New aspects on Current enhancement in Brownian motors driven by non Gaussian noises

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Abstract

Recent studies on Brownian motors driven by colored non Gaussian noises have shown that the departure of the noise distribution from Gaussian behavior induces an enhancement of its current and efficiency. Here we discuss some new aspects of this phenomenon focusing in some analytical results based in an adiabatic approximation, and in the analysis of the long probability distribution tails' role.

Key words: Brownian motors, non Gaussian noises, current enhancement
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1 Introduction

Recent studies on the role of non Gaussian noises on several "noise-induced phenomena", like stochastic resonance, resonant trapping, and noise-induced transitions [1,2,3,4,5,6,7], have shown the existence of strong effects on the system's response. The form of the noise source used was based on the nonextensive statistics [8,9,10] with a probability distribution that depends on q, a parameter indicating the departure from Gaussian behavior: for $q = 1$ the distribution is Gaussian, while different non Gaussian distributions result for $q > 1$ or $q < 1$. What was observed was an enhancement of the signal-to-noise ratio in stochastic resonance, an enhancement of the trapping current in resonant trapping, and a marked shift in the transition line of noise-induced transitions.

Those studies motivated us to also analyze the effect of non Gaussian noises on the behavior and transport properties of Brownian motors [11]. As is well known the study of noise induced transport by "ratchets" has attracted in recent years the attention of an increasing number of researchers due to its biological interest and also to its potential technological applications [12,13,14,15].

In [11] we have shown that, under certain general conditions, non Gaussian noise induces current and efficiency enhancement, without the need of a fine tuning of the parameters. Also we have found the phenomenon of current inversion as the parameter q (associated to the non Gaussian properties of the noise) is varied. The analysis of the effects of a colored non Gaussian noise source on the transport properties of a Brownian motor was done by considering two alternative points of view: one which can be interpreted as more directly connected to technological applications and the other more related to biological or natural systems [11]. In the model analyzed, these two visions correspond to consider as control parameters, different (but related) constants that are associated to correlation times and noise amplitudes. In this work we will focus on the second point of view, which considers the non Gaussian noise as the primary or direct forcing of the Brownian motor.

In [11] we have analyzed the system considering a long correlation time and a small amplitude of the stochastic forcing. In this contribution we extend the results found in [11] by relaxing the condition of small amplitude of the noise. In this way we find new regimes, where the dynamics of the Brownian motor is affected in different ways by the non Gaussianity of the noise. We also delve deeper into the analytical study of the system in the adiabatic approximation (which assumes a large correlation time of the forcing) and we give a more complete understanding of some of the phenomena studied in [11]. Finally, we analyze the role played by the long tails of the distribution, and compare the results of truncated non Gaussian distributions for different cutoff values but having the same width. In this way we show the crucial role played by those long tails in determining the enhancement of the Brownian motor response.

2 Model and approximations

We consider the following model for a particle in a rocked ratchet, which is the same studied in Ref. [11] but considering the overdamped limit

$$\frac{dx}{dt} = -V'(x) - F + \xi(t) + \eta(t), \tag{1}$$

where $V(x)$ is the ratchet potential, F is a constant "load" force, and $\xi(t)$ the thermal noise satisfying $\langle \xi(t)\xi(t')\rangle = 2T\delta(t-t')$. Finally, $\eta(t)$ is the time correlated forcing (with zero mean) that allows the rectification of the motion, keeping the system out of thermal equilibrium even for $F = 0$. For the ratchet potential we consider the same form as in [16] and [11] (for instance, see Fig. 3 in [11])

$$V(x) = V_1(x) = -\int^{x} dx' \left(\frac{\exp[\alpha \cos(x')]}{J_0(i\alpha)} - 1 \right), \tag{2}$$

where $J_0(i\alpha)$ is the Bessel function, and $\alpha = 16$.

We will consider the dynamics of $\eta(t)$ as described by the following Langevin equation [1,2]

$$\frac{d\eta}{dt} = -\frac{1}{\tau}\frac{d}{d\eta}V_q(\eta) + \frac{1}{\tau}\zeta(t), \tag{3}$$

with $\langle \zeta(t) \rangle = 0$ and $\langle \zeta(t)\zeta(t') \rangle = 2D\delta(t-t')$, and

$$V_q(\eta) = \frac{D}{\tau(q-1)} \ln[1 + \frac{\tau}{D}(q-1)\frac{\eta^2}{2}].$$

For $q = 1$, the process η coincides with the Ornstein-Uhlenbeck (OU) one (with a correlation time equal to τ), while for $q \neq 1$ it is a non Gaussian process. As shown in [1,2], the form of the probability distribution function (pdf), for q within the range $-\infty < q < 3$, is

$$P_q(\eta) = \frac{1}{Z_q}\left[1 + \frac{\tau(q-1)\eta^2}{2D}\right]^{\frac{1}{1-q}}, \tag{4}$$

that for $1 < q < 3$, extents along the interval $-\infty < \eta < \infty$, and decays as a power law (slower than a Gaussian distribution). For $q < 1$, the sign inside the brackets is changed, and the pdf has a bounded support with a cut-off at $|\eta| = \omega \equiv [(1-q)\tau/(2D)]^{-\frac{1}{2}}$. Z_q is a normalization constant. For $q > 3$ this distribution can not be normalized.

The main characteristic introduced by this non Gaussian form of the forcing is that, for $q > 1$, the distribution decays as a power law. This leads to the appearance of arbitrary strong "kicks" on the ratchet particle with relatively high probability when compared, for example, with the Gaussian OU noise and, of course, with the dichotomic non Gaussian process. For a picture of the typical form of this pdf for different values of q, we refer to Fig. 1 in [11].

In [1,2] it was shown that the second moment of the distribution, which we interpret as the "intensity" of the non Gaussian noise, which is given by

$$D_{ng} \equiv \langle \eta^2 \rangle = \frac{2D}{\tau(5-3q)}, \tag{5}$$

diverges for $q \geq 5/3$. For the correlation time τ_{ng} of the process $\eta(t)$, defined in detail in [1,2] it was not possible to find an analytical expression. However, it is known [1,2] that for $q \to 5/3$ it diverges as $\sim (5-3q)^{-1}$. In our analysis, we will consider values of q in the range $0.5 < q < 5/3 \simeq 1.66$. For this interval we have studied numerically the dependence of τ_{ng} on q, and have found the following analytical approximation

$$\tau_{ng} \simeq 2\frac{[1+4(q-1)^2]\,\tau}{(5-3q)}, \tag{6}$$

that fits very accurately the results. This fitting will be the one we will consider when analyzing the dependence of the transport properties of the ratchet system on the intrinsic parameters of the non Gaussian noise, D_{ng} and τ_{ng}.

As mentioned in the introduction, in [11] we have studied the current and efficiency of the system as function of q from two points of view. One corresponds to consider D and τ as control parameters (in addition to q) and the other is to consider D_{ng} and τ_{ng} instead. The first point of view is the more direct one when thinking on tailor made technological devices [17,18,19]. The second one is the more natural from the point of view of biological systems, as it consider η as a primary source of noise characterized by its intensity and correlation time. As indicated in the introduction, in this work we focus on this second view.

Considering the first point of view, in [11] we have found that a departure from Gaussian behavior (particularly for $q > 1$), induces a remarkable increase of the current together with an enhancement of the motor efficiency. The efficiency shows in addition an optimum value for a given degree of departure from the Gaussian behavior and decays due to the enhancement of fluctuations when the correlation of the non Gaussian noise diverges. When inertia is taken into account we have also found a considerable increment in the mass separation capability of the system.

The main results in [11] corresponding to the second point of view will be discussed in the following sections in connection with the new results we will show. In the next section, we extend the analysis made in [11] – which involves only relatively low intensities of the non Gaussian noise – providing analytical results for a wide range of D_{ng}. Different regions of this parameter are found where q_o, the optimum value of q that maximizes the current, changes from $q_o > 1$ to $q_o < 1$.

3 Analytical results within the adiabatic approximation.

In the overdamped regime we are able to give an approximate analytical solution for the problem, which is valid in the large correlation time regime ($\frac{\tau}{D} \gg 1$): we perform the adiabatic approximation of solving the Fokker-Planck equation associated to Eq. (1) assuming a constant value of η [20], see e.g. [11]. This leads us to obtain an η–dependent value of the current $J(\eta)$ that is then averaged over η using the distribution $P_q(\eta)$ in Eq. (4) [11] with the desired values of q, D and τ

$$J = \int d\eta\, J(\eta)\, P_q(\eta).$$

We remark that, although the Fokker-Planck equation is solved in the $\tau/D \to \infty$ limit, the solution we find depends on D and τ through the $P_q(\eta)$ distribution. Note that, in order to perform the adiabatic approximation, it is essential to consider a non vanishing temperature, since this gives the random ingredient that leads to a Fokker-Planck equation. In Fig. 1 we show a typical curve for $J(\eta)$.

Now we study the dependence of the current J on the parameters D_{ng} and q. We consider the range $0 < D_{ng} < 200$ and $0.5 < q < 1.66$. (For $D_{ng} > 100$ the current decays as a consequence of the "excess" of noise, as is typical in most of the noise induced phenomena. Hence, the behavior of the system for higher values of D_{ng} is not considered to be relevant.) For the rest of the parameters we consider values similar to those in [16] and [11] where interesting transport phenomena have been observed. We fix $T = 0.5$, $F = 0.1$ and $\tau_{ng} = 100/(2\pi) \sim 15.9$. The relevance of this parameter region was discussed in [11]. As we show in the next section, the value set for τ_{ng} is high enough to make the adiabatic approximation valid.

In Fig. 2 we show results for the averaged current J as function of D_{ng} for different values of q. Parts (a) and (b) of the figure show the same curves on different scales in order to better appreciate the crossings' details. At a first glance, one may observe that for $D_{ng} \lesssim 0.4$ the current increases with q, for $0.4 \lesssim D_{ng} \lesssim 30$ the current decreases with q, and finally, for $D_{ng} > 30$ the current increases with q again. However, what actually occurs is a little more complicated. The results should be carefully read, as we are not showing the curves for all the possible values of q. In [11] we have analyzed the behavior of the system in the region of small D_{ng}. It was shown that, for a fixed value of $D_{ng} < 0.4$, there is an optimum value of $q > 1$, q_o, that maximizes the

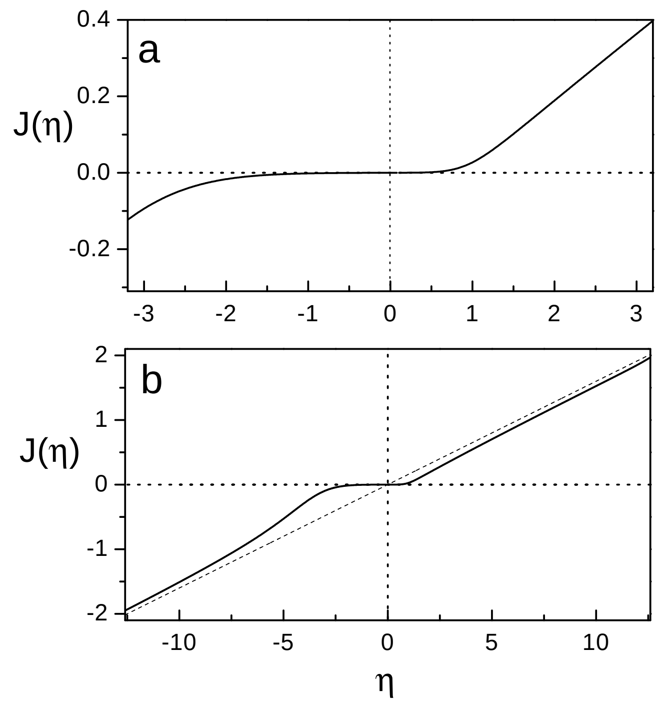

Fig. 1. Results for $J(\eta)$ for $F = 0.1$ and $T = 0.5$. In both parts of the figure the same curve is plotted at different scales in order to better show the behavior. In part (b) we have also shown the line to which $J(\eta)$ converges for $\eta \rightarrow \pm\infty$.

current. What happens in Fig. 2a for very low values of D_{ng} is that q_o is larger than the highest value plotted for q. However, approximately at $D_{ng} = 0.1$ a crossing between the curves for $q = 1.55$ and $q = 1.4$ occurs. This means that the optimum value became lower than $q = 1.55$.

Now, considering that, as discussed in [11], the current should vanish for very low values of q, since the bounds for the distribution of η are reduced, and that for $q \rightarrow 5/3$ the current decreases due to the increasing of the fluctuations of η, an optimum value of q should always be expected. Hence, the results in Fig. 2 should be interpreted as follows. For every value of D_{ng} there is an optimum value of $q \equiv q_o$ that maximizes the current. For $D_{ng} \lesssim 0.4$ we have $q_o > 1$, for $0.4 \lesssim D_{ng} \lesssim 30$ we have $q_o < 1$, and for $40 < D_{ng}$ we have again $q_o > 1$.

In the following section we present results from numerical simulations that validate these predictions of the adiabatic approximation.

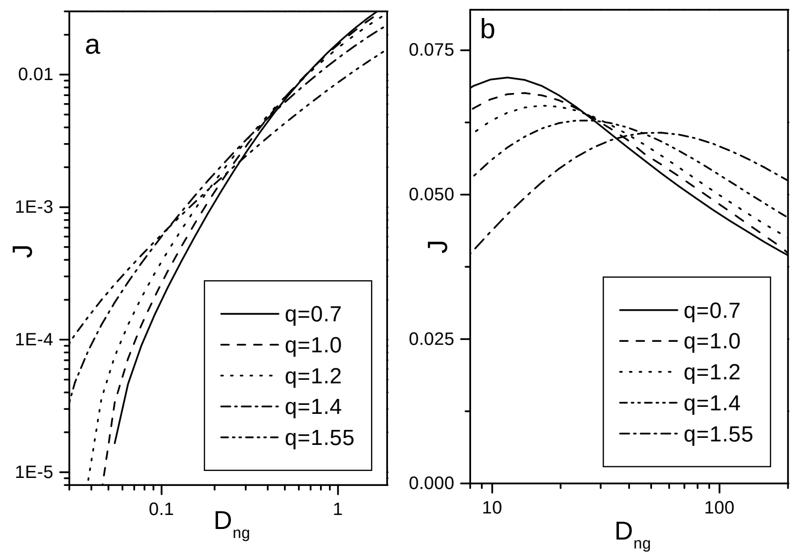

Fig. 2. Current vs D_{ng} for different values of q, calculated within the adiabatic approximation. Both parts of the figure show the same curves in different ranges of D_{ng}.

4 Numerical simulations

We have analyzed numerically the evolution of Eqs. (1) and (3) by numerical integration of those equations using a second order stochastic Runge–Kutta type algorithm [21]. The current is defined as $J = \frac{\langle \dot{x} \rangle}{L}$ where $L = 2\pi$ is the period of the ratchet potential, and $\langle \rangle$ indicates temporal averaging.

Here, in order to appreciate the maxima predicted in the previous section, we present results for the current as function of q for different values of D_{ng}. We have only considered values of $D_{ng} \leq 20$, as higher values of this parameter requires considerable computational effort. In Fig. 3 we show the results coming from simulations together with the curves obtained analytically from the adiabatic theory. It can be seen that, for the value of τ_{ng} considered, concerning the general behavior of the results as function of q and D_{ng} there is a rather good agreement between theory and simulations. Better accuracy from the adiabatic theory can only be obtained for larger values of the correlation time.

The results in Fig 3. confirm the predictions of the previous section (at least for the values of $D_{ng} \leq 30$ here considered), as the optimum value of q appears to be $q_o > 1$ for $D_{ng} \lesssim 0.4$ and $q_o < 1$ for $0.4 \lesssim D_{ng} \lesssim 30$. It should be mentioned that some of the results on this figure (those for $D_{ng} < 1$) have been presented in [11]. However, we want to remark that in that work we have

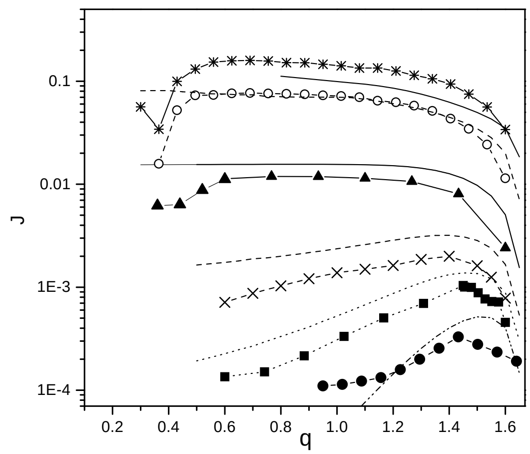

Fig. 3. Current as a function of q for fixed $\tau_{ng} = 100/(2\pi)$ and different fixed values of D_{ng}. The lines with symbols corresponds to simulations and the lines without symbols to the adiabatic theory. From top to bottom, the curves are for $D_{ng} = 20$ (solid line for theory and solid line with stars for simulations); $D_{ng} = 5$ (dashed line for theory and dashed line with hollow circles for simulations); $D_{ng} = 1$ (solid line for theory and solid line with triangles for simulations); $D_{ng} = 0.35$ (dashed line for theory and dashed line with crosses for simulations); $D_{ng} = 0.2$ (dotted line for theory and dotted line with squares for simulations); and $D_{ng} = 0.1$ (dash-dot-dot line for theory and dash-dot-dot line with solid circles for simulations). All calculations are for $T = 0.5$ and $F = 0.1$.

only explored the region of low D_{ng} while now, we have a more panoramic view of the system's behavior as function of that parameter that ranges from 0 to ∞.

Finally, in order to analyze the relevance of the long tails of the Non Gaussian distributions for $q > 1$ in determining the value of the current, we have done some special calculations. We have simulated the dynamics of the Brownian motor forced by a different -but related- non Gaussian noise. In Eq. (1), instead of $\eta(t)$ we consider the precess $\eta_u(t)$ which is defined as $\eta_u(t) = \lambda_u(t)\eta(t)$, where $\lambda_u(t)$ is 1 if $|\eta(t)| < u$, and $\lambda_u(t) = 0$ if $|\eta(t)| > u$. Here, $u > 0$ is a parameter that plays the role of a threshold for the non Gaussian noise, and indicates the maximum value of the noise that can be "feel" by the particle. For $u \to \infty$ the process $\eta_u(t)$ converges to $\eta(t)$. Note that, in practice, in the simulations we calculate the complete evolution for $\eta(t)$ as in the normal case, but we change the way in which this noise couples to the Brownian particle.

In Fig.4.a we show results for the current as function of u for different values of q. It is apparent that the threshold u needed to obtain the asymptotic value of the current (corresponding to the one obtained with the process $\eta(t)$) increases with q. This means that the tails of the distribution are relevant up

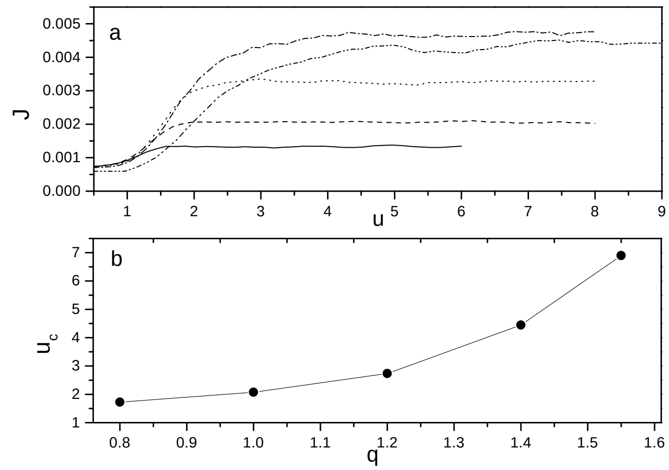

Fig. 4. (a) Current as a function of the threshold u for different values of q: Solid line for $q = 8$, dashed line for $q = 1.$, dotted line for $q = 1.2$, dash-dotted line for $q = 1.4$, and dash–dot–dotted line for $q = 1.55$.

to higher values of η as q is increased. In Fig. 4.b we plot, as a function of q, the value of the threshold u_c at which the asymptotic value of the current is reached with an error lower than 5%.

5 Conclusions

We have here further extended the study of the effect of non Gaussian noises on the behavior and transport properties of Brownian motors initiated in [11]. We have focused on two aspects: (a) the adiabatic approximation (valid for a high correlation time of the forcing) [11], showing some analytical results for the current J as a function of D_{ng} for different q; and (b) have analyzed the role played by the long tails of the distribution, by analyzing the results of truncated non Gaussian distributions.

By means of the adiabatic approximation as well as related numerical simulations, we found that there is an "optimal" value of the parameter q, yielding the maximum possible value of the current J for a given value of D_{ng}. Also, that such an optimum value can change from $q_o > 1$ to $q_o < 1$ for different regions of values of D_{ng}. Regarding the analysis of truncated non Gaussian distributions, we have shown the crucial role played by the long tails in determining the enhancement of the Brownian motor response.

These results complement those of [11] and supports the finding of a strong influence of non Gaussian noises on the response of Brownian ratchets, as was previously found for other noise induced phenomena.

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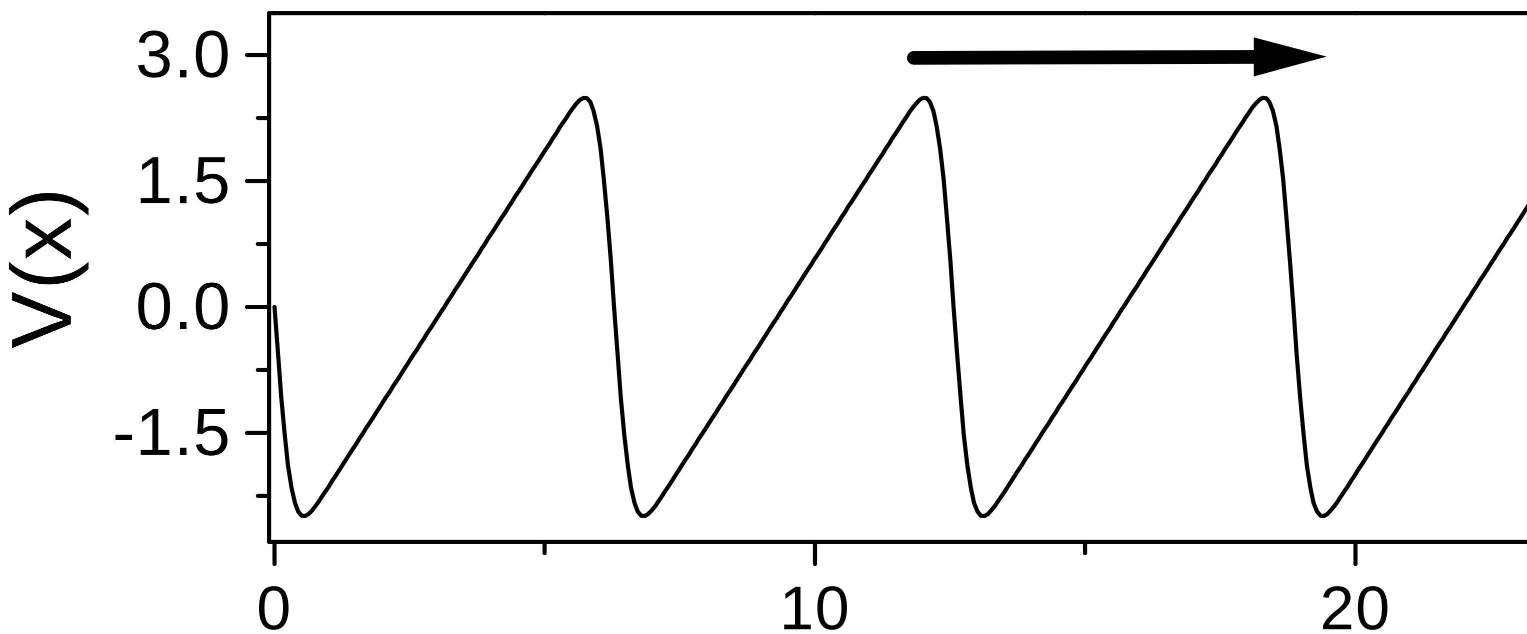

V(x)
3.0
1.5
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-1.5
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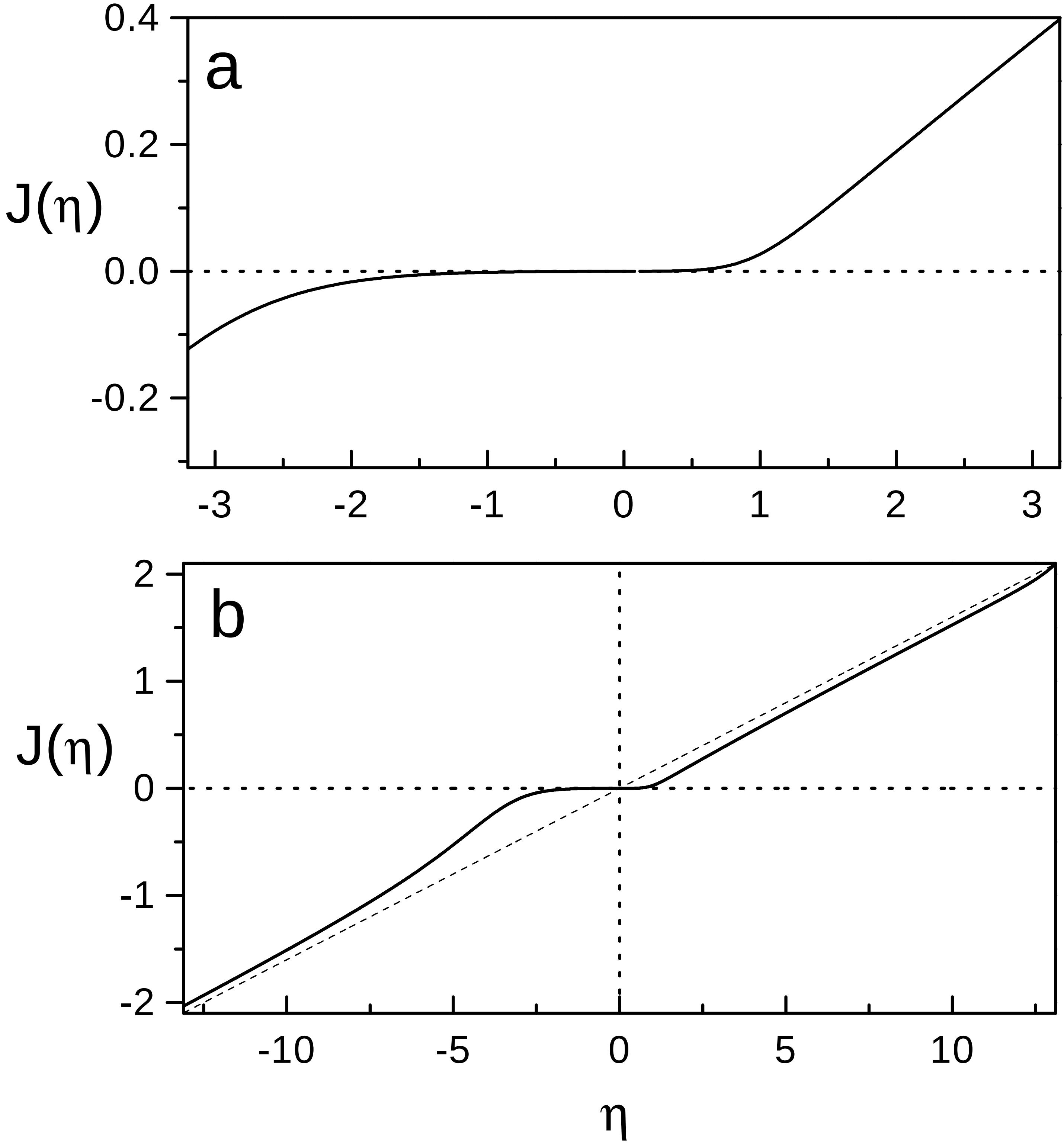
10
20



a
b
J(η)
η
0.4
0.2
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-0.2
-3
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-2
-10
-5
0
5
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