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New approach to solve symmetric fully fuzzy linear systems

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Abstract. In this paper, we present a method to solve fully fuzzy linear systems with symmetric coefficient matrix. The symmetric coefficient matrix is decomposed into two systems of equations by using Cholesky method and then a solution can be obtained. Numerical examples are given to illustrate our method.

Keywords. Fully fuzzy linear system; fuzzy number; fuzzy linear systems; symmetric fuzzy linear systems; fuzzy set.

1. Introduction

System of simultaneous linear equations plays a major role in various areas such as Mathematics, Physics, Statistics, Engineering and Social sciences. Since in many applications at least some of the system's parameters and measurements are represented by fuzzy rather than crisp numbers, it is important to develop mathematical models and numerical procedures that would appropriately treat general fuzzy linear systems. In practice, the parameters of the mathematical model are not always exactly known and stable. This imprecision may follow from the lack of exact information or may be a consequence of a certain flexibility the given enterprise has at planning capacities. A frequently used means to express the imprecision are the fuzzy numbers.

A general model for solving a fuzzy linear system whose coefficient matrix is crisp and the right hand side column is an arbitrary fuzzy vector, was first proposed by Friedman *et al* (1998) and many authors consider these models for their studies (Nasseri & Khorramizadeh (2007), Senthilkumar & Rajendran (2009)). There is a difference between a fuzzy linear system and the fully fuzzy linear system. The coefficient matrix is treated as crisp in the fuzzy linear system, but in the fully fuzzy linear system all the parameters are considered to be fuzzy numbers. Many authors worked the case in which all parameters in a fuzzy linear system are fuzzy numbers, which we call it a fully fuzzy linear system of equations, for example see Dehghan *et al* (2006), Muzzioli & Reynaerts (2006), Rao & Chen (1998). Dehghan *et al* have studied some direct methods for solving fully fuzzy linear system. They have represented fuzzy numbers in LR (left-to-right) form and applied approximately operators between fuzzy numbers. In their approach, finding the solution of fully fuzzy linear systems is transformed to find the solutions of three

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crisp systems. Also the result of multiplying two triangular fuzzy numbers is a triangular fuzzy number which is not a good approximation. Muzzioli & Reynaerts (2006) have considered fully fuzzy linear systems only in the form of $A_1x + b_1 = A_2x + b_2$ with A_1 and A_2 are two square matrices of fuzzy numbers and b_1 and b_2 are fuzzy number vectors.

We will solve $\tilde{A} \otimes \tilde{x} = \tilde{b}$, where $\tilde{A}$ is a symmetric fuzzy matrix $\tilde{x}$ and $\tilde{b}$ are fuzzy vectors. We will use fuzzy matrix defined in Dubois & Prade (1980). This class of fuzzy matrices consists of applicable matrices, which can model uncertain aspects and the works for them are too limited. Some of the interesting works on these matrices can be seen in Matinfar *et al* (2008), Senthilkumar & Rajendran (2008).

The structure of this paper is organized as follows: In section 2, we first give some basic concepts of fuzzy set theory and then define a fully fuzzy linear system of equations. A numerical method for computing the solution of the fully fuzzy linear system is designed in section 3. Numerical examples are given in section 4 to examine our method.

2. Preliminaries

In this section, we review some necessary backgrounds and notions of fuzzy sets theory (Dubois & Prade (1980), Matinfar *et al* (2008)).

Definition 2.1. A fuzzy subset $\tilde{A}$ of R is defined by its membership function $\mu_{\tilde{A}}: R \to [0, 1]$, which assigns a real number $\mu_{\tilde{A}}$ in the interval [0, 1], to each element $x \in R$, where the value of $\mu_{\tilde{A}}$ at x shows the grade of membership of x in $\tilde{A}$.

Definition 2.2. A fuzzy set with the following membership function is named as the triangular fuzzy number and in this paper we will use these fuzzy numbers.

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - \dfrac{m-x}{\alpha}, m - \alpha \le x < m, \alpha > 0, \\ 1 - \dfrac{x-m}{\beta}, m \le x < m + \beta, \beta > 0, \\ \quad 0, otherwise \end{cases} \tag{1}$$

Definition 2.3. A fuzzy number $\tilde{A}$ is called positive (negative), denoted by $\tilde{A} > 0$ ($\tilde{A} < 0$), if its membership function $\mu_{\tilde{A}}(x)$ satisfies $\mu_{\tilde{A}}(x) = 0, \forall x \le 0$ ($\forall x \ge 0$).

Using its mean value and left and right spreads, and shape functions, such a fuzzy number $\tilde{A}$ is symbolically written $\tilde{A} = (m, \alpha, \beta)$. Clearly, $\tilde{A} = (m, \alpha, \beta)$ is positive, if and only if $m - \alpha \ge 0$.

Remark 2.1. We consider $\tilde{0} = (0, 0, 0)$ as a zero triangular fuzzy number.

Remark 2.2. We show the set of all triangular fuzzy numbers by F(R).

Definition 2.4. (Equality in fuzzy numbers). Two fuzzy numbers $M = (m, \alpha, \beta)$ and $N = (n, \gamma, \delta)$ are said to be equal, if and only if $m = n$, $\alpha = \gamma$ and $\beta = \delta$.

Definition 2.5. For two fuzzy numbers M = (m, α, β) and N = (n, γ, δ) the formula for the extended addition becomes:

$$(\mathrm{m}, \alpha, \beta) \oplus (\mathrm{n}, \gamma, \delta) = (\mathrm{m} + \mathrm{n}, \alpha + \gamma, \beta + \delta)\,. \tag{2}$$

The formula for the extended opposite becomes:

$$-\mathrm{M} = -(\mathrm{m},\ \alpha,\ \beta) = (-\mathrm{m},\ \beta,\ \alpha)\,. \tag{3}$$

The approximate formula for the extended multiplication of two fuzzy numbers can be summarized as follows as given in Muzzioli & Reynaerts (2006):

$$\text{If } \mathrm{M} > 0 \text{ and } \mathrm{N} > 0, \text{ then } (\mathrm{m},\ \alpha,\ \beta) \otimes (\mathrm{n},\ \gamma,\ \delta) = (\mathrm{mn}, \mathrm{m}\gamma + \mathrm{n}\alpha, \mathrm{m}\delta + \mathrm{n}\beta)\,. \tag{4}$$

For scalar multiplication:

$$\lambda \otimes (\mathrm{m},\ \alpha,\ \beta) = \begin{cases} (\lambda \mathrm{m},\ \lambda\alpha,\ \lambda\beta)\,, \lambda \geq 0 \\ (\lambda \mathrm{m},\ -\lambda\alpha,\ -\lambda\beta)\,, \lambda < 0. \end{cases} \tag{5}$$

Definition 2.6. A matrix $\tilde{A} = (\tilde{a}_{ij})$ is called a fuzzy matrix, if each element of $\tilde{A}$ is a fuzzy number. A fuzzy matrix $\tilde{A}$ will be positive and denoted by $\tilde{A} > 0$, if each element of $\tilde{A}$ be positive. We may represent n × n fuzzy matrix $\tilde{A} = (\tilde{a}_{ij})_{n\,x\,n}$, such that $\tilde{a}_{\mathrm{ij}} = (\mathrm{a}_{ij}, \alpha_{ij}, \beta_{ij})$, with the new notation A = (A, M, N), where A = $(\mathrm{a}_{\mathrm{ij}})$, M = (α_{ij}) and N = (β_{ij}) are three n × n crisp matrices.

Definition 2.7. A square fuzzy matrix $\tilde{A} = (\tilde{a}_{ij})$ will be an upper triangular fuzzy matrix, if $\tilde{a}_{ij} = \tilde{0} = (0, 0, 0), \forall i > j$, and a square fuzzy matrix $\tilde{A} = (\tilde{a}_{ij})$ will be a lower triangular fuzzy matrix, if $\tilde{a}_{ij} = \tilde{0} = (0, 0, 0), \forall i < j$.

Definition 2.8. Consider the n × n fuzzy linear system of equations (Dubois & Prade (1980), Nasseri & Khorramizadeh (2007)):

$$\begin{array}{c} (\tilde{a}_{11} \otimes \tilde{x}_1) \oplus (\tilde{a}_{12} \otimes \tilde{x}_2) \oplus \ldots\ldots \oplus (\tilde{a}_{1n} \otimes \tilde{x}_n) = \tilde{b}_1, \\ (\tilde{a}_{21} \otimes \tilde{x}_1) \oplus (\tilde{a}_{22} \otimes \tilde{x}_2) \oplus \ldots\ldots \oplus (\tilde{a}_{2n} \otimes \tilde{x}_n) = \tilde{b}_2, \\ . \\ . \\ (\tilde{a}_{n1} \otimes \tilde{x}_1) \oplus (\tilde{a}_{n2} \otimes \tilde{x}_2) \oplus \ldots\ldots \oplus (\tilde{a}_{nn} \otimes \tilde{x}_n) = \tilde{b}_n. \end{array} \tag{6}$$

The matrix form of the above equations is

$$\tilde{A} \otimes \tilde{\mathrm{x}} = \tilde{b},$$

where the coefficient matrix $\tilde{A} = (\tilde{a}_{ij})$, 1 ≤ i, j ≤ n is a nxn fuzzy matrix and $\tilde{x}_{\mathrm{j}}, \tilde{b}_{\mathrm{j}} \in \mathrm{F(R)}$. This system is called a fully fuzzy linear system.

In this paper, we are going to obtain a positive solution of a fully fuzzy linear system $\tilde{A}\,\tilde{\mathrm{x}} = \tilde{b}$, where $\tilde{A}$ = (A,M,N) > 0, $\tilde{b}$ = (b, g, h) > 0 and $\tilde{x}$ = (x, y, z) > 0. So we have (A, M, N) ⊗ (x, y, z) = (b, g, h).
Then by using Eq. (4) we have (Ax, Ay + Mx, Az + Nx) = (b, h, g).

Therefore, Definition 2.4 concludes that (Nasseri *et al* (2008))

$$\begin{aligned} Ax &= b, \\ Ay + Mx &= g \\ Az + Nx &= h. \end{aligned} \tag{7}$$

So, by assuming that A is a non-singular matrix we have

$$\begin{aligned} &Ax = b && => \ x = A^{-1}b, \\ &Ay = g - Mx && => \ y = A^{-1}(g - Mx), \\ &Az = h - Nx && => \ z = A^{-1}(h - Nx). \end{aligned}$$

3. A decomposition method for solving fully fuzzy linear systems

Theorem 3.1
Let A be a n $\times$ n symmetric matrix with all non-zero leading principal minors. Then A has a unique factorization: $A = LL^T$, where L is a unit lower triangular matrix.

Assume that $\tilde{A} = (A,M,N)$, where A is a full rank crisp matrix. Then if we let $(L, 0, 0) \otimes (L^T, P, Q) = (A, M,N)$.

Then,

$$\begin{aligned} &LL^T = A => L^T = L^{-1}A \\ &LP = M => P = L^{-1}M \\ &LQ = N => Q = L^{-1}N, \ \text{where L is a lower triangular crisp matrix.} \end{aligned}$$

Now, again consider the fully fuzzy linear system

$$\begin{aligned} &(A,\ M,\ N) \otimes (x,\ y,\ z) \ = \ (b,\ g,\ h) \\ &\left(LL^T,\ LP,\ LQ\right) \otimes (x,\ y,\ z) = (b,\ g,\ h) \\ &\left(LL^Tx,\ LL^Ty + LPx,\ LL^Tz + LQx\right) = (b,\ g,\ h). \end{aligned}$$

By solving, we get

$$\begin{aligned} x &= \left(L^T\right)^{-1} L^{-1}b \\ y &= \left(L^T\right)^{-1} L^{-1}(g - LPx) \\ z &= \left(L^T\right)^{-1} L^{-1}(h - LQx). \end{aligned} \tag{8}$$

A new algorithm to solve the fully fuzzy linear system with symmetric coefficient matrix is presented below.

3.1 *Algorithm*

Step 1.
Assume that $\tilde{A} = (A,M,N)$, where A is a full rank crisp matrix. Compute decomposition for crisp symmetric matrix A as $A = L\,L^T$.

Step 2.
Compute $P = L^{-1}M$.

Step 3.

Compute $Q = L^{-1}N$.

Step 4.

Compute the solution of the fully fuzzy linear system $(A, M, N) \otimes (x, y, z) = (b, g, h)$ as follows:

$$\begin{aligned} x &= \left(L^T\right)^{-1} L^{-1} b \\ y &= \left(L^T\right)^{-1} L^{-1} (g - LPx) \\ z &= \left(L^T\right)^{-1} L^{-1} (h - LQx) \,. \end{aligned}$$

4. Numerical examples

In this section, we apply our algorithm for solving two fully fuzzy linear systems to illustrate our method.

Example 1. The omega manufacturing company has decided to produce three products namely Product 1, Product 2 and Product 3. The available capacity of the machines that might limit output is summarized below.

Machine type	Available time (Machine hours per month)
Milling Machine	(124,178,320)
Lathe	(495,741,1222)
Grinder	(890,1349,2164)

The number of machine hours required for each unit of the respective product is given below.

Product coefficient (in machine hours per unit)			
Machine type	Product 1	Product 2	Product 3
Milling Machine	(4,2,6)	(12,12,14)	(18,16,20)
Lathe	(12,10,14)	(45,45,50)	(78,74,80)
Grinder	(18,16,18)	(78,75,80)	(146,146,150)

Now, to determine how much of each product should produce to utilize the entire available time.

To represent the above problem as fully fuzzy linear system, we represent x as a quantity of the product 1 that will be produced during the month. Similarly, y and z represent the quantity of products 2 and 3 respectively.

The corresponding fully fuzzy linear system for the above problem as

$$\begin{pmatrix} (4,\ 2,\ 6) & (12,\ 12,\ 14) & (18,\ 16,\ 20) \\ (12,\ 10,\ 14) & (45,\ 45,\ 50) & (78,\ 74,\ 80) \\ (18,\ 16,\ 18) & (78,\ 75,\ 80) & (146,\ 146,\ 150) \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} (124,\ 178,\ 320) \\ (495,\ 741,\ 1222) \\ (890,\ 1349,\ 2164) \end{pmatrix}.$$

Now, we solve the above system by using the algorithm 3.1.

First, we obtain decomposition for symmetric matrix A as follows:

$$\begin{pmatrix} 4 & 12 & 18 \\ 12 & 45 & 78 \\ 18 & 78 & 146 \end{pmatrix} = LL^T = \begin{pmatrix} 2 & 0 & 0 \\ 6 & 3 & 0 \\ 9 & 8 & 1 \end{pmatrix} \begin{pmatrix} 2 & 6 & 9 \\ 0 & 3 & 8 \\ 0 & 0 & 1 \end{pmatrix}.$$

So, we obtain matrices P and Q from steps 2 and 3 in Algorithm as follows:

$$\mathrm{P} = \mathrm{L}^{-1}\mathrm{M} = \begin{pmatrix} 1 & 6 & 8 \\ 1.333 & 2.9985 & 8.6642 \\ -3.667 & -3.0015 & 4.6642 \end{pmatrix}$$

$$\mathrm{Q} = \mathrm{L}^{-1}\mathrm{N} = \begin{pmatrix} 3 & 7 & 10 \\ 1.3338 & 2.665 & 6.664 \\ 1.6662 & -4.335 & 6.664 \end{pmatrix}.$$

Therefore, Equation (8) concludes that

$$\tilde{x} = \left(4,\ 1,\ 3\right) \tag{9}$$

$$\tilde{y} = \left(3,\ 4,\ 6\right) \tag{10}$$

$$\tilde{z} = \left(4,\ 1,\ 5\right). \tag{11}$$

Example 2. A person loves Steaks, Potatoes and Milk product. Therefore he has decided to go on a steady diet of only these three foods for all his meals. He realizes that this is not the healthiest diet, so he wants to make sure that he eats the right quantities of the three foods to satisfy some key nutritional requirements. He has obtained the following nutritional and cost information.

Ingredient	Grams of ingredient per serving			Daily requirement (Grams)
	Steak	*Potatoes*	*Milk product*	
Carbohydrates	(4,1,4)	(6,2,2)	(10,2,1)	(58,30,60)
Protein	(6,8,20)	(25,12,15)	(23,8,10)	(142,139,257)
Fat	(10,2,1)	(23,8,10)	(33,19,24)	(316,297,514)

Now, to determine the number of daily servings of steak, potatoes and milk products that will meet the above requirements.

To represent the above problem as fully fuzzy linear system, we represent x as a quantity of steak that will be incurred daily. Similarly, y and z represent the quantity of potatoes and milk product incurred daily respectively.

$$\begin{pmatrix} (4,\ 1,\ 4) & (6,\ 2,\ 2) & (10,\ 2,\ 1) \\ (6,\ 8,\ 20) & (25,\ 12,\ 15) & (23,\ 8,\ 10) \\ (10,\ 2,\ 1) & (23,\ 8,\ 10) & (33,\ 19,\ 24) \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} (58,\ 30,\ 60) \\ (142,\ 139,\ 257) \\ (316,\ 297,\ 514) \end{pmatrix}$$

First we obtain decomposition for symmetric matrix A as follows:

$$\begin{pmatrix} 4 & 6 & 10 \\ 6 & 25 & 23 \\ 10 & 23 & 33 \end{pmatrix} = LL^T = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 5 & 2 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 & 5 \\ 0 & 4 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

So, we obtain matrices P and Q from steps 2 and 3 in Algorithm as follows:

$$\mathrm{P} = \mathrm{L}^{-1}\mathrm{M} = \begin{pmatrix} 0.5 & 1 & 1 \\ 1.625 & 2.25 & 1.25 \\ -1.875 & -0.75 & 5.75 \end{pmatrix}$$

$$\mathrm{Q} = \mathrm{L}^{-1}\mathrm{N} = \begin{pmatrix} 2 & 1 & 0.5 \\ 3.5 & 3 & 2.125 \\ -8 & -0.5 & 8.625 \end{pmatrix}.$$

Therefore, Equation (8) concludes that

$$\begin{aligned} \tilde{x} &= (\,-53.4346,\;\; 174.4964,\;\; 521.6268\,) \\ \tilde{y} &= (\,-14.5,\;\; 81.9897,\;\; 199.9839\,) \\ \tilde{z} &= (\,35.875,\;\; -114.9231,\;\; -301.9493\,). \end{aligned}$$

5. Conclusion

Fully fuzzy linear systems can be solved by Linear programming approach, Gauss elimination method, Cramer's rule, etc. (Dehghan *et al* 2006). These computational methods have various disadvantages like number of iterations, triangularisation, finding more number of determinants, etc. to solve the fully fuzzy linear system. To overcome these drawbacks, we have introduced a new method for solving fully fuzzy linear systems based on Cholesky decomposition method. In this method, we decomposed the symmetric coefficient matrix into two matrices and then the solution was obtained within three steps.

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