New Analytical Approach to Two-Dimensional Viscous Flow with a Shrinking Sheet via Sumudu Transform

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Abstract

In this paper, a new analytical approach based on homotopy perturbation Sumudu transform method (HPSTM) to a two-dimensional viscous flow with a shrinking sheet is presented. The series solution is obtained by HPSTM coupled with Padé approximants to handle the condition at infinity. The HPSTM is a combined form of the Sumudu transform method, homotopy perturbation method and He's polynomials. This scheme finds the solution without any discretization or restrictive assumptions and avoids round-off errors. The numerical solutions obtained by the proposed method indicate that the approach is easy to implement and computationally very attractive.

Keywords: Sumudu transform, homotopy perturbation method, He's polynomials, Padé approximants, Shrinking sheet, Similarity transformations

Introduction

The boundary layer viscous flow induced by stretching surface movement with a certain velocity in an otherwise quiescent fluid medium often occurs in several engineering processes. It has attracted considerable interest during the last few decades. Such flows have important applications in industries, for example in the extrusion of a polymer sheet from a die or in the drawing of plastic films. During the manufacture of these sheets, the melt issues from a slit and is subsequently stretched to achieve the desired thickness. The mechanical properties of the final product strictly depend on the stretching and cold drawing rates in the process. Pioneering work in this area was conducted by Sakiadis [1,2], and the boundary layer flow on a continuously stretching surface with a constant speed was investigated by several researchers in the field. Specifically, Crane [3] found a closed form solution for the flow of an incompressible viscous fluid past a stretching plate. Later basic stretching solutions which differ appreciably from Crane's followed. Gupta and Gupta [4] added suction or injection on the surface. The flow inside a stretching channel or tube was considered by Brady and Acrivos [5] and the flow outside a stretching tube by Wang [6]. The three-dimensional and axisymmetric stretching flat surface was studied by Wang [7]. The unsteady stretching sheet was investigated by Wang [8] and Usha and Sridharan [9].

In recent years, the boundary layer flow due to a shrinking sheet has attracted considerable attention. The unsteady viscous flow induced by a shrinking sheet was first studied by Wang [8]. The proof of the existence and (non)uniqueness, and the exact solutions, both numerical and in closed form, were given by Miklavcic and Wang [10] for the steady viscous hydrodynamic flow due to a shrinking sheet for a specific value of the suction parameter. Miklavcic and Wang [10] concluded that the solution for shrinking sheets may not be unique at certain suction rates for both 2-dimensional and axisymmetric flows. Wang [11] investigated the stagnation flow towards a shrinking sheet and found for the first time that non-alignment of the stagnation flow and the shrinking of the sheet destroys the symmetry and complicates the flow field. Furthermore, Faraz *et al*. [12] obtained the analytical solution of a 2-dimensional viscous flow due to a shrinking sheet by using variational iteration algorithm-II (VIM-II) and Adomian decomposition method (ADM).

In this paper, the HPSTM basically illustrates how the Sumudu transform can be used to approximate the solutions of the nonlinear equations by manipulating the homotopy perturbation method (HPM). The perturbation methods which are generally used to solve nonlinear problems have some limitations e.g., the approximate solution involves a series of small parameters, which causes difficulty since the majority of nonlinear problems have no small parameters at all. Although appropriate choices of small parameters sometime leads to ideal solutions, in most cases, unsuitable choices lead to serious effects in the solutions. The HPM was first introduced by He [13]. The HPM was also studied by many authors to handle linear and nonlinear equations arising in physics and engineering [14-19]. In a recent paper, Singh, Kumar and Sushila [20] have paid attention in studying the solutions of linear and nonlinear partial differential equations by using a combined form of HPM and Sumudu transform named homotopy perturbation Sumudu transform method (HPSTM). The HPSTM is a combination of Sumudu transform method, HPM and He's polynomials, and credit for it is mainly due to Ghorbani [21,22].

The objective of this paper is to present a simple recursive algorithm based on the new HPSTM which produces the series solution of the two-dimensional viscous flow due to a shrinking sheet. The difficulty of the condition at infinity is overcome by the use of Padé approximants. The velocity profiles given by HPSTM are in good agreement with the VIM-II solutions given in [12]. For best approximation, the resulting series is best manipulated by Padé approximants. The advantage of this technique is its capability of combining two powerful methods for obtaining exact and approximate analytical solutions for nonlinear equations. It is worth mentioning that the proposed method is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result; the size reduction amounts to an improvement of the performance of the approach.

Padé approximants

Padé approximants constitute the best approximation of a function by a rational function of a given order. The Padé approximants developed by Henri Padé, often provide better approximation of a function than does truncating its Taylor Series, and they may still work in cases in which the Taylor Series does not converge. Due to these reasons, Padé approximants are used extensively in computer calculations, and it is now well known that these approximants have the advantage of being able to manipulate polynomial approximation into the rational functions of polynomials. In addition, power series are not useful for large values of a variable, say $n \to \infty$, which can be attributed to the possibility of the radius of convergence not being sufficiently large to contain the boundaries of the domain. To provide an effective tool that can handle boundary value problems on an infinite or semi-infinite domain, it is therefore essential to combine the series solution, which is obtained by the iteration method or any other series solution method, with the Padé approximants [23].

Sumudu transform

In early 90's, Watugala [24] introduced a new integral transform, named the Sumudu transform, and applied it to the solution of ordinary differential equations in control engineering problems. The Sumudu transform is defined over the set of functions.

$$A = \{f(t) \mid \exists M, \tau_1, \tau_2 > 0, \mid f(t) \mid < M \, e^{|t|/\tau_j}, \text{if } t \in (-1)^j \times [0, \infty)\}$$

by the following formula.

$$\bar{f}(u) = S[f(t)] = \int_0^\infty f(ut) \, e^{-t} \, dt, u \in (-\tau_1, \tau_2). \tag{1}$$

Some of the properties of the Sumudu transform were established by Asiru [25]. Further, fundamental properties of this transform were established by Belgacem *et al.* [26]. It was shown that there

is strong relationship between Sumudu and other integral transforms, see Kilicman *et al.* [27]. In particular the relation between Sumudu transforms and Laplace transforms was proved in Kilicman and Eltayeb [28]. The Sumudu transform has scale and unit preserving properties, so it can be used to solve problems without resorting to a new frequency domain.

Basic idea of HPSTM

In order to illustrate the solution procedure of the HPSTM, the following general form of 3rd order non-homogenous nonlinear ordinary differential equation with the initial conditions is given by;

$$f''' + b_1(x)f'' + b_2(x)f' + b_3(x)f = g(f), \tag{2}$$

$$f(0) = \alpha, \ f'(0) = \beta, \ f''(0) = \gamma. \tag{3}$$

By applying the Sumudu transform on both sides of Eq. (2) gives;

$$S[f'''] + S[b_1(x)f'' + b_2(x)f' + b_3(x)f] = S[g(f)]. \tag{4}$$

Using the differentiation property of the Sumudu transform gives;

$$S[f] = \alpha + \beta u + \gamma u^2 + u^3 S[g(f)] - u^3 S[b_1(x)f'' + b_2(x)f' + b_3(x)f]. \tag{5}$$

Operating with the Sumudu inverse on both sides of Eq. (5) gives;

$$f = G(x) + S^{-1}\left[u^3 S[g(f)]\right] - S^{-1}\left[u^3 S[b_1(x)f'' + b_2(x)f' + b_3(x)f]\right], \tag{6}$$

where $G(x)$ represents the term arising from the source term and the prescribed initial conditions. Now the HPM is applied.

$$f = \sum_{n=0}^{\infty} p^n f_n, \tag{7}$$

and the nonlinear term can be decomposed as;

$$g(f) = \sum_{n=0}^{\infty} p^n H_n, \tag{8}$$

and for some He's polynomials H_n [22,29] that are given by;

$$H_n = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[g\left(\sum_{i=0}^{\infty} p^i f_i\right)\right]_{p=0}, \ n = 0,1,2,3,... \tag{9}$$

Substituting Eq. (7) and Eq. (8) in Eq. (6) gives;

$$\sum_{n=0}^{\infty} p^n f_n = G(x) + p\left(S^{-1}\left[u^3 S\left[\sum_{n=0}^{\infty} p^n H_n \right] \right] \right)$$

$$- p\left(S^{-1}\left[u^3 S\left[b_1(x)\sum_{n=0}^{\infty} p^n f_n^{"} + b_2(x)\sum_{n=0}^{\infty} p^n f_n^{'} + b_3(x)\sum_{n=0}^{\infty} p^n f_n \right] \right] \right), \qquad (10)$$

which is the coupling of the Sumudu transform and the HPM using He's polynomials. Comparing the coefficients of like powers of p, the following approximations are obtained.

$$p^0 : f_0 = G(x),$$

$$p^1 : f_1 = S^{-1}\left[u^3 S\left[H_0 \right] \right] - S^{-1}\left[u^3 S \left[b_1(x) f_0^{"} + b_2(x) f_0^{'} + b_3(x) f_0 \right] \right], \qquad (11)$$

$$p^2 : f_2 = S^{-1}\left[u^3 S[H_1] \right] - S^{-1}\left[u^3 S \left[b_1(x) f_1^{"} + b_2(x) f_1^{'} + b_3(x) f_1 \right] \right],$$

$$\vdots$$

Proceeding in this same manner, the rest of the components $f_n(\eta)$ can be completely obtained and the series solution is thus entirely determined. Finally, the analytical solution $f(\eta)$ by truncated series is obtained.

$$f(\eta) = \lim_{N \to \infty} \sum_{n=0}^{N} f_n(\eta).$$

The above series solutions generally converge very rapidly. A classical approach of convergence of this type of series has already been presented by Abbaoui and Cherruault [30].

Mathematical formulation

In this paper, the 2 basic equations of fluid mechanics in Cartesian coordinates are considered. The continuity equation and momentum equations for viscous flow are;

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0, \qquad (12)$$

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right), \qquad (13)$$

$$U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + W\frac{\partial V}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right), \qquad (14)$$

$$U\frac{\partial W}{\partial x} + V\frac{\partial W}{\partial y} + W\frac{\partial W}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} \right), \qquad (15)$$

where $\nu = \mu/\rho$ is the kinematic viscosity.

The boundary conditions applicable to the present flow are;

$$U = -ax, \quad V = -a(m-1)y, \quad W = -w \quad at \quad y = 0,$$
$$U \to 0 \quad as \quad y \to \infty. \tag{16}$$

For shrinking phenomenon, a > 0 is the shrinking constant and w is the suction velocity. m = 1 when the sheet shrinks in the x-direction and m=2 when it shrinks axisymmetrically. Introducing the following similarity transformations;

$$U = axf^{'}(\eta), \qquad V = a(m-1)yf^{'}(\eta), \qquad \eta = \sqrt{\frac{a}{v}}z. \tag{17}$$

Eq. (12) is identically satisfied. Eq. (15) can be integrated to give;

$$\frac{p}{\rho} = v\frac{\partial W}{\partial z} - \frac{W^2}{2} + Cons\tan t. \tag{18}$$

Eqs. (13), (14) and (16) are reduced to the boundary value problem;

$$f^{'''} - (f^{'})^2 + mff^{''} = 0, \tag{19}$$

$$f = s, \quad f^{'} = -1 \quad at \quad \eta = 0,$$
$$f^{'} \to 0 \quad as \quad \eta \to \infty, \tag{20}$$

where $s = w/m\sqrt{av}$.

HPSTM solution and discussion

In this section, the HPSTM is applied to obtain an approximate analytical solution of (19) - (20). By applying the Sumudu transform on both sides of Eq. (19),

$$S[f(\eta)] = 2 - u + \alpha u^2 + u^3 S[(f^{'})^2 - 2ff^{''}], \tag{21}$$

is obtained, where $f^{''}(0) = \alpha, m = 2 \ and \ s = 2$.

The inverse Sumudu transform implies that;

$$f(\eta) = 2 - \eta + \frac{\alpha\eta^2}{2} + S^{-1}\left[u^3 S\left[(f')^2 - 2ff^{''}\right]\right] \tag{22}$$

Now applying the HPM,

$$\sum_{n=0}^{\infty} p^n f_n(\eta) = 2 - \eta + \frac{\alpha\eta^2}{2} + p\left(S^{-1}\left[u^3 S\left[\left(\sum_{n=0}^{\infty} p^n H_n(\eta)\right) - 2\left(\sum_{n=0}^{\infty} p^n H_n^{'}(\eta)\right)\right]\right]\right), \tag{23}$$

is obtained, where $H_n(\eta)$ and $H_n^{'}(\eta)$ are He's polynomials [22,29] that represents the nonlinear terms. So, He's polynomials are given by;

$$\sum_{n=0}^{\infty} p^n H_n(\eta) = (f')^2(\eta). \tag{24}$$

The first few components of He's polynomials are given by;

$$H_0(\eta) = (f_0')^2(\eta),$$
$$H_1(\eta) = 2f_0'(\eta)f_1'(\eta),$$
$$H_2(\eta) = (f_1')^2(\eta) + 2f_0'(\eta)f_2'(\eta),$$
$$H_3(\eta) = 2f_1'(\eta)f_2'(\eta) + 2f_0'(\eta)f_3'(\eta), \tag{25}$$
$$\vdots$$
$$H_n(\eta) = \sum_{i=0}^{n} f_i'(\eta)f_{n-i}'(\eta).$$

And for $H_n'(\eta)$;

$$\sum_{n=0}^{\infty} p^n H_n'(\eta) = f(\eta)f''(\eta), \tag{26}$$

$$H_0'(\eta) = f_0(\eta)f_0''(\eta),$$

$$H_1'(\eta) = f_0(\eta)f_1''(\eta) + f_1(\eta)f_0''(\eta),$$

$$H_2'(\eta) = f_0(\eta)f_2''(\eta) + f_1(\eta)f_1''(\eta) + f_2(\eta)f_0''(\eta), \tag{27}$$

$$H_3'(\eta) = f_0(\eta)f_3''(\eta) + f_1(\eta)f_2''(\eta) + f_2(\eta)f_1''(\eta) + f_3(\eta)f_0''(\eta),$$

$$\vdots$$

$$H_n'(\eta) = \sum_{i=0}^{n} f_i(\eta)f_{n-i}''(\eta).$$

is obtained. Comparing the coefficients of like powers of p, the following is seen.

$$p^0 : f_0 = 2 - \eta + \frac{\alpha\eta^2}{2}, \tag{28}$$

$$p^1 : f_1 = \frac{\eta^3}{6} - \frac{2\alpha\eta^3}{3}, \tag{29}$$

$$p^2 : f_2 = -\frac{\eta^4}{6} + \frac{2\alpha\eta^4}{3} + \frac{\eta^5}{60} - \frac{\alpha\eta^5}{15} + \frac{\alpha\eta^6}{360} - \frac{\alpha^2\eta^6}{90}, \tag{30}$$

$$p^3 : f_3 = \frac{2\eta^5}{15} - \frac{8\alpha\eta^5}{15} - \frac{\eta^6}{30} + \frac{2\alpha\eta^6}{15} + \frac{\eta^7}{504} - \frac{\alpha\eta^7}{315} - \frac{2\alpha^2\eta^7}{105}$$
$$- \frac{\alpha\eta^8}{5040} + \frac{\alpha^2\eta^8}{1260} - \frac{\alpha^2\eta^9}{9072} + \frac{\alpha^3\eta^9}{2268}, \tag{31}$$

$$p^4 : f_4 = -\frac{4\eta^6}{45} + \frac{16\alpha\eta^6}{45} + \frac{4\eta^7}{105} - \frac{16\alpha\eta^7}{105} - \frac{\eta^8}{360} - \frac{\alpha\eta^8}{126}$$

$$+\frac{8\alpha^2\eta^8}{105}-\frac{\eta^9}{90720}+\frac{37\alpha\eta^9}{11340}-\frac{73\alpha^2\eta^9}{5670}$$

$$-\frac{47\alpha\eta^{10}}{30400}+\frac{37\alpha^2\eta^{10}}{56700}-\frac{\alpha^3\eta^{10}}{8100}-\frac{\alpha^2\eta^{11}}{178200}$$

$$+\frac{\alpha^3\eta^{11}}{44550}+\frac{\alpha^3\eta^{12}}{213840}-\frac{\alpha^4\eta^{12}}{53460},\tag{32}$$

$\vdots$

and so on. In this way, the remaining terms of the HPSTM series can be calculated.

The series solution is given by;

$$f = f_0 + f_1 + f_2 + f_3 + f_4 + \cdots.\tag{33}$$

Substituting Eqs. (28)- (32) into Eq. (33), the following series solution is obtained.

$$f(\eta)=2-\eta+\frac{\alpha\eta^2}{2}+\frac{\eta^3}{6}-\frac{2\alpha\eta^3}{3}-\frac{\eta^4}{6}+\frac{2\alpha\eta^4}{3}+\frac{3\eta^5}{20}-\frac{3\alpha\eta^5}{5}$$

$$-\frac{11\eta^6}{90}+\frac{59\alpha\eta^6}{120}-\frac{\alpha^2\eta^6}{90}+\frac{101\eta^7}{2520}-\frac{7\alpha\eta^7}{45}-\frac{2\alpha^2\eta^7}{105}$$

$$-\frac{\eta^8}{360}-\frac{41\alpha\eta^8}{5040}+\frac{97\alpha^2\eta^8}{1260}-\frac{\eta^9}{90720}+\frac{37\alpha\eta^9}{11340}$$

$$-\frac{589\alpha^2\eta^9}{45360}+\frac{\alpha^3\eta^9}{2268}-\frac{47\alpha\eta^{10}}{302400}+\frac{37\alpha^2\eta^{10}}{56700}-\frac{\alpha^3\eta^{10}}{8100}$$

$$-\frac{\alpha^2\eta^{11}}{178200}+\frac{\alpha^3\eta^{11}}{44550}+\frac{\alpha^3\eta^{12}}{213840}-\frac{\alpha^4\eta^{12}}{53460}+\cdots.\tag{34}$$

Table 1 The numerical values for $f^{''}=\alpha$ using Padé approximation.

Padé approximation	$f^{''}=\alpha$ **for VIM-II [12]**	$f^{''}=\alpha$ **for HPSTM**
[1/1]	0.224748	0.292893
[2/2]	Complex number	Complex number
[3/3]	0.294748	Complex number
[4/4]	0.308086	0.247723
[5/5]	0.249556	0.24893

Software packages such as Maple or Mathematica can be used to solve the polynomials $f'(\eta)$ to calculate the value of α with the help of boundary condition $f'(\eta)\to 0$, for $n\to\infty$. By using the table above, the value of α = $f'(\eta)\to 0 = 0.249243$ can be chosen for HPSTM and VIM-II solutions, which is an average value of [5/5] Padé approximation (**Table 1**).

The numerical results of HPSTM are depicted in **Figure 1**. **Figure 2** compares the solutions obtained by the present method and VIM-II [12].
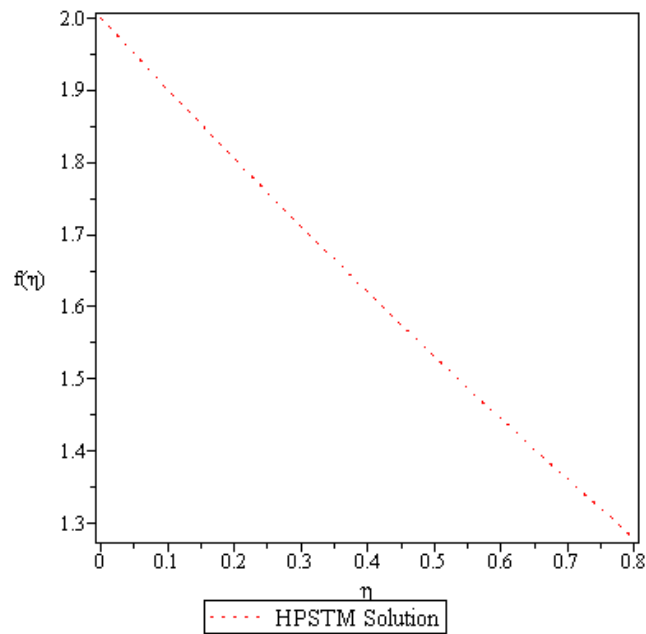
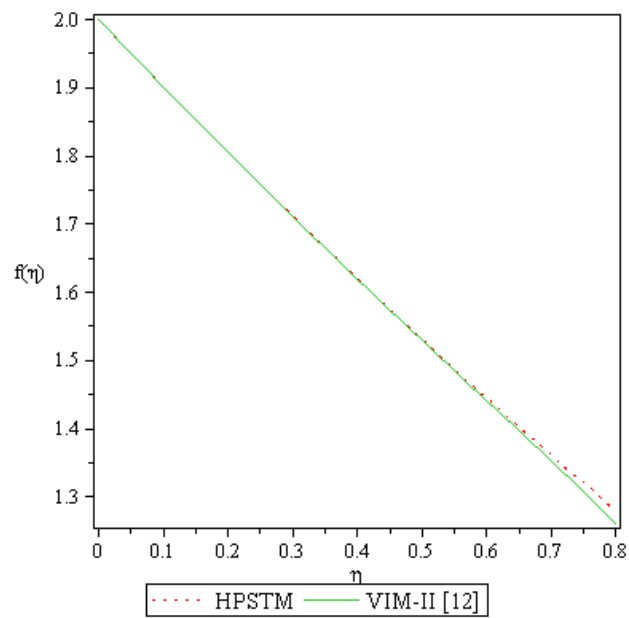**Figure 1** Graphical presentation of HPSTM solution, when α = 0.249243, m = 2 and s = 2.

Figure 2 Comparison of HPSTM solution and VIM-II solution [12], when α = 0.249243, m = 2 and s = 2.

Conclusions

In this paper, a simple algorithm based on the HPSTM-Padé approach has been successfully applied to solve the viscous flow due to a shrinking sheet. The method is applied here in direct manner without the use of linearization, transformation, discretization, perturbation, or restrictive assumptions. This study has considered only an axisymmetrically shrinking sheet by taking m = 2. The results compare very well with the results obtained by VIM-II [12]. The results show that the HPSTM is a powerful and efficient technique in finding exact and approximate solutions for nonlinear differential equations. The proposed method's ability to solve nonlinear problems without the use of Adomian polynomials is evidence of its clear advantage over the decomposition method. In conclusion, the HPSTM could be a promising tool for solving more complex boundary equations than the one studied in this paper.

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