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## New Bäcklund Transformations and Superposition Principle for Gravitational Fields with Symmetries

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Vector Bäcklund transformations which relate solutions of the vacuum Einstein equations having two commuting Killing fields are introduced. Such transformations generalize those found by Pohlmeyer in connection with the nonlinear  $\sigma$  model. A simple algebraic superposition principle, which permits the combination of Bäcklund transforms in order to get new solutions, is given. The superposition preserves the asymptotic flatness condition, and the whole scheme is manifestly  $O(2,1)$  invariant.

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In recent years, several methods have been proposed in order to construct physically meaningful exact solutions of the Einstein equations of general relativity. Much work has been done in the special case of vacuum gravitational fields possessing two Killing vector fields which commute<sup>1-7</sup>; this includes, in particular, stationary systems with axial symmetry. A technique to produce solutions with an appropriate asymptotic behavior, and a prescribed mass and angular momentum multipole structure, was found in Ref. 7. Alternative approaches to the subject include the application of the inverse scattering technique to this case,<sup>8</sup> and the use of Bäcklund transformations.<sup>9</sup> It is the purpose of this Letter to introduce a new type of vector Bäcklund transformation, which admits a rather simple, closed expression, and an associated superposition principle. The latter provides an easy way of algebraically combining two Bäcklund transforms of a given solution in order to get a new solution. It is found that, in the case of stationary axisymmetric systems, such a superposition preserves the asymptotic flatness condition automatically.

The computations involved in getting a new solution through this method are quite simple, and the whole scheme is manifestly  $O(2,1)$  invariant.

If the gravitational field admits a two-parameter Abelian group of isometries, and if one assumes the existence of two-surfaces orthogonal to the group orbits ("orthogonal transitivity"), then the Einstein equations  $R_{\mu\nu}=0$  separate into two sets: One of them is a system of integrable equations for a single scalar function, and the remaining set may be expressed as a single partial differential equation for a three-vector  $q$ . In case the two Killing fields are spacelike (e.g., cylindrical symmetry), the resulting equation is<sup>6</sup>

$$q_u \cdot q_v + (\tau_u/2\tau)q_v + (\tau_v/2\tau)q_u = (q_u \cdot q_v)q, \quad (1)$$

$$q^2 = -1$$

with

$$\tau(u, v) = U(u) + V(v), \quad (2)$$

where  $U$  and  $V$  are arbitrary. [Here and in the sequel, subscripts denote ordinary partial derivatives; the scalar product of two vectors  $q = (q^i$ ,

$q^2, q^3$ ) and  $p = (p^1, p^2, p^3)$  is  $p \cdot q = (p^1 q^1 + p^2 q^2 - p^3 q^3)$ ;  $u = \frac{1}{2}(\rho + t)$ , and  $v = \frac{1}{2}(\rho - t)$ .] It should be stressed that the components of the vector  $q$  appearing in Eq. (1) are *algebraic* combinations of the coefficients of the space-time metric.

The constraint  $q^2 = -1$  in Eq. (1) may be solved by means of a single complex function  $f$ , parametrizing  $q$  as

$$q = (f + f^*)^{-1}(i(f - f^*), 1 - ff^*, 1 + ff^*). \quad (3)$$

With use of (3), Eq. (1) reduces to

$$f_{uv} + \frac{\tau_u}{2\tau} f_v + \frac{\tau_v}{2\tau} f_u = \frac{2f_u f_v}{f + f^*}. \quad (4)$$

Equation (4) is formally identical to the Ernst equation<sup>1</sup> in the hyperbolic case. It should be noticed, however, that  $f$  is *not* the Ernst potential  $\epsilon$  of the system. The relation between  $f$  and  $\epsilon$  is easily worked out by writing the metric in the Lewis-Papapetrou form<sup>10</sup>

$$ds^2 = h(dz + \omega d\varphi)^2 + S^2 h^{-1} d\varphi^2 + e^{\nu} (d\rho^2 - dt^2),$$

where  $h$ ,  $\omega$ , and  $\nu$  are functions of  $\rho$  and  $t$ , and  $S$  may be taken as  $S = \rho = u + v$ . The functions  $f$  and  $\epsilon$  are then

$$f = Sh^{-1} + i\omega, \quad \epsilon = h + i\psi,$$

where  $\psi$  is related to  $\omega$  by

$$\psi_u = S^{-1} h^2 \omega_u, \quad \psi_v = -S^{-1} h^2 \omega_v.$$

For stationary axisymmetric systems, the relevant equation [analogous to Eq. (1)] may be shown to be<sup>6</sup>

$$q_{\zeta \zeta^*} + \frac{\tau_{\zeta}}{2\tau} q_{\zeta^*} + \frac{\tau_{\zeta^*}}{2\tau} q_{\zeta} = -(q_{\zeta} \cdot q_{\zeta^*}) q, \quad q^2 = 1, \quad (5)$$

where  $\zeta = \frac{1}{2}(\rho + iz)$  and  $\tau(\zeta, \zeta^*) = \eta(\zeta) + \eta^*(\zeta^*)$ , with

$$q_u + p_u = [1/2(U + \lambda)][(U + V)q_u \cdot p - U_u]q + [1/2(U + \lambda)][(U + V)q \cdot p_u - U_u]p, \quad (8a)$$

$$q_v - p_v = -[1/2(V - \lambda)][(U + V)q_v \cdot p + V_v]q + [1/2(V - \lambda)][(U + V)q \cdot p_v + V_v]p, \quad (8b)$$

together with the compatible constraints

$$q^2 = p^2 = -1, \quad p \cdot q = (U - V + 2\lambda)/(U + V), \quad (9)$$

where  $\lambda$  is a real constant, with  $U$  and  $V$  the functions appearing in (2). Given a solution  $q$  of Eq. (1), a new solution  $p$  may be found by integrating Eqs. (8a) and (8b). [The integrability condition for (8a) and (8b), considered as a system of partial differential equations for  $p$ , is satisfied by virtue of Eq. (1).] The corresponding transformation for Eq. (5) is

$$q_{\zeta} + ip_{\zeta} = [1/2(\eta + i\lambda)][(\eta + \eta^*)p_{\zeta} \cdot q - i\eta_{\zeta}]p - [i/2(\eta + i\lambda)][(\eta + \eta^*)q_{\zeta} \cdot p - i\eta_{\zeta}]q \quad (10)$$

with

$$q^2 = 1, \quad p^2 = -1, \quad q \cdot p = [i(\eta - \eta^*) - 2\lambda]/(\eta + \eta^*). \quad (11)$$

$\eta(\zeta)$  arbitrary. In the present case, a solution of the constraint  $q^2 = 1$ , closely resembling (3), is

$$q = (f + g)^{-1}(-f + g, 1 + fg, 1 - fg), \quad (6)$$

where  $f$  and  $g$  are now *real* functions. Equation (5) reduces to the following pair of equations:

$$f_{\zeta \zeta^*} + \frac{\tau_{\zeta}}{2\tau} f_{\zeta^*} + \frac{\tau_{\zeta^*}}{2\tau} f_{\zeta} = \frac{2f_{\zeta} f_{\zeta^*}}{f + g}, \quad (7a)$$

$$g_{\zeta \zeta^*} + \frac{\tau_{\zeta}}{2\tau} g_{\zeta^*} + \frac{\tau_{\zeta^*}}{2\tau} g_{\zeta} = \frac{2g_{\zeta} g_{\zeta^*}}{f + g}. \quad (7b)$$

With the chosen parametrization (3) and (6), the  $O(2,1)$  invariance of Eqs. (1) and (5) is reflected in the following  $SL(2, R)$  invariance transformation of Eq. (4):

$$f \rightarrow (af + ib)/(-icf + d)$$

and of Eqs. (7a) and (7b):

$$f \rightarrow (af + b)/(cf + d), \quad g \rightarrow (ag - b)/(-cg + d),$$

where  $a$ ,  $b$ ,  $c$ , and  $d$  are real constants such that  $ad - bc = 1$ .

Bäcklund transformations have been introduced for the Ernst equations corresponding to gravitational fields of the type considered in this Letter,<sup>9</sup> and solutions of the Einstein field equations have been generated through their use.<sup>11, 12</sup> On the other hand, it seems desirable to have a solution-generating technique directly applicable to the vector equations (1) and (5), as  $q$  may be simply read off the metric. By the exploiting of the close connection between such equations and the field equation for an  $O(2,1)$ -invariant nonlinear  $\sigma$  model in two dimensions, vector Bäcklund transformations generalizing those introduced by Pohlmeyer in Ref. 13 may be found. For Eq. (1), the transformation is

Equations (8)–(11) are manifestly invariant under  $p \rightarrow Rp$ ,  $q \rightarrow Rq$ , with a constant  $R \in O(2, 1)$ . Notice that, while Eqs. (8a) and (8b) transform a solution of Eq. (1) into another solution of the same equation, Eq. (10) carries a solution of (5) into a solution of the equation

$$p_\xi \zeta^* + (\tau_\xi/2\tau)p_{\zeta^*} + (\tau_{\zeta^*}/2\tau)p_\xi = (p_\xi \cdot p_{\zeta^*})p, \quad (12)$$

$$p^2 = -1.$$

Thus, in order to go back to solutions of Eq. (5), a second Bäcklund transformation (10) must be used. Fortunately, no further integrations are necessary, because of the existence of a permutability property,<sup>14</sup> by means of which the second Bäcklund transform may be computed algebraically: Suppose Eq. (10) has been integrated for two different values  $\lambda$  and  $\mu$ , giving the transformed vectors  $p$  and  $s$ , respectively. It may be easily shown that there exists a vector  $w$ , such that  $w$  is simultaneously a Bäcklund transform of  $p$  with parameter  $\mu$ , and of  $s$  with parameter  $\lambda$ , and such that  $w$  satisfies Eq. (5); furthermore, there exists a simple algebraic relation among  $q$ ,  $p$ ,  $s$ , and  $w$ :

$$w = q + \frac{2(\lambda - \mu)}{\eta + \eta^*} \frac{1}{1 + p \cdot s} (s - p). \quad (13)$$

Equation (13) may be considered as a kind of non-linear superposition principle. A detailed derivation of Eq. (13) will be published elsewhere. A similar relation holds for Eqs. (8a) and (8b).

The following important aspect should now be noticed: As there exists a manifest duality be-

tween  $p$  and  $q$  in Eqs. (10) and (11), one may take the point of view that the equation to be solved is Eq. (12). This is motivated by the fact that the Ernst equation for the potential  $\epsilon$  of a stationary axisymmetric system is<sup>1</sup>

$$\epsilon_\xi \zeta^* + (\tau_\xi/2\tau)\epsilon_{\zeta^*} + (\tau_{\zeta^*}/2\tau)\epsilon_\xi = 2\epsilon_\xi \epsilon_{\zeta^*} / (\epsilon + \bar{\epsilon}) \quad (14)$$

which may be cast in precisely the form (12) by defining

$$p = (\epsilon + \epsilon^*)^{-1} (i(\epsilon - \epsilon^*), 1 - \epsilon\epsilon^*, 1 + \epsilon\epsilon^*).$$

The starting point will now be the vector  $p$ ; a new solution  $\epsilon'$  of Eq. (14) may be obtained by means of Eq. (10). As a result of its symmetry, the superposition (13) is equally applicable in this case. It may now be written as

$$s = p + [2(\lambda - \mu)/(\eta + \eta^*)(w \cdot q - 1)]^{(w \cdot q)}, \quad (15)$$

where  $q$  (respectively,  $w$ ) is a Bäcklund transform of  $p$  with parameter  $\lambda$  (respectively,  $\mu$ ).

The new Ernst potential is

$$\epsilon' = (1 - i s^1)/(s^2 + s^3),$$

where  $s = (s^1, s^2, s^3)$ . The superposition formula (15) has the additional interest of preserving the right asymptotic conditions. In order to see this, it is convenient to find the general solution of (15) starting from Minkowski space, described by  $p = (0, 0, 1)$ . An asymptotically flat space will approach this value, and the conclusions that follow will also apply in that case. Integrating (10) with  $\eta(\xi) = \xi$ , so that  $\tau = \rho$  (using Weyl canonical coordinates  $\rho$  and  $z$ ), one gets

$$q = (\xi + \zeta^*)^{-1} (2 \cos \alpha (\xi + i\lambda)^{1/2} (\zeta^* - i\lambda)^{1/2}, 2 \sin \alpha (\xi + i\lambda)^{1/2} (\zeta^* - i\lambda)^{1/2}, -i(\xi - \zeta^*) + 2\lambda), \quad (16)$$

where  $\alpha$  is a constant of integration. Similarly, we find  $w$  by substituting  $\lambda \rightarrow \mu$ ,  $\alpha \rightarrow \beta$  in (16).

After computing  $s$  according to (15) we get the following expression:

$$\frac{1 + \epsilon'}{1 - \epsilon'} = \frac{s^2 - i s^1}{s^3 - 1} = \frac{i}{a - b} (e^{i\beta} r_1 - e^{i\alpha} r_2), \quad (17)$$

where  $r_1 = (r^2 + b^2 + 2br \cos \theta)^{1/2}$ ,  $r_2 = (r^2 + a^2 + 2ar \times \cos \theta)^{1/2}$ ,  $\rho = r \sin \theta$ ,  $z = r \cos \theta$ ,  $a = 2\lambda$ , and  $b = 2\mu$ . From (17), one obtains the following expansion for  $\epsilon'$  in terms of the radial coordinate  $r$ , showing the correct asymptotic behavior:

$$\epsilon' = 1 + \frac{2(a - b)}{i(e^{i\beta} - e^{i\alpha})} \frac{1}{r} + O\left(\frac{1}{r^2}\right).$$

In order to get a real coefficient in the  $1/r$  term, we may either set  $\beta = -\alpha$  or subject  $\epsilon'$  to an Eh-

lers-type transformation,

$$\epsilon'' = \frac{\cos \sigma \epsilon' + i \sin \sigma}{i \sin \sigma \epsilon' + \cos \sigma},$$

with an appropriately chosen  $\sigma$ . Both procedures leave the zeroth-order term invariant. The Kerr solution,  $(1 + \epsilon')(1 - \epsilon')^{-1} = x \cos \nu + i y \sin \nu$ , is obtained as a special case of (16) by setting  $a = -b = k$ ,  $\alpha = -\beta = \nu - \pi/2$ , and defining  $x + y = k^{-1}(z^2 + \rho^2 + k^2 + 2kz)^{1/2}$ ,  $x - y = k^{-1}(z^2 + \rho^2 + k^2 - 2kz)^{1/2}$ .

Further work on other applications of the present method, and on its relations with existing techniques for generating solutions of the Einstein equations and of the equations for non-Abelian gauge fields, is currently in progress.

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## Data on the Gross-Llewellyn Smith Sum Rule as a Function of $q^2$

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Data are presented on the Gross-Llewellyn Smith sum rule obtained from combined narrow-band neon and Freon bubble-chamber neutrino-antineutrino experiments. Remarkably no significant deviation from the parton-model prediction for the sum rule is observed at very low values of  $q^2 \lesssim 1 \text{ GeV}^2$ . Limits on the effective QCD scale parameter  $\Lambda$  and on the magnitude of the twist-4 correction are set. The best fit, neglecting higher-twist contributions, gives  $\Lambda = 92_{-38}^{+20} \text{ MeV}$ .

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In the quark-parton model the neutrino-nucleon structure function  $F_3$  measures the difference of the  $x$  distributions of the quarks and the anti-quarks in the nucleon:

$$F_3(x)_{q^2 \rightarrow \infty} = \frac{dN_q}{dx} - \frac{dN_{\bar{q}}}{dx}, \quad (1)$$

where  $x$  is the usual Bjorken scaling variable. The integral of  $F_3$  measures the number of "valence quarks" per nucleon, equal to three in the quark model:

$$\int_0^1 F_3(x) dx_{q^2 \rightarrow \infty} = N_q - N_{\bar{q}} = 3 \text{ valence quarks.} \quad (2)$$