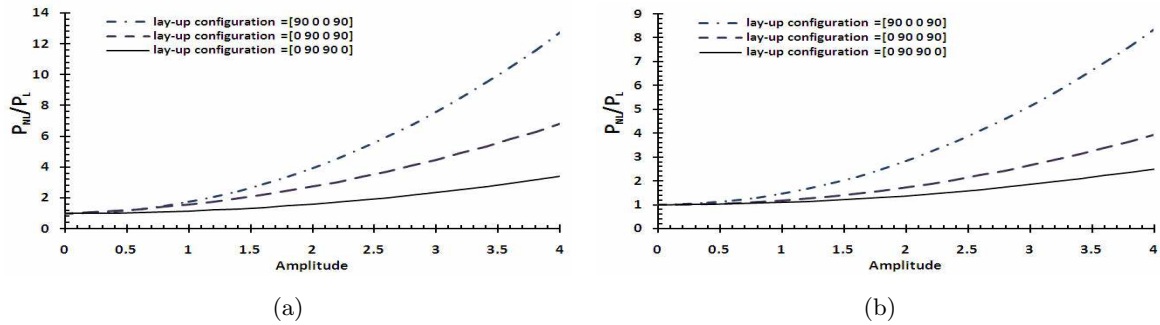


Figure 4 The effect of cross-ply lay-up configuration on the post buckling to the buckling load ratio (Left) Simply supported, (Right) Fixed-Fixed boundary conditions

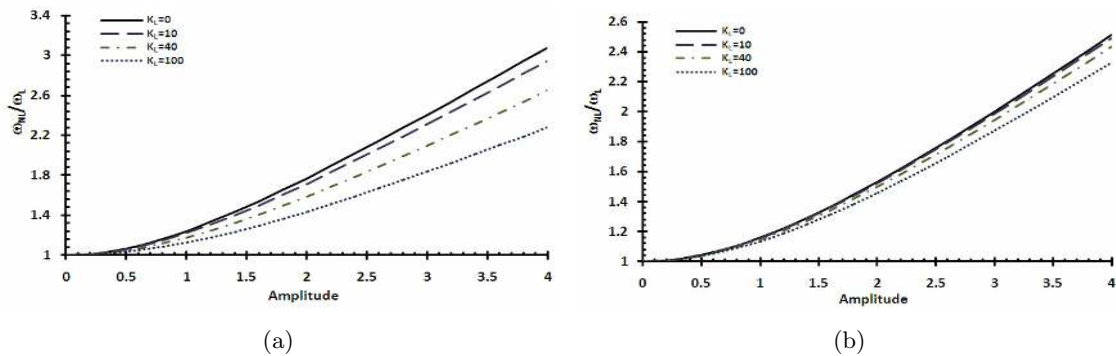


Figure 5 The effect of the linear stiffness  $K_L$  on the nonlinear to the linear frequency ratio (Left) Simply supported, (Right) Fixed-Fixed boundary conditions

161 In the next step, the nonlinear behavior of the considered LCB due to elastic foundation is  
 162 investigated. As the [90/0/0/90] lay-up has the most critical nonlinear behavior, this configu-  
 163 ration is selected for the rest of the paper. Figure 5 to 7 demonstrate the effects of different  
 164 stiffness values of  $K_L$ ,  $K_{NL}$  and  $K_{Sh}$  on  $\omega_{NL}/\omega_L$  ratio for both boundary conditions, corre-  
 165 spondingly. It can be seen that an increase in the linear and shearing layer stiffness of the  
 166 foundation leads to decrement of the nonlinear to linear frequency ratio and also an increase  
 167 in nonlinear stiffness augments this ratio. Also, the shearing layer stiffness has the strongest  
 168 effect.

169 Now, the axial loading is applied. Figure 8 shows the variation of the nonlinear to the  
 170 linear frequency ratio  $\omega_{NL}/\omega_L$  due to change in the axial loading  $P$ . It shows that axial  
 171 loading amplifies the nonlinear frequency ratio of the LCB.

172 Finally, the thermal loading is considered. As it is seen in Figure 9, thermal loading  
 173 increases the nonlinear to the linear frequency of the considered LCB. The results show that the  
 174 linear and nonlinear natural frequencies decrease by increasing the thermal loading however,  
 175 the decreasing rate of nonlinear frequency is less than linear natural frequency.

176 In the previous steps, the effect of each factor was studied, independently. So in the last

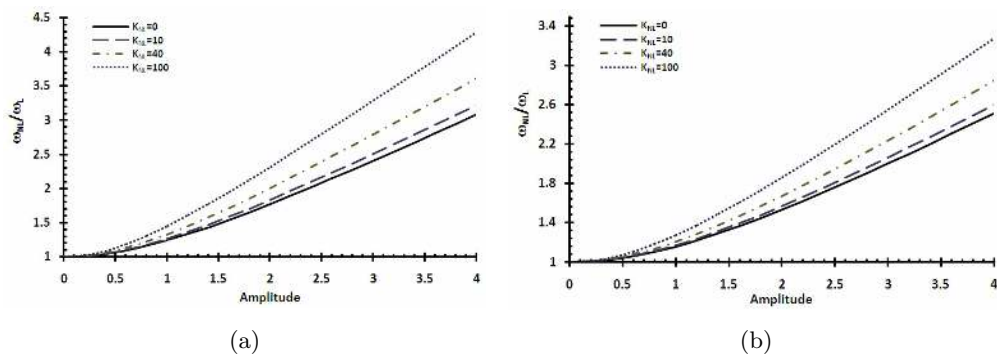


Figure 6 The effect of the nonlinear stiffness  $K_{NL}$  on the nonlinear to the linear frequency ratio (Left) Simply supported, Right) Fixed-Fixed boundary conditions

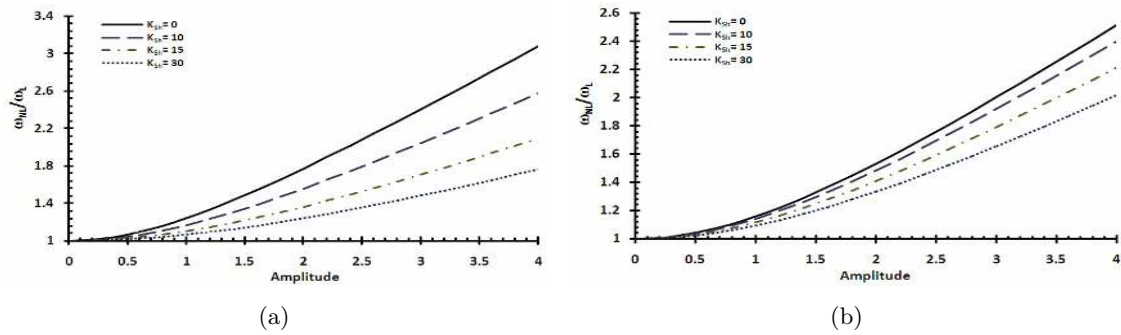


Figure 7 The effect of the shear stiffness  $K_{Sb}$  on the nonlinear to the linear frequency ratio (Left) Simply supported, Right) Fixed-Fixed boundary conditions

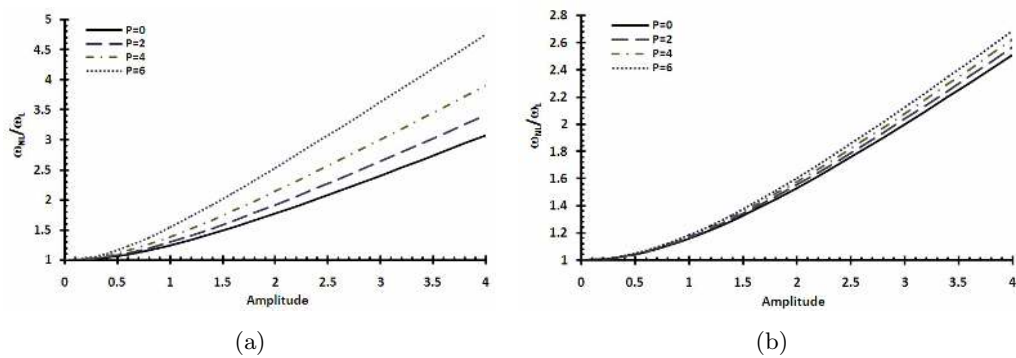


Figure 8 The effect of the axial loading on the nonlinear to the linear frequency ratio (Left) Simply supported, Right) Fixed-Fixed boundary conditions

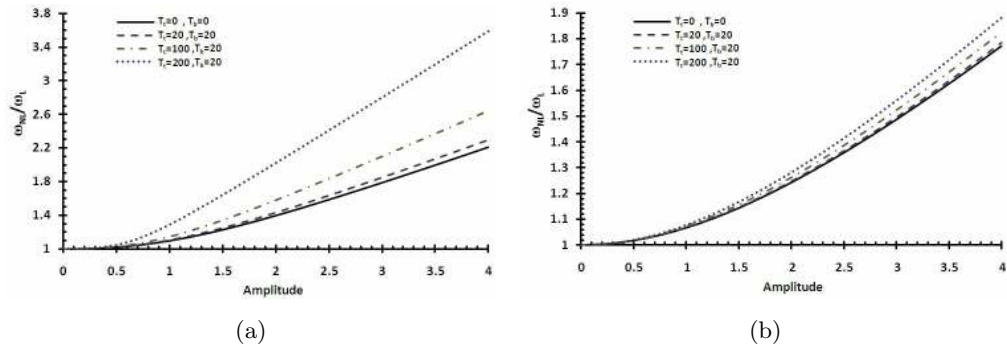


Figure 9 The effect of the thermal loading on the nonlinear to the linear frequency ratio Left) Simply supported, Right) Fixed-Fixed boundary conditions For [0/90/0/90] lay-up configuration and L/h=50

177 walk, effects of all factors are implemented simultaneously. Table 3 and 4 show the results for  
 178 cases with simply supported and fixed-fixed boundary conditions, respectively.

Table 3 Comparison of nonlinear frequency ( $\omega_{NL}$ ) and nonlinear to linear frequency ratio ( $\omega_{NL}/\omega_L$ ) due to change of different factors for S-S LCB, [0/90/0/90] lay-up configuration, A=2 and L/h=50

$(T_t, T_b)$	P	$K_L$	Nonlinear Frequency				Nonlinear to linear Frequency				
			$K_{NL}$		$K_{NL}$		$K_{NL}$		$K_{NL}$		
			0	50	0	50	0	50	0	50	
			$K_{sh}$	$K_{sh}$	$K_{sh}$	$K_{sh}$	$K_{sh}$	$K_{sh}$	$K_{sh}$	$K_{sh}$	$K_{sh}$
			0	25	0	25	0	25	0	25	25
(0,0)	0	0	13.796	20.771	17.622	23.439	1.3978	1.1197	1.7855	1.2635	
		50	15.423	21.935	18.919	24.468	1.2703	1.1049	1.5583	1.2325	
	3	0	12.771	20.051	16.821	22.810	1.5510	1.1306	2.0428	1.2862	
		50	14.475	21.253	18.159	23.864	1.3336	1.1131	1.6730	1.2499	
(50,50)	0	0	13.047	20.244	17.036	22.978	1.5001	1.1275	1.9587	1.2798	
		50	14.731	21.436	18.363	24.025	1.3142	1.1108	1.6382	1.2450	
	3	0	11.994	19.506	16.217	22.337	1.7676	1.1400	2.3900	1.3054	
		50	13.749	20.738	17.585	23.411	1.4030	1.1201	1.7944	1.2645	
(150,50)	0	0	11.935	19.464	16.171	22.301	1.7915	1.1408	2.4273	1.3071	
		50	13.693	20.698	17.541	23.376	1.4095	1.1206	1.8055	1.2656	
	3	0	10.864	18.696	15.328	21.642	2.8264	1.1561	3.9877	1.3383	
		50	12.664	19.976	16.738	22.745	1.5735	1.1318	2.0797	1.2887	

Table 4 Comparison of nonlinear frequency ( $\omega_{NL}$ ) and nonlinear to linear frequency ratio ( $\omega_{NL}/\omega_L$ ) due to change of different factors for F-F LCB, [0/90/0/90] lay-up configuration, A=2 and L/h=50

$(T_t, T_b)$	P	$K_L$	Nonlinear Frequency				Nonlinear to linear Frequency			
			$K_{NL}$		50		$K_{NL}$		50	
			$K_{sh}$	0	$K_{sh}$	25	$K_{sh}$	0	$K_{sh}$	25
(0,0)	0	0	27.778	32.868	32.425	36.899	1.2416	1.1562	1.4493	1.2980
		50	28.668	33.622	33.195	37.575	1.2218	1.1477	1.4147	1.2827
	3	0	27.102	32.300	31.844	36.392	1.2586	1.1631	1.4788	1.3104
		50	28.014	33.067	32.628	37.077	1.2361	1.1539	1.4396	1.2938
(50,50)	0	0	27.283	32.451	31.999	36.527	1.2539	1.1612	1.4706	1.3071
		50	28.189	33.215	32.779	37.210	1.2321	1.1522	1.4327	1.2908
	3	0	26.594	31.876	31.410	36.015	1.2729	1.1686	1.5034	1.3203
		50	27.523	32.653	32.205	36.707	1.2478	1.1587	1.4601	1.3026
(150,50)	0	0	26.555	31.844	31.376	35.986	1.2740	1.1690	1.5053	1.3211
		50	27.485	32.621	32.173	36.679	1.2487	1.1591	1.4617	1.3033
	3	0	25.846	31.257	30.775	35.466	1.2963	1.1771	1.5435	1.3356
		50	26.802	32.049	31.587	36.169	1.2669	1.1663	1.4931	1.3162

179 **6 CONCLUSION**

180 In this paper, the effects of different parameters such as vibration amplitude, nonlinear elastic  
 181 foundation, axial and thermal loading on the nonlinear behavior of the LCBs such as natu-  
 182 ral frequency and buckling load were investigated. For this purpose and to solve nonlinear  
 183 governing equation, a new approach based on the Laplace transform method which is called  
 184 LIM was implemented. This technique provides the ability for parametric study of the consid-  
 185 ered problem. Results revealed that the presented method offers accurate solution with low  
 186 computational effort.

187 Moreover, the presented expression is valid for a wide range of vibration amplitudes while  
 188 predictions of the other analytical techniques such as perturbation methods are valid for small  
 189 amplitudes. Comparison between the results of the present study and other methods available  
 190 in the literature shows the accuracy of the method. Results reveal that decreasing linear  
 191 and shear parameters and increasing nonlinear parameters of foundation lead to increasing  
 192 frequency and buckling load ratios. Furthermore, increasing axial force decreases absolute  
 193 values of both linear and nonlinear frequencies as well as natural frequency ratio.

194 **Appendix:  $I_i$  coefficients in the second approximation of deflection.**

$$I_0 = 990904320A((\Pi_1 + \frac{21}{16}\Pi_3 - 2\Pi_4)A^2 + (-\frac{1}{32}\Pi_3^2 + (-\frac{49}{512}\Pi_5 + \frac{41}{48}\Pi_4)\Pi_3 - \frac{2}{3}\Pi_4^2)A + (\frac{13}{4096}\Pi_5^2 + \frac{31}{72}\Pi_4^2)\Pi_3 - \frac{23}{36}\Pi_4^3)$$

$$I_1 = 1981808640A^4 + (-54190080\Pi_5 - 1052835840\Pi_3 - 1101004800\Pi_1 - 7741440\Pi_2 + 1761607680\Pi_4)A^3 + (26512128\Pi_3^2 + (-595574784\Pi_4 + 101007360\Pi_5)\Pi_3 - 7741440\Pi_5^2 + 399114240\Pi_4^2)A^2 + ((-457900032\Pi_4^2 - 3740640\Pi_5^2)\Pi_3 + 690880512\Pi_4^3 + 574560\Pi_5^3)A - 14364\Pi_5^4$$

$$I_2 = 110100480A((\Pi_1 - 3\Pi_3 + 2\Pi_4)A^2 + (-\frac{1}{64}\Pi_3^2 + (-2\Pi_4 - \frac{51}{1024}\Pi_5)\Pi_3 + 2\Pi_4^2)A + (\frac{7}{16}\Pi_4^2 + \frac{3}{512}\Pi_5^2)\Pi_3 - \frac{2}{3}\Pi_4^3)$$

$$I_3 = (7741440\Pi_2 + 54190080\Pi_5 + 61931520\Pi_3)A^3 + \left( \begin{matrix} 8547840\Pi_3^2 + (3870720\Pi_5 - 45158400\Pi_4)\Pi_3 \\ + 5806080\Pi_5^2 + 41287680\Pi_4^2 \end{matrix} \right) A^2 + ((-181440\Pi_5^2 - 16629760\Pi_4^2)\Pi_3 - 544320\Pi_5^3 + 13762560\Pi_4^3)A + 15120\Pi_5^4$$

$$I_4 = 20643840A \left( A^2\Pi_3 - \frac{1}{10}\Pi_3 \left( \frac{49}{16}\Pi_5 - 6\Pi_4 + \Pi_3 \right) A + \left( -\frac{2}{45}\Pi_4^2 + \frac{3}{320}\Pi_5^2 \right) \Pi_3 + \frac{4}{45}\Pi_4^3 \right)$$

$$I_5 = -483840A \left( \left( \Pi_3^2 + \left( -\frac{8}{3}\Pi_5 - \frac{40}{9}\Pi_4 \right) \Pi_3 - 4\Pi_5^2 \right) A + \left( -\frac{32}{27}\Pi_4^2 + \frac{\Pi_5^2}{24} \right) \Pi_3 + \frac{\Pi_5^3}{8} \right)$$

$$I_6 = 147456A\Pi_3 \left( \left( \Pi_3 + \frac{51}{16}\Pi_5 \right) A - \frac{3}{8}\Pi_5^2 + \frac{4}{9}\Pi_4^2 \right)$$

195

$$I_7 = 26880\Pi_3^2A^2 + 10080\Pi_5^2(3\Pi_5 + \Pi_3)A - 945\Pi_5^4 \quad I_8 = 3840\Pi_5^2A\Pi_3 \quad I_9 = 189\Pi_5^4$$

196

Where

$$\Pi_1 = \frac{\lambda_1 A^2 \lambda_2}{\omega^4}, \quad \Pi_2 = \frac{\lambda_1 A^3 \lambda_3}{\omega^4}, \quad \Pi_3 = \frac{\lambda_2 A^4 \lambda_3}{\omega^4}, \quad \Pi_4 = \frac{\lambda_2 A^2}{\omega^2}, \quad \Pi_5 = \frac{\lambda_3 A^3}{\omega^2}$$

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