# New black hole solutions in three-dimensional $f(R)$ gravity 

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#### Abstract

We construct two new classes of analytical solutions in three-dimensional spacetime and in the framework of $f(R)$ gravity. The first class represents a non-rotating black hole (BH) while the second class corresponds to a rotating BH solution. The Ricci scalar of these BH solutions have non-trivial values and are described by the gravitational mass $M$, two angular momentums $J$ and $J_{1}$, and an effective cosmological constant $\Lambda_{\text {eff }}$. Moreover, these solutions do not restore the 3 -dimensional Bañados-Teitelboim-Zanelli (BTZ) solutions of general relativity (GR) which implies the novelty of the obtained BHs in $f(R)$ gravity. Depending on the range of the parameters, these solutions admit rotating/non-rotating asymptotically $\mathrm{AdS} / \mathrm{dS}$ BH interpretation in spite that the field equation of $f(R)$ has no cosmological constant. Interestingly enough, we observe that in contrast to BTZ solution which has only causal singularity and scalar invariants are constant everywhere, the scalar invariants of these solutions indicate strong singularity for the spacetime. Furthermore, we construct the forms of the $f(R)$ function showing that they behave as polynomial functions. Finally, we show that the obtained solutions are stable from the viewpoint that heat capacity has a positive value, and also from the condition of Ostrogradski which state that the second derivative of $f(R)$ should have a positive value.


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## I. INTRODUCTION

The investigations on GR solutions in $(2+1)$-dimensions provide a powerful background to study the physical properties of gravity and examine its viability in lower spacetime dimensions. In particular, it can be simulated as a toy model of quantum gravity. This property has been discovered after the arguments presented connecting the possible links between $(2+1)$-dimensional gravitation and the Chern-Simons theory [1, 2]. A great step put forwarded by the authors of [3] who discovered a novel solution of GR in $(2+1)$-dimensions. They constructed their solution's using the $\mathrm{SO}(2,2)$ gauge group with a negative cosmological constant. This BH solution, looks like the features of the $(3+1)$-dimensional Schwarzschild and Kerr black holes which mean that it has a credible physical significance.

The BTZ BH solution has been given further improvements, modifications and generalizations [4-10], and later, Witten has calculated the entropy of the BTZ BHs [11]. The BTZ BH solution has been studied by different fulfillments, from different physical viewpoints. For example, the construction of geodesic equations of the uncharged BTZ BHs [12], the quasi-normal modes [13-16], the scattering process of test particles [17, 18], the hydrostatic equilibrium conditions for finite distributions [19], and solutions for fluid distributions matching the exterior BTZ spacetime $[20-24]$. Moreover, by taking into account a non-constant coupling parameter with the energy-scale, the scale-dependent version of the BTZ solution has been derived and analyzed [25-29]. It has been also debated that, if the energy-momentum complexes of Landau-Lifshitz and Weinberg are used for a rotating BTZ BH, the same energy distribution is obtained [30].

The rotating BTZ spacetime has been generalized through the addition of terms connected to the non-linear electrodynamics [31, 32] and the conformal group [33]. Moreover, concerning the BH thermodynamics, the BTZ BH solutions have been analyzed by using their critical behavior and phase transitions. Another viewpoint of studying the thermodynamics of the BTZ BH was carried out by calculating the equilibrium thermodynamic fluctuations [34], in which the extremal BTZ BH with angular momentum serves as the critical point, and the density of the states in the micro-and grand-canonical ensembles has been calculated [35]

Moreover, if the cosmological constant is dealt with as a thermodynamical parameter, the AdS Kerr and the BTZ BH solutions have been discussed in [36]. The quantum corrections to the enthalpy and the equation of state of the uncharged BTZ BH solutions have been investigated [37]. A general class of BTZ BH solution has been studied from the viewpoint of Ruppeiner geometry of the thermodynamic state space, and it is shown that such a geometry is a flat one for the rotating BTZ and the BTZ-Chern-Simons BHs, in the canonical ensemble [38]. The establishment of geometrothermodynamics and the one introduced in [39], is a method that is used to specify a flat $(1+1)$-dimensional space of equilibrium states, which is endowed with a thermodynamic metric. A generalization of geometrothermodynamics is discussed [40], in which the thermodynamics of the charged BTZ BH is explained in the frame of the Weinhold and Ruppeiner geometries where it was shown that such geometry cannot describe, the BH thermodynamics. To tackle this issue [41], a new metric (the HPEM metric) was inserted by a particular formalism and it was shown that the corresponding Ricci scalar was able to bring together, different types of phase transitions. The HPEM metric was proved to give a consistent formalism to study the thermodynamics of the BTZ BH solutions [42]. The introduction of quantum scalar fields in the study of BH thermodynamics was carried out [43] and yields the introduction of entanglement thermodynamics for mass-less scalar fields. It has been also shown that the thermodynamics of BTZ BHs can be deformed in the frame of gravity's rainbow, however, the Gibbs free energy remains unchanged [44]. The gravity of the rainbow has been utilized to study the BH heat capacity and phase transition of BTZ BHs [45, 46]. There are also some other extensions of the BTZ BHs that are obtained as alternative theories of gravity for example the Noether symmetries of the rotating BTZ BH in $f(R)$ gravitation has been employed to create new BTZ-type solutions [47]. In the same direction some thermodynamic features of the BTZ BHs, such as the Carnot heat engine, are investigated in the frame of massive gravity [48]. Moreover, Horndeski's action is considered as the source field of the BTZ BH, and reduce it to the familiar Einstein-Hilbert action involving a cosmological constant has been studied where the usual 3-dimensional Smarr formula by using a scaling symmetry of this reduced action [49]. Additionally, the rotating BTZ BHs, have been proven to display no kind of superradiance, if we considered Dirac fields to vanish at infinity [50]. Some thermodynamic aspects of the rotating BTZ BH have been investigated in [51].

The modified gravitational theory, $f(R)$, was introduced in the scientific community as an attempt to prescribe the early and late cosmological story of the universe [52-62]. From the recent results of the cosmological observations, cosmological models of $f(R)$ gravity were used to explain the transmission of deceleration and acceleration. This yields to impose limitations on the $f(R)$ cosmological models to allow for viable choices of $f(R)$ [63]. The theories of $f(R)$ eliminate the contributions of any curvature invariants except the Ricci scalar, $R$ and they avoid the Ostrogradski instability [64] that us usually exists in higher derivative gravitational theories [65]. Many BH solutions in the theories of $f(R)$ were derived and they either are deviations from the well-known BH solutions of GR , or they have new properties that could be discussed. Static and spherically symmetric BH solutions in $(3+1)$ and ( $2+1$ )-dimensions are derived and analyzed [66-69], while charged and rotating BH solutions were explained in [70-76]. Meanwhile
static and spherically symmetric BH solutions were discussed with constant curvature, with/without electric charge and cosmological constant [77, 78]. It is the aim of this study, to extend the above catalog with a new family of 3-dimensional in $f(R)$ modified theory and derive analytic generalizations of the BTZ rotating/non-rotating metric describing BHs. The new BH solutions display one or several horizons and have satisfactory thermodynamical results.

The arrangement of this study is as follows: In Sec. II we give the action and field equations of $f(R)$ gravitational theory. In Sec. III we apply the field equations of $f(R)$ gravity to 3-dimensional spacetime, rotating one, having three unknown metric potentials, $b(r), b_{1}(r)$ and $b_{2}(r)$ which is responsible for rotation. The resulting differential equations are classified into four different cases:(i) $f_{R}(r)=$ const. and $b_{2}(r)=0$, (ii) $f_{R}(r)=$ const. and $b_{2}(r) \neq 0$, (iii) $f_{R}(r)=c_{0}+\frac{c_{1}}{r^{2}}$ and $b_{2}(r)=0$, and (iv) $f_{R}(r)=c_{0}+\frac{c_{1}}{r^{2}}$ and $b_{2}(r) \neq 0{ }^{1}$. Note that here $f_{R}=d f(R) / d R$ and we use the chain rule i.e., $f_{R}=d f(R) / d r \times d r / d R$. The first two cases do not give any new results different from the BTZ BH rotating/non-rotating of Einstein GR. The last two cases are discussed in detail regarding their analytic solutions and their asymptote. The most amazing thing is that our starting point of the field equation has no cosmological constant and our derived solutions behave as AdS/dS spacetimes. Also, we calculate the invariants, Kretschmann scalar, the Ricci tensor square, and the Ricci scalar, showing that the trace of $f(R)$ gravity on such invariants makes the singularity stronger than those of 3-dimensional GR BHs because of the non-triviality of the Ricci scalar associated with those solutions. In Sec. IV we derive the form of $f(R)$ and its second derivative, of the last two cases, and draw them showing that their behaviors have a positive manner which means that our solutions avoid the Ostrogradski instability. In Sec. V we calculate the thermodynamical quantities of the two cases (iii) and (iv) showing their horizons, entropy, Hawking temperature, and heat capacity analytically and graphically. In the final section, we conclude our study with the main novel results.

## II. FIELD EQUATIONS OF $f(R)$ GRAVITY

In this section, we consider a three-dimensional action of $f(R)$ gravity and construct the corresponding field equations in three spacetime dimensions. It is ingredient to mention that $f(R)$ gravity is an amended of Einstein gravity and restores the general relativity in the limiting case $f(R)=R$ and when $f(R) \neq R$ then the theory becomes different from GR. The action of $f(R)$ theory can be written as [79-86]

$$
\begin{equation*}
I_{G}=\frac{1}{2 \kappa^{2}} \int d^{3} x \sqrt{-g} f(R) \tag{1}
\end{equation*}
$$

where $\kappa^{2}=8 \pi G$, and $G$ is the Newtonian gravitational constant and $g$ is the determinant of the metric.
Varying the above action with respect to the metric $g_{\mu \nu}$ yields the vacuum field equations of $f(R)$ as [87]

$$
\begin{equation*}
R_{\mu \nu} f_{R}-\frac{1}{2} g_{\mu \nu} f(R)+\left[g_{\mu \nu} \nabla^{2}-\nabla_{\mu} \nabla_{\nu}\right] f_{R}=0 \tag{2}
\end{equation*}
$$

where $\nabla^{2} \equiv \nabla_{\mu} \nabla^{\mu}$. Taking the trace of the field equations (2) in 3D yields

$$
\begin{equation*}
2 \nabla^{2} f_{R}+R f_{R}-\frac{3 f(R)}{2}=0 \tag{3}
\end{equation*}
$$

Using Eq. (3) yields the function $f(R)$ in 3-dimensional in the following form

$$
\begin{equation*}
f(R)=\frac{2}{3}\left[2 \nabla^{2} f_{R}+R f_{R}\right] \tag{4}
\end{equation*}
$$

Substituting Eq. (4) in Eq. (2) yields the field equations of $f(R)$ as

$$
\begin{equation*}
R_{\mu \nu} f_{R}-\frac{1}{3} g_{\mu \nu} R f_{R}+\frac{1}{3} g_{\mu \nu} \nabla^{2} f_{R}-\nabla_{\mu} \nabla_{\nu} f_{R}=0 \tag{5}
\end{equation*}
$$

Thus, it is important to check Eqs. (3) and (5) to a spherically symmetric ansatz having two unknown functions [89].

[^1]
## III. THE 3-DIMENSION BLACK HOLE SOLUTIONS

The line element of the rotating 3-dimensional spacetime in the coordinates $(t, r, \phi)$ can be written as [90]

$$
\begin{equation*}
d s^{2}=-\left[b(r)-r^{2} b_{2}^{2}(r)\right] d t^{2}+2 r^{2} b_{2}(r) d r d \phi+\frac{d r^{2}}{b_{1}(r)}+r^{2} d \phi^{2} \tag{6}
\end{equation*}
$$

with $b(r), b_{1}(r)$, and $b_{1}(r)$ are functions depending on the radial coordinate $r$. The Ricci scalar of the metric (6) figured out as

$$
\begin{equation*}
R(r)=-\frac{2 b^{2} b_{1}^{\prime}-r^{3} b b_{1} b_{2}^{\prime 2}+r b_{1}^{\prime} b b^{\prime}+2 r b b_{1} b^{\prime \prime}-r b_{1} b_{1}^{2}+2 b_{1} b^{\prime} b}{2 b^{2} r} \tag{7}
\end{equation*}
$$

where $b \equiv b(r), b_{1} \equiv b_{1}(r), b_{2} \equiv b_{2}(r), b^{\prime}=\frac{d b}{d r}, b^{\prime \prime}=\frac{d^{2} b}{d r^{2}}, b_{1}^{\prime}=\frac{d b_{1}}{d r}$ and $b_{2}^{\prime}=\frac{d b_{2}}{d r}$. Plugging Eqs. (3), (5) with Eq. (6) and by using Eq. (7) into the field equations (5), we get the following non-linear differential equations, in the vacuum case

$$
\begin{align*}
& \mathrm{E}_{t}{ }^{t}=\frac{1}{12 b^{2} r}\left\{F\left[b_{2}^{\prime}\left(3 b r^{3} b_{2} b_{1}^{\prime}-3 r^{3} b_{2} b_{1} b^{\prime}+18 b r^{2} b_{2} b_{1}+4 b r^{3} b_{1} b_{2}^{\prime}\right)-b^{\prime}\left(2 b_{1} b-r b_{1} b^{\prime}+r b_{1}^{\prime} b\right)+4 b_{1}^{\prime} b^{2}-2 r b b_{1} b^{\prime \prime}+6 b r^{3} b_{2} b_{1} b_{2}^{\prime \prime}\right]\right. \\
& \left.+F^{\prime}\left[4 b_{1} b^{2}-4 b b_{1} r b^{\prime}+2 b_{1}^{\prime} b^{2} r+6 b_{1} b r^{3} b_{2} b_{2}^{\prime}\right]+4 F^{\prime \prime} b_{1} b^{2} r\right\}=0, \\
& \mathrm{Ł}_{t}^{\phi}=\frac{1}{4 b^{2} r}\left\{2 b_{2} F r b b_{1} b^{\prime \prime}-2 r F b b_{1}\left(r^{2} b_{2}^{2}+b\right) b_{2}^{\prime \prime}-4 F b r^{3} b_{1} b_{2} b_{2}^{\prime 2}-\left(r^{2} b_{2}^{2}+b\right)\left\{\left(F r b_{1}^{\prime}+2 b_{1}\left(3 F+r F^{\prime}\right)\right) b-r F b_{1} b^{\prime}\right\} b_{2}^{\prime}\right. \\
& \left.+b_{2}\left[r b\left(2 F^{\prime} b_{1}+F b_{1}^{\prime}\right) b^{\prime}-2 b^{2}\left(2 F^{\prime} b_{1}+F b_{1}^{\prime}\right)-F r b_{1} b^{\prime 2}\right]\right\}=0, \\
& \mathrm{Ł}_{r}{ }^{r}=\frac{F r b_{1} b^{\prime 2}-8 F^{\prime \prime} b_{1} b^{2} r-2 F r b b_{1} b^{\prime \prime}-\left\{F r b_{1}^{\prime}-2 b_{1}\left(2 F+r F^{\prime}\right)\right\} b b^{\prime}+2 b\left[2 b_{1}\left(b F^{\prime}+F r^{3} b_{2}^{\prime 2}\right)-\left(F+2 r F^{\prime}\right) b b_{1}^{\prime}\right]}{12 r b^{2}}=0, \\
& \mathrm{Ł}_{\phi}{ }^{r}=\frac{r\left[2 F b r b_{1} b_{2}^{\prime \prime}+b_{2}^{\prime}\left(F r b_{1}^{\prime} b-\left\{-2 b F^{\prime} r+F\left(r b^{\prime}-6 b\right)\right\} b_{1}\right)\right]}{4 b^{2}}=0, \\
& \mathrm{£}_{\phi}^{\phi}=\frac{1}{12 r b^{2}}\left\{4 F^{\prime \prime} b_{1} b^{2} r-6 F b r^{3} b_{2} b_{1} b_{2}^{\prime \prime}+4 F r b b_{1} b^{\prime \prime}-8 F b r^{3} b_{1} b_{2}^{2}-3\left[\left(F r b_{1}^{\prime}+2 b_{1}\left\{3 F+r F^{\prime}\right\}\right) b-r F b_{1} b^{\prime}\right] b_{2} r^{2} b_{2}^{\prime}\right. \\
& \left.-2 F r b_{1} b^{\prime 2}+2\left[F r b_{1}^{\prime}+b_{1}\left(r F^{\prime}-F\right)\right] b b^{\prime}+2\left[\left(r F^{\prime}-F\right) b_{1}^{\prime}-4 F b_{1}\right] b^{2}\right\}=0, \tag{8}
\end{align*}
$$

where $F \equiv F(r)=\frac{d f(R(r))}{d R(r)}=\frac{d f(r)}{d r} \times \frac{d r}{d R}, F^{\prime}=\frac{d F(r)}{d r}$, and $F^{\prime \prime}=\frac{d^{2} F(r)}{d r^{2}}$. Since we are dealing with a spherical symmetry spacetime we assume $f(R)=f(r)$. Finally, the form of the trace equation given by $(3)$ takes the following form
$\mathrm{£}=\frac{F\left[b r^{3} b_{1} b_{2}^{\prime 2}-2 b_{1}^{\prime} b^{2}-r b_{1}^{\prime} b b^{\prime}-2 r b b_{1} b^{\prime \prime}+r b_{1} b^{\prime 2}-2 b_{1} b^{\prime} b\right]+F^{\prime}\left[2 b b_{1} r b^{\prime}+2 b_{1}^{\prime} b^{2} r+4 b_{1} b^{2}\right]+4 F^{\prime \prime} b_{1} b^{2} r-3 f b^{2} r}{2 r b^{2}}=0$.

Now, we are going to study special cases of the above differential equations given by Eqs. (8) and (9), trying to find analytical solutions:

## A. When $F(r)=c_{0}$ and $b_{2}=0^{2}$

When $F(r)=$ constant $=c_{0}$ and $b_{2}=0$, the differential equations (8) reduce to the well-known BTZ differential equation and in that case $b_{1}$ and $b_{2}$ take the following form [16]:

$$
\begin{equation*}
b(r)=b_{1}(r)=\Lambda r^{2}-m, \tag{10}
\end{equation*}
$$

where $\Lambda$ and $m$ are integration constants.

[^2]
## B. When $F(r)=c_{0}$ and $b_{2} \neq 0$

When $b_{2} \neq 0$ and $F(r)=c_{0}$ the differential equations (8) coincide with those presented in [38] and we get the solution of these differential equations after rescaling the constants as:

$$
\begin{equation*}
b(r)=b 1(r)=\Lambda r^{2}-m+\frac{J}{r^{2}}, \quad b 2(r)=\Lambda+\frac{\sqrt{J}}{r^{2}} \tag{11}
\end{equation*}
$$

## C. When $F(r) \neq$ constant and $b_{2}=0$

When $F(r) \neq$ constant, i.e., when, for example, $F(r)=c_{0}+\frac{c_{1}}{r^{2}}$ and when $b_{2}=0$ we get after rescaling the constants ${ }^{3}$ :

$$
\begin{align*}
& b(r)=C\left[r^{2} \ln \left(c_{0}+\frac{c_{1}}{r^{2}}\right)-\frac{7 c_{1}}{8 c_{0}}-\frac{C_{1} r^{2}}{24 C_{2} c_{0}^{3}}+\frac{c_{1}^{2}}{4 r^{2} c_{0}^{2}}-\frac{c_{1}^{3}}{24 c_{0}^{3} r^{4}}\right] \\
& b_{1}(r)=\frac{1}{\left(c_{0}-\frac{c_{1}}{r^{2}}\right)^{6}}\left[C_{1} r^{2}-C_{2}\left\{r^{2} c_{0}^{3} \ln \left(c_{0}+\frac{c_{1}}{r^{2}}\right)-21 c_{1} c_{0}^{2}+\frac{6 c_{1}^{2} c_{0}}{r^{2}}-\frac{c_{1}^{3}}{r^{4}}\right\}\right] \tag{12}
\end{align*}
$$

The line-element of solution (12) takes the form:

$$
\begin{align*}
& d s^{2}=-\left[\frac{24 c_{0}{ }^{3} r^{2} \ln \left(c_{0}+\frac{c_{1}}{r^{2}}\right)}{c_{1}}-3 c_{0}\left[7 c_{0}-\frac{2 c_{1}}{r^{2}}\right]-\frac{c_{1}^{2}}{r^{4}}+c_{2} r^{2}\right] d t^{2} \\
& +\frac{\left(c_{0}-\frac{c_{1}}{r^{2}}\right)^{6} d r^{2}}{c_{1}}+r^{2} d \phi^{2}  \tag{13}\\
& \frac{24 c_{0}{ }^{3} r^{2} \ln \left(c_{0}+\frac{c_{1}}{r^{2}}\right)}{c_{1}}-3 c_{0}\left[7 c_{0}-\frac{2 c_{1}}{r^{2}}\right]-\frac{c_{1}^{2}}{r^{4}}+c_{2} r^{2}
\end{align*}
$$

where we have assumed $C=\frac{24 c_{0}{ }^{3}}{c_{1}}, C_{1}=c_{2}$, and $C_{2}=-\frac{1}{c_{1}}{ }^{4}$. The above line-element, (13), shows clearly that the constant $c_{1}$ cannot equal to zero which means that this solution can not coincides with the BTZ GR BH solutions and this yields that such BH solution is a new one. Moreover, the metric (13) asymptotes AdS/dS spacetime when $c_{0}=0$. Now using Eq. (13) to calculate the invariants of GR we obtain the following forms:

$$
\begin{equation*}
R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma}=R_{\mu \nu} R^{\mu \nu} \approx C_{3}+\frac{C_{4}}{r^{2}}+\frac{C_{5}}{r^{4}}, \quad R \approx C_{6}+\frac{C_{7}}{r^{2}}+\frac{C_{8}}{r^{4}} \tag{14}
\end{equation*}
$$

where the constants $C_{i}, i=3 \ldots 8$ are combinations of $c_{0}, c_{1}$, and $c_{2}$, i.e.,

$$
\begin{align*}
& C_{3}=\frac{12}{c_{0}^{12}}\left(c_{2}+24 c_{0}{ }^{3} \ln c_{0}\right)^{2}, \quad C_{4}=\frac{4 C_{3} c_{1}}{c_{0}}, \quad C_{5}=-\frac{24 c_{1}^{2}}{c_{0}^{14}}\left(c_{2}+8 c_{0}{ }^{3}+24 c_{0}{ }^{3} \ln c_{0}\right)\left(c_{2}+24 c_{0}{ }^{3} \ln c_{0}\right), \\
& C_{6}=-\frac{6}{c_{0}{ }^{6}}\left(c_{2}-24 c_{0}{ }^{3} \ln c_{0}\right), \quad C_{7}=\frac{2 C_{6} c_{1}}{c_{0}}, \quad C_{8}=\frac{6 c_{1}^{2}}{c_{0}^{8}}\left(7 c_{2}-8 c_{0}^{3}-168 c_{0}{ }^{3} \ln c_{0}\right) . \tag{15}
\end{align*}
$$

[^3]Here $\left(R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma}, R_{\mu \nu} R^{\mu \nu}, R\right)$ are the Kretschmann scalar, the Ricci tensor square, the Ricci scalar, respectively and all have a true singularity at $r=0$. Moreover, the above equations show that $c_{0}$ must not equal zero. It is important to stress on the fact that the constant $c_{1}$ is the main source for the deviation of the above results from GR that has the following values $\left(R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma}, R_{\mu \nu} R^{\mu \nu}, R\right)=\left(12 \Lambda^{2}, 12 \Lambda^{2}, \mp 8 \Lambda\right)$. Equation (14) indicates that the leading term of the invariants $\left(R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma}, R_{\mu \nu} R^{\mu \nu}, R\right)$ is $\left(C_{3}, C_{3}, C_{6}\right)$. Therefore, Eq. (14) indicates that scalar invariants of our solutions are stronger than the BTZ spacetime of GR.

## D. When $F(r) \neq$ constant and $b_{2} \neq 0$

Now let us turn our attention to the general case, i.e., when $F(r) \neq$ constant $F(r)=c_{0}+\frac{c_{1}}{r^{2}}$ and when $b_{2} \neq 0$ we get the following solutions:

$$
\begin{align*}
& b(r)=\frac{C_{9}{ }^{2}}{36 r^{10} c_{1}{ }^{2} C_{11}}\left(576 C_{11} c_{0}{ }^{6} r^{12}\left[\left(\ln \left(c_{0} r^{2}+c_{1}\right)\right)^{2}+4(\ln r)^{2}\right]+c_{2} r^{12}+21 C_{10} r^{10} c_{0}{ }^{2} c_{1}-6 c_{1}{ }^{2} c_{0} r^{8} C_{10}+C_{11} c_{1}{ }^{6}\right. \\
& +24 \ln \left(c_{0} r^{2}+c_{1}\right)\left[12 r^{8} c_{0}{ }^{4} C_{11} c_{1}{ }^{2}-42 r^{10} c_{0}{ }^{5} C_{11} c_{1}-2 r^{6} c_{0}{ }^{3} c_{1}{ }^{3} C_{11}-r^{12} c_{0}{ }^{3} C_{10}-96 r^{12} c_{0}{ }^{6} C_{11} \ln r\right]+441 c_{1}{ }^{2} c_{0}{ }^{4} r^{8} C_{11} \\
& \left.+48 c_{0}{ }^{3} r^{6} \ln r\left[r^{6} C_{10}+42 c_{0}{ }^{2} r^{4} C_{11} c_{1}-12 c_{0} r^{2} C_{11} c_{1}{ }^{2}+2 c_{1}{ }^{3} C_{11}\right]-6 c_{0}{ }^{3} C_{11} r^{2}\left[42 r^{4} c_{0}{ }^{3}+13 r^{2} c_{0}{ }^{2} c_{1}-2 c_{0} c_{1}{ }^{2}\right]+c_{0}{ }^{3} r^{6} C_{10}\right), \\
& b_{1}(r)=\frac{r^{2}}{\left(c_{0} r^{2}-c_{1}\right)^{6}}\left(576 C_{11} c_{0}{ }^{6} r^{12}\left[\left(\ln \left(c_{0} r^{2}+c_{1}\right)\right)^{2}+4(\ln r)^{2}\right]+c_{2} r^{12}+21 C_{10} r^{10} c_{0}{ }^{2} c_{1}-6 c_{1}^{2} c_{0} r^{8} C_{10}\right. \\
& +24 c_{0}^{3} r^{6} \ln \left(c_{0} r^{2}+c_{1}\right)\left[12 r^{2} c_{0} C_{11} c_{1}{ }^{2}-2 c_{1}{ }^{3} C_{11}-r^{6} C_{10}-42 r^{4} c_{0}{ }^{2} C_{11} c_{1}-r^{6} c_{0}{ }^{3} \ln r C_{11}\right]+C_{11} c_{1}^{6}+78 C_{11} r^{4} c_{0}{ }^{2} c_{1}{ }^{4} \\
& \left.+48 c_{0}{ }^{3} r^{6} \ln r\left[r^{6} C_{10}+C_{11}\left\{42 c_{0}{ }^{2} r^{4} c_{1}-12 c_{0} r^{2} c_{1}{ }^{2}+2 c_{1}^{3}\right\}\right]+C_{11} r^{2} c_{0} c_{1}{ }^{2}\left[441 c_{0}^{3} r^{6}-252 c_{1} r^{4} c_{0}{ }^{2}-12 c_{1}^{3}\right]+c_{1}^{3} r^{6} C_{10}\right) \\
& b_{2}(r)=\frac{6 c_{4} c_{1} r^{6}+48 C_{9} c_{0}{ }^{3} r^{6} \ln r-24 C_{9} r^{6} c_{0}{ }^{3} \ln \left(c_{0} r^{2}+c_{1}\right)+21 C_{9} r^{4} c_{0}^{2} c_{1}-6 C_{9} r^{2} c_{0} c_{1}{ }^{2}+C_{9} c_{1}^{3}}{6 c_{1} r^{6}}, \tag{16}
\end{align*}
$$

where $c_{3}, c_{4}, C_{9}, C_{10}$ and $C_{11}$ are constants. Re-scaling the constants by putting $C_{9}=6 c_{1}, C_{10}=\frac{1}{c_{1}}$ and $C_{11}=1$ in Eq. (16) we get

$$
\begin{align*}
b(r)= & c_{2} r^{2}+21 c_{0}{ }^{2}-\frac{c_{0}{ }^{3} \ln \left(c_{0}+\frac{c_{1}}{r^{2}}\right)}{c_{1}}\left[24\left\{1-24 c_{0}{ }^{3} \ln \left(c_{0}+\frac{c_{1}}{r^{2}}\right)\right\} r^{2}+1008 c_{0}{ }^{2} c_{1}{ }^{2}-\frac{288 c_{0} c_{1}{ }^{3}}{r^{2}}+\frac{48 c_{1}^{4}}{r^{4}}\right] \\
& -\frac{3 c_{0} c_{1}\left[2-147 c_{1} c_{0}{ }^{3}\right]}{r^{2}}+\frac{c_{1}{ }^{2}\left[1-252 c_{0}{ }^{3} c_{1}\right]}{r^{4}}+\frac{78 c_{0}{ }^{2} c_{1}{ }^{4}}{r^{6}}-\frac{12 c_{0} c_{1}{ }^{5}}{r^{8}}+\frac{c_{1}{ }^{6}}{r^{10}} \\
b_{1}(r)= & \frac{b(r)}{\left(c_{0}-\frac{c_{1}}{r^{2}}\right)}, \quad b_{2}(r)=c_{4}-24 c_{0}{ }^{3} \ln \left(c_{0}+\frac{c_{1}}{r^{2}}\right)+\frac{21 c_{0}^{2} c_{1}}{r^{2}}-\frac{6 c_{0} c_{1}{ }^{2}}{r^{4}}+\frac{c_{1}^{3}}{r^{6}} . \tag{17}
\end{align*}
$$

The line-element of the BH solution (17) takes the following form:

$$
\begin{align*}
& d s^{2}= \\
& -\left[\frac{\left.\frac{24 r^{6} c_{0}{ }^{3}}{c_{1}}\left(1-2 c_{4} c_{1}\right) \ln \left(c_{0}+\frac{c_{1}}{r^{2}}\right)+2 c_{1}^{3} c_{4}-\left(1+12 c_{4} c_{0} r^{2}\right) c_{1}^{2}+6 c_{0} r^{2}\left(1+7 c_{4} c_{0} r^{2}\right) c_{1}+\left(\left(c_{4}^{2}-c_{2}\right) r^{2}-21 c_{0}^{2}\right) r^{4}\right]}{r^{4}}\right] d t^{2} \\
& +\frac{\left(c_{0}-\frac{c_{1}}{r^{2}}\right)^{6} r^{4} d r^{2}}{\frac{24 r^{6} c_{0}{ }^{3}}{c_{1}}\left(1-2 c_{4} c_{1}\right) \ln \left(c_{0}+\frac{c_{1}}{r^{2}}\right)+2 c_{1}^{3} c_{4}-\left(1+12 c_{4} c_{0} r^{2}\right) c_{1}^{2}+6 c_{0} r^{2}\left(1+7 c_{4} c_{0} r^{2}\right) c_{1}+\left(\left(c_{4}^{2}-c_{2}\right) r^{2}-21 c_{0}^{2}\right) r^{4}} \\
& +r^{2} d \phi^{2}+\frac{c_{4} r^{6}+48 c_{0}{ }^{3} r^{6} \ln r-24 c_{0}^{3} \ln \left(c_{0} r^{2}+c_{1}\right) r^{6}+21 r^{4} c_{0}^{2} c_{1}-6 r^{2} c_{0} c_{1}^{2}+c_{1}^{3}}{r^{4}} d t d \phi \tag{18}
\end{align*}
$$

Now use Eq. (17) in order to calculate the invariants as in the case of non-rotating case we obtain the following expressions:

$$
\begin{equation*}
R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma}=R_{\mu \nu} R^{\mu \nu} \approx C_{12}+\frac{C_{13}}{r^{2}}+\frac{C_{14}}{r^{4}}, \quad R \approx C_{15}+\frac{C_{16}}{r^{2}}+\frac{C_{17}}{r^{4}} \tag{19}
\end{equation*}
$$

where $C_{12} \cdots, C_{17}$ are defined as:
$C_{12}=\frac{12}{c_{0}{ }^{12} c_{1}^{2}}\left(c_{2} c_{1}-24 c_{0}{ }^{3} \ln c_{0}\left[1-24 c_{0}{ }^{3} c_{1} \ln c_{0}\right]\right)^{2}, \quad C_{13}=\frac{4 C_{3} c_{1}}{c_{0}}$
$C_{14}=-\frac{24\left(c_{2} c_{1}-24 c_{0}{ }^{3} \ln c_{0}\left[1-24 c_{0}{ }^{3} c_{1} \ln c_{0}\right]\right)\left(576 c_{0}{ }^{6} c_{1}\left(\ln \left(c_{0}\right)\right)^{2}+\left(-24 c_{0}{ }^{3}+384 c_{0}{ }^{6} c_{1}\right) \ln \left(c_{0}\right)+c_{1} c_{2}-8 c_{0}{ }^{3}\right)}{c_{0}{ }^{14}}$,
$C_{15}=-\frac{6}{c_{0}{ }^{6} c_{1}}\left(c_{2} c_{1}-24 c_{0}{ }^{3} \ln c_{0}\left[1-24 c_{0}{ }^{3} c_{1} \ln c_{0}\right]\right), \quad C_{16}=\frac{2 C_{0} c_{1}}{c_{0}}$,
$C_{17}=\frac{6 c_{0}}{c_{0}^{8}}\left(7 c_{2} c_{1}-8 c_{0}^{3}+24 c_{0}{ }^{3} \ln c_{0}\left[16 c_{0}{ }^{3}-7 c_{3}+168 c_{0}{ }^{3} \ln c_{0}\right]\right)$.
Equation (19) shows that all the invariants have a true singularity at $r=0$. Moreover, the above equations show that $c_{1}$ must not equal zero. It is important to stress on the fact that the constant $c_{1}$ is the main source for the deviation of the above results from the BH BTZ of GR .

## IV. PHYSICAL PROPERTIES OF THE BH SOLUTIONS (12) AND (17)

In this section, we are going, to understand the physical properties of the BH solutions (12) and (17).

## A. Physical properties of the non-rotating BH solution, Eq. (12)

For the BH solution (12), we write the asymptote behaviors of the metric potentials, $b(r)$, and $b_{1}(r)$, given by Eq. (12) and get the following expressions:

$$
\begin{align*}
& g_{t t}=\frac{1}{g_{r r}}=\frac{c_{2}-c_{0}{ }^{3} \ln \left(c_{0}+\frac{c_{1}}{r^{2}}\right)}{c_{1}} r^{2}+21 c_{0}^{2}-\frac{6 c_{0} c_{1}}{r^{2}}-\frac{c_{1}{ }^{2}}{r^{4}} \\
& g_{t t}(r \rightarrow \infty) \approx r^{2} \Lambda_{e f f}-M+\frac{\mathcal{J}}{r^{2}}-\frac{\mathcal{J}_{1}^{2}}{r^{4}}-\mathcal{O}\left(r^{-6}\right) \\
& g_{t t}(r \rightarrow 0) \approx\left\{\Lambda_{e f f}-\frac{24 c_{0}^{3}}{c_{1}}\left[\ln c_{0}+c_{1} \ln \left(\frac{r^{2}}{c_{1}}\right)\right]\right\} r^{2}-7 M+\frac{4 J c_{0}^{3} r^{4}}{c_{1}}-\mathcal{O}\left(r^{6}\right), \tag{21}
\end{align*}
$$

where $M=3 c_{0}{ }^{2}, \quad \Lambda_{e f f}=\frac{c_{2} c_{1}-24 c_{0}{ }^{3} \ln c_{0}}{c_{1}}, \mathcal{J}=6 c_{0} c_{1}, \mathcal{J}_{1}{ }^{2}=7 c_{1}{ }^{2}$. Using Eq. (21) in (6), when $b_{2}=0$, we get

$$
\begin{equation*}
d s^{2} \approx-\left[r^{2} \Lambda_{e f f}-M+\frac{\mathcal{J}}{r^{2}}-\frac{\mathcal{J}_{1}^{2}}{r^{4}}\right] d t^{2}+\frac{d r^{2}}{\left[r^{2} \Lambda_{e f f}-M+\frac{\mathcal{J}}{r^{2}}-\frac{\mathcal{J}_{1} 2}{r^{4}}\right]}+r^{2} d \phi^{2} \tag{22}
\end{equation*}
$$

The line element (22) is asymptotically approaching AdS/dS spacetime and does not coincide with the BTZ spacetime due to the contribution of the extra terms that come mainly from the constant $c_{1}$ whose source is the effect of higherorder curvature terms of $f(R)$ [16]. Moreover, Eq. (21) shows in a clear way that the constant $c_{1}$ cannot take the value zero which indeed indicate that the BH solution (12) cannot rerun to GR. This means that the BH solution (12) is a new one in the $f(R)$ modified theory.

Now we are going to use Eq. (12) in Eq.(7) to calculate the Ricci scalar and get:

$$
\begin{align*}
& R(r)=\frac{r^{8}}{6\left(c_{0} r^{2}+c_{1}\right)^{2}\left(c_{0} r^{2}-c_{1}\right)^{7}}\left(24 r^{4} c_{0}{ }^{3} \ln \left(c_{0} r^{2}+c_{1}\right)\left\{r^{6} c_{0}^{3}-3 c_{1} r^{4} c_{0}^{2}-9 c_{1}^{2} r^{2} c_{0}-5 c_{1}^{3}\right\}-3 c_{0}^{2} r^{2} c_{1}\left[c_{2} r^{6}+12 c_{1}{ }^{3}\right]\right. \\
& +48 c_{0}{ }^{3} r^{4} \ln r\left\{3 c_{0}{ }^{2} r^{4} c_{1}-r^{6} c_{0}{ }^{3}+9 c_{0} r^{2} c_{1}{ }^{2}+5 c_{1}{ }^{3}\right\}-24 c_{1} r^{8} c_{0}{ }^{5}+76 c_{1}{ }^{2} r^{6} c_{0}{ }^{4}+c_{0}{ }^{3} c_{2} r^{10}+172 r^{4} c_{0}{ }^{3} c_{1}{ }^{3}-9 c_{0} c_{2} r^{6} c_{1}{ }^{2} \\
& \left.-4 c_{0} c_{1}{ }^{5}-5 c_{2} c_{1}^{3} r^{4}\right), \\
& R(r \rightarrow \infty) \approx-\frac{6}{c_{0}{ }^{6}}\left(c_{2}-24 c_{0}{ }^{3} \ln c_{0}\right)-\frac{12 c_{1}}{c_{0}{ }^{7} r^{2}}\left(c_{2}-24 c_{0}{ }^{3} \ln c_{0}\right)+\frac{6 c_{1}{ }^{2}}{r^{4} c_{0}^{8}}\left(7 c_{2}-8 c_{0}{ }^{3}-168 c_{0}{ }^{3} \ln c_{0}\right)+\mathcal{O}\left(r^{-6}\right), \\
& R(r \rightarrow 0) \approx \frac{24 c_{0} r^{8}}{c_{1}^{4}}-\frac{96 c_{0}^{2} r^{10}}{c_{1}{ }^{5}}+\mathcal{O}\left(r^{12}\right), \\
& r(R) \approx \pm 2 \sqrt{\frac{3\left(c_{2} c_{1}-24 c_{0}{ }^{3} \ln c_{0}\right) c_{1}}{c_{0}\left(144 c_{0}{ }^{3} \ln c_{0}-R c_{0}{ }^{6} c_{1}-6 c_{2} c_{1}\right)}}, \quad r \rightarrow \infty . \tag{23}
\end{align*}
$$

Eq. (23) shows that

$$
c_{0}>0, \quad \text { and } \quad c_{1}>0, \quad \text { and } \quad c_{2} c_{1}-24 c_{0}{ }^{3} \ln c_{0}>0, \quad \text { and } \quad 144 c_{0}{ }^{3} \ln c_{0}-R c_{0}{ }^{6} c_{1}-6 c_{2} c_{1}>0
$$

or $\quad c_{1}<0, \quad$ and $\quad c_{2} c_{1}-24 c_{0}{ }^{3} \ln c_{0}<0, \quad$ and $\quad 144 c_{0}{ }^{3} \ln c_{0}-R c_{0}{ }^{6} c_{1}-6 c_{2} c_{1}>0$,
or $\quad c_{1}<0, \quad$ and $\quad c_{2} c_{1}-24 c_{0}{ }^{3} \ln c_{0}>0, \quad$ and $\quad 144 c_{0}{ }^{3} \ln c_{0}-R c_{0}{ }^{6} c_{1}-6 c_{2} c_{1}<0$,
or $\quad c_{1}>0, \quad$ and $\quad c_{2} c_{1}-24 c_{0}{ }^{3} \ln c_{0}<0, \quad$ and $\quad 144 c_{0}{ }^{3} \ln c_{0}-R c_{0}{ }^{6} c_{1}-6 c_{2} c_{1}<0$,
otherwise we will have an imaginary quantity. Also we stress that the constant $c_{0}$ must take a positive value as Eq. (23) shows. The form of $f(r)$, that reproduces solution (12), after using the form of $F(r)=c_{0}+\frac{c_{1}}{r^{2}}$, takes the following form:

$$
\begin{align*}
& f(r)=\frac{1}{c_{0}{ }^{5}\left(c_{0} r^{2}-c_{1}\right)^{7}\left(c_{0} r^{2}+c_{1}\right) c_{1}}\left\{96 r^{10} c_{0}{ }^{8} c_{1}\left(c_{0} r^{2}+c_{1}\right)\left(4 c_{1} c_{0} r^{2}+9 c_{1}{ }^{2}-c_{0}{ }^{2} r^{4}\right) \ln \left(c_{0}+\frac{c_{1}}{r^{2}}\right)-c_{0}{ }^{13} r^{16}\right. \\
& +96 c_{0}{ }^{3} c_{1}\left(c_{0} r^{2}+c_{1}\right)\left(c_{0} r^{2}-c_{1}\right)^{7} \ln \left(\frac{2}{3}\right)-14 c_{0}{ }^{11} r^{12} c_{1}{ }^{2}+\left(96 c_{1}{ }^{2} r^{14}+14 r^{10} c_{1}{ }^{3}\right) c_{0}{ }^{10}-288 c_{1}{ }^{3} c_{0}{ }^{9} r^{12}+6 c_{0}{ }^{12} r^{14} c_{1} \\
& -\left(14 r^{6} c_{1}{ }^{5}+1088 c_{1}^{4} r^{10}\right) c_{0}{ }^{8}+\left(14 r^{4} c_{1}{ }^{6}-12 c_{1}^{2} r^{14} c_{2}-288 c_{1}^{5} r^{8}\right) c_{0}{ }^{7}+\left(-6 c_{1}{ }^{7} r^{2}+32 r^{6} c_{1}{ }^{6}+108 c_{1}{ }^{3} r^{12} c_{2}\right) c_{0}{ }^{6} \\
& \left.+\left(c_{1}{ }^{8}-20 c_{1}^{4} r^{10} c_{2}\right) c_{0}^{5}+56 r^{6} c_{0}{ }^{3} c_{1}{ }^{6} c_{2}-56 c_{1}{ }^{7} c_{2} r^{4} c_{0}{ }^{2}+24 c_{1}{ }^{8} c_{2} r^{2} c_{0}-4 c_{1}{ }^{9} c_{2}\right\}, \\
& f(r \rightarrow 0) \approx \frac{96 c_{1} \ln \left(\frac{2}{3 c_{0}}\right)-c_{0}^{2}}{c_{0}^{2} c_{1}}-\frac{12 c_{1}\left(c_{2}+24 c_{0}{ }^{3} \ln c_{0}\right)}{c_{0}{ }^{6} r^{2}}+\frac{12 c_{1}^{2}\left(3 c_{2}+4 c_{0}^{3}+72 c_{0}{ }^{3} \ln c_{0}\right)}{c_{0}{ }^{7} r^{4}}+\mathcal{O}\left(r^{-6}\right), \\
& f(r \rightarrow 0) \approx \frac{96 c_{0}{ }^{3} c_{1} \ln \left(\frac{2}{3}\right)-c_{0} 65+4 c_{1} c_{2}}{c_{0} 65 c_{1}}-\frac{32 c_{0}}{c_{1}{ }^{3}} r^{6}+\mathcal{O}\left(r^{8}\right) . \tag{25}
\end{align*}
$$

The use of the last equation of $(23), r(R)$, in the first equation of (25) we get:

$$
\begin{equation*}
f(R \rightarrow \infty) \approx d_{1}+d_{2} R+d_{3} R^{2} \tag{26}
\end{equation*}
$$

where $d_{i}$ are constants that have the form:

$$
\begin{align*}
d_{1} & =-\frac{1}{729} \frac{3072 \ln c_{0} c_{1} c_{0}{ }^{3}+729 c_{0}{ }^{5}-73056 c_{1} c_{0}{ }^{3} \ln 2+69984 c_{1} c_{0}{ }^{3} \ln 3+256 c_{1} c_{0}{ }^{3}-2788 c_{1} c_{2}}{c_{0}{ }^{5} c_{1}} \\
d_{2} & =-\frac{64}{6561} \frac{c_{1} c_{0}\left(408 \ln c_{0} c_{0}{ }^{3}-408 c_{0}{ }^{3} \ln 2+358 c_{0}^{3}+17 c_{2}\right)}{24 \ln c_{0} c_{0}{ }^{3}-c_{1} c_{2}} \\
d_{3} & =-\frac{32}{59049} \frac{c_{0}{ }^{7} c_{1}{ }^{2}\left(1704 \ln c_{0} c_{0}{ }^{3}-1704 c_{0}{ }^{3} \ln 2+952 c_{0}^{3}+71 c_{2}\right)}{\left(24 \ln c_{0} c_{0}{ }^{3}-c_{1} c_{2}\right)^{2}} \tag{27}
\end{align*}
$$

To avoid the tachyonic instability, we check the Dolgov-Kawasaki stability criterion [52-55] which states that the second derivative of the gravitational model $f_{R R}$ must be always positive. Using the chain rule we get

$$
\begin{equation*}
f_{R}=\frac{d f(R)}{d R}=\frac{d f(r)}{d r} \frac{d r}{d R}=c_{0}+\frac{c_{1}}{r^{2}} \tag{28}
\end{equation*}
$$

$f_{R R}=\frac{d^{2} f(R)}{d R^{2}}=\frac{d f_{R}}{d R}=\frac{d f_{R}(r)}{d r} \frac{d r}{d R}$
$=\left(c_{0} r^{2}-c_{1}\right)^{8}\left(c_{0} r^{2}+c_{1}\right)^{3} c_{1}\left\{12 r^{10}\left[24 \ln \left(c_{0} r^{2}+c_{1}\right) r^{12} c_{0}{ }^{7}-288 \ln \left(c_{0} r^{2}+c_{1}\right) c_{1} r^{10} c_{0}{ }^{6}-1008 \ln \left(c_{0} r^{2}+c_{1}\right) c_{1}{ }^{2} r^{8} c_{0}{ }^{5}\right.\right.$
$-1056 \ln \left(c_{0} r^{2}+c_{1}\right) c_{1}{ }^{3} r^{6} c_{0}{ }^{4}-360 \ln \left(c_{0} r^{2}+c_{1}\right) c_{1}{ }^{4} r^{4} c_{0}{ }^{3}-48 \ln r r^{12} c_{0}{ }^{7}+576 c_{0}{ }^{6} r^{10} \ln r c_{1}-32 c_{1} r^{10} c_{0}{ }^{6}$
$+2016 c_{0}^{5} r^{8} c_{1}^{2} \ln r+280 c_{1}^{2} r^{8} c_{0}^{5}+2112 c_{0}{ }^{4} r^{6} c_{1}^{3} \ln r+816 c_{1}^{3} r^{6} c_{0}^{4}+656 c_{1}{ }^{4} r^{4} c_{0}{ }^{3}+720 c_{0}^{3} r^{4} c_{1}{ }^{4} \ln +80 c_{1}^{5} r^{2} c_{0}{ }^{2}$
$\left.\left.-8 c_{1}{ }^{6} c_{0}-c_{2} r^{12} c_{0}{ }^{4} c_{1}+12 c_{2} c_{1}^{2} r^{10} c_{0}{ }^{3}+42 c_{1}{ }^{3} c_{2} r^{8} c_{0}{ }^{2}+44 c_{1}{ }^{4} c_{2} r^{6} c_{0}+15 c_{1}{ }^{5} c_{2} r^{4}\right]\right\}^{-1}$.
The behavior of the Ricci scalar, $f(r), f_{R}$ and $f_{R R}$ are given in Figure 1. As figure 1 (a)-1 (d) shows that the Ricci scalar, $f(r)$, the first derivative of $f(R)$ and the second derivative of $f(R)$ all of them have positive value which means that the condition of stability given by Dolgov-Kawasaki is satisfied [52-55].


Figure 1. Systematic plots of; (a) the Ricci scalar given by Eq. (23); (b) the analytic function f(r) given (25); (c) the derivative function $f_{R}$ given by Eq. (28), and (d) the second derivative $f_{R R}$ given by Eq. (29). All the figures are plotted using the following values of the constants, $M=1, c_{0}=0.1, c_{1}=10, c_{2}=-10^{5}$. These values satisfy the constrains given by Eq. (24).

## B. Physical properties of the rotating BH solution, Eq. (17)

Now, let us turn our attention to the rotating case to where the asymptote behaviors of the metric potentials, $g_{t t}(r)$, $g_{r \phi}(r)$, and $g_{r r}(r)$ are given by Eq. (17) and get
$g_{t t}=\frac{\frac{24 r^{6} c_{0}{ }^{3}}{c_{1}}\left(1-2 c_{4} c_{1}\right) \ln \left(c_{0}+\frac{c_{1}}{r^{2}}\right)+2 c_{1}^{3} c_{4}-\left(1+12 c_{4} c_{0} r^{2}\right) c_{1}^{2}+6 c_{0} r^{2}\left(1+7 c_{4} c_{0} r^{2}\right) c_{1}+\left(\left(c_{4}^{2}-c_{2}\right) r^{2}-21 c_{0}^{2}\right) r^{4}}{r^{4}}$,
$g_{t t}(r \rightarrow \infty) \approx r^{2} \Lambda_{1_{e f f}}-\mathcal{M}+\frac{\mathcal{J}}{r^{2}}-\frac{\mathcal{J}_{1}^{2}}{r^{4}}-\mathcal{O}\left(r^{-6}\right)$,
$g_{t t}(r \rightarrow 0) \approx\left(\Lambda_{1 e f f}-48 c_{0}{ }^{3}\left(1-2 c_{1} c_{4}\right) \ln r\right) r^{2}-7 \mathcal{M}-\frac{24\left[2 c_{1} c_{4}-1\right] c_{0}{ }^{4} r^{4}}{c_{1}{ }^{2}}+\mathcal{O}\left(r^{6}\right)$,
$g_{r r}=\frac{g_{t t}}{\left(c_{0}-\frac{c_{1}}{r^{2}}\right)^{6}}$,
$g_{r r}(r \rightarrow \infty) \approx \frac{c_{1} c_{0}{ }^{6}}{\left(c_{1} c_{2}-24 c_{0}{ }^{3} \ln c_{0}\left[1-24 c_{0}{ }^{3} c_{1} \ln c_{0}\right]\right) r^{2}}+\frac{3 c_{1}{ }^{2} c_{0}{ }^{5}\left(2 c_{1} c_{2}-c_{0}{ }^{3}\left[1-48 \ln c_{0}\left\{1+c_{0}{ }^{3} c_{1}\left(1+24 \ln c_{0}\right)\right\}\right]\right)}{\left(c_{1} c_{2}-24 c_{0}{ }^{3} \ln c_{0}\left[1-24 c_{0}{ }^{3} c_{1} \ln c_{0}\right]\right)^{2} r^{4}}$
$+\mathcal{O}\left(r^{-6}\right)$,
$g_{r r}(r \rightarrow 0) \approx\left(\Lambda_{1 e f f}-48 c_{0}{ }^{3}\left(1-2 c_{1} c_{4}\right) \ln r\right) r^{2}-7 \mathcal{M}-\frac{24\left[2 c_{1} c_{4}-1\right] c_{0}{ }^{4} r^{4}}{c_{1}{ }^{2}}-\mathcal{O}\left(r^{6}\right)$,
$g_{t \phi}=\frac{c_{2} r^{6}-24 c_{0}{ }^{3} r^{6} \ln \left(c_{0}+\frac{c_{1}}{r^{2}}\right)+21 c_{0}{ }^{2} c_{1} r^{4}-6 c_{0} c_{1}{ }^{2} r^{2}+c_{1}{ }^{3}}{r^{4}}$,
$g_{t \phi}(r \rightarrow \infty) \approx\left(c_{4}-24 c_{0}{ }^{3} \ln c_{0}\right) r^{2}-3 c_{0}{ }^{2} c_{1}+\frac{6 c_{0} c_{1}{ }^{2}}{r^{2}}-\frac{7 c_{1}{ }^{3}}{r^{4}}-\mathcal{O}\left(r^{-6}\right)$
$g_{t \phi}(r \rightarrow 0) \approx 21 c_{0}{ }^{2} c_{1}+\left[c_{4}+24 c_{0}{ }^{3} \ln \left(\frac{r^{2}}{c_{1}}\right)\right] r^{2}-\frac{24 c_{0}{ }^{4} r^{4}}{c_{1}}+\mathcal{O}\left(r^{6}\right)$.
where $\mathcal{M}=-3 c_{0}^{2}\left(1-2 c_{1} c_{4}\right), \quad \Lambda_{1 e f f}=\frac{24 c_{0}{ }^{3}}{c_{1}}\left(1-2 c_{4} c_{1}\right) \ln c_{0}+c_{4}^{2}-c_{2}, \mathcal{J}=-6 c_{0} c_{1}\left(1-2 c_{1} c_{4}\right), \mathcal{J}_{1}^{2}=-7 c_{1}^{2}\left(1-2 c_{1} c_{4}\right)$.
Using Eq. (21) in (6), when $b_{2} \neq 0$, we get

$$
\begin{align*}
& d s^{2} \approx-\left[r^{2} \Lambda_{1 e f f}-\mathcal{M}+\frac{\mathcal{J}}{r^{2}}-\frac{\mathcal{J}_{1}^{2}}{r^{4}}\right] d t^{2}+\frac{d r^{2}}{\frac{c_{1} c_{0}{ }^{6}}{\left(c_{1} c_{2}-24 c_{0}{ }^{3} \ln c_{0}\left[1-24 c_{0}{ }^{3} c_{1} \ln c_{0}\right]\right) r^{2}}+\frac{3 c_{1}{ }^{2} c_{0}{ }^{5}\left(2 c_{1} c_{2}-c_{0}{ }^{3}\left[1-48 \ln c_{0}\left\{1+c_{0}{ }^{3} c_{1}\left(1+24 \ln c_{0}\right)\right\}\right]\right)}{\left(c_{1} c_{2}-24 c_{0}{ }^{3} \ln c_{0}\left[1-24 c_{0}{ }^{3} c_{1} \ln c_{0}\right]\right)^{2} r^{4}}} \\
& +r^{2} d \phi^{2}+\frac{c_{4} r^{6}+48 r^{6} c_{0}{ }^{3} \ln r-24 c_{0}{ }^{3} \ln \left(c_{0} r^{2}+c_{1}\right) r^{6}+21 r^{4} c_{0}{ }^{2} c_{1}-6 r^{2} c_{0} c_{1}{ }^{2}+c_{1}{ }^{3}}{r^{4}} d t d \phi \tag{31}
\end{align*}
$$

The line element (22) is asymptotically approaching AdS spacetime and as we discussed in the non-rotating case it does not coincide with the rotating BTZ spacetime [39] due to the contribution of the extra terms that come from
the higher-order curvature terms of $f(R)$ [16]. Now we are going to use Eq. (17) in Eq. (7) and get

Equation (23) shows that when the constant $c_{0}$ must not equal to zero also Eq. (23) shows that either
$c_{0}>0, \quad$ and $\quad c_{1}>0, \quad$ and $\quad\left(c_{2} c_{1}+576 c_{0}{ }^{6} c_{1}\left(\ln c_{0}\right)^{2}-24 c_{0}{ }^{3} \ln c_{0}\right)>0$, and $\quad\left(144 c_{0}{ }^{3} c_{4} \ln c_{0}-R c_{0}{ }^{6} c_{1}-6 c_{2} c_{1}-3456 c_{0}{ }^{6} c_{1}\left(\ln c_{0}\right)^{2}\right)>0$,

$$
\text { or } \quad c_{1}<0, \quad \text { and } \quad\left(c_{2} c_{1}+576 c_{0}{ }^{6} c_{1}\left(\ln c_{0}\right)^{2}-24 c_{0}{ }^{3} \ln c_{0}\right)<0, \quad \text { and }
$$

$$
\left(144 c_{0}{ }^{3} c_{4} \ln c_{0}-R c_{0}{ }^{6} c_{1}-6 c_{2} c_{1}-3456 c_{0}{ }^{6} c_{1}\left(\ln c_{0}\right)^{2}\right)>0
$$

$$
\text { or } \quad c_{1}<0, \quad \text { and } \quad\left(c_{2} c_{1}+576 c_{0}{ }^{6} c_{1}\left(\ln c_{0}\right)^{2}-24 c_{0}{ }^{3} \ln c_{0}\right)>0, \quad \text { and }
$$

$$
\left(144 c_{0}{ }^{3} c_{4} \ln c_{0}-R c_{0}{ }^{6} c_{1}-6 c_{2} c_{1}-3456 c_{0}{ }^{6} c_{1}\left(\ln c_{0}\right)^{2}\right)<0
$$

$$
\text { or } \quad c_{1}>0, \quad \text { and } \quad\left(c_{2} c_{1}+576 c_{0}{ }^{6} c_{1}\left(\ln c_{0}\right)^{2}-24 c_{0}{ }^{3} \ln c_{0}\right)<0, \quad \text { and }
$$

$$
\begin{equation*}
\left(144 c_{0}^{3} c_{4} \ln c_{0}-R c_{0}{ }^{6} c_{1}-6 c_{2} c_{1}-3456 c_{0}{ }^{6} c_{1}\left(\ln c_{0}\right)^{2}\right)<0 \tag{33}
\end{equation*}
$$

$$
\begin{align*}
& R(r)=\frac{6}{c_{1}\left(c_{0} r^{2}+c_{1}\right)^{2}\left(c_{1}-c_{0} r^{2}\right)^{7}}\left\{c_{0}{ }^{3} c_{2} r^{18} c_{1}-3 c_{0}{ }^{2} c_{2} r^{16} c_{1}{ }^{2}-9 c_{0} c_{2} r^{14} c_{1}{ }^{3}+48 c_{0}{ }^{6} \ln r r^{18}-1968 r^{12} c_{1}{ }^{4} c_{0}{ }^{6}-4 c_{1}{ }^{10}\right. \\
& +576 r^{14} c_{1}{ }^{3} c_{0}{ }^{7}-172 r^{12} c_{1}{ }^{3} c_{0}{ }^{3}-76 c_{0}{ }^{4} r^{14} c_{1}{ }^{2}+268 c_{0}{ }^{4} r^{8} c_{1}{ }^{6}+24 c_{0}{ }^{5} r^{16} c_{1}-2940 c_{0}{ }^{5} r^{10} c_{1}{ }^{5}+20 c_{0} r^{2} c_{1}{ }^{9}-5 c_{2} r^{12} c_{1}{ }^{4} \\
& +24 r^{8} \ln r\left[96 c_{0}{ }^{8} c_{1}{ }^{2} r^{8}-304 c_{0}{ }^{7} c_{1}{ }^{3} r^{6}-688 c_{0}{ }^{6} c_{1}{ }^{4} r^{4}-144 c_{0}{ }^{5} c_{1}{ }^{5} r^{2}-18 c_{0}{ }^{4} c_{1}{ }^{2} r^{6}-10 c_{0}{ }^{3} c_{1}{ }^{3} r^{4}-6 c_{0}{ }^{5} c_{1} r^{8}+16 c_{0}{ }^{4} c_{1}{ }^{6}\right] \\
& +24 c_{0}{ }^{4} r^{8} c_{1} \ln \left(c_{0} r^{2}+c_{1}\right)\left[9 r^{6} c_{1}+152 r^{6} c_{1}{ }^{2} c_{0}{ }^{3}+344 r^{4} c_{1}{ }^{3} c_{0}{ }^{2}+72 c_{0} r^{2} c_{1}{ }^{4}-192 c_{1}{ }^{5}+3 c_{0} r^{8}+5 r^{4} c_{1}{ }^{2}-c_{0}^{2} r^{10}\right] \\
& -576 c_{0}{ }^{6} c_{1} r^{12} \ln \left(\left(c_{0} r^{2}+c_{1}\right)\right)^{2}\left[3 r^{4} c_{0}{ }^{2} c_{1}+9 r^{2} c_{1}{ }^{2} c_{0}+5 c_{1}{ }^{3} r^{2} c_{0}{ }^{3}\right]-24 c_{0}{ }^{3} r^{6} c_{1}{ }^{7}-24 c_{0}{ }^{2} r^{4} c_{1}{ }^{8}-36 c_{0}{ }^{2} r^{10} c_{1}{ }^{4}+4 c_{0} r^{8} c_{1}{ }^{5} \\
& -2304 c_{0}{ }^{6} c_{1}(\ln r)^{2} r^{12}\left[3 c_{0}{ }^{2} c_{1} r^{4}+9 c_{0} c_{1}{ }^{2} r^{2}+5 c_{1}{ }^{3}-r^{6} c_{0}{ }^{3}\right]+1152 r^{8} c_{0}{ }^{7} c_{1} \ln \left(c_{0} r^{2}+c_{1}\right) \ln r\left[18 c_{1}{ }^{2} r^{6}+6 c_{0} c_{1} r^{8}-2 r^{10} c_{0}{ }^{2}\right. \\
& \left.\left.-r^{6} c_{0}^{3}+10 c_{1}^{3} r^{4}\right]\right\}, \\
& R(r \rightarrow \infty) \approx-\frac{6\left[c_{2} c_{1}+576 c_{0}{ }^{6}\left(\ln c_{0}\right)^{2}-24 c_{0}{ }^{3} c_{3} \ln c_{0}\right]}{c_{0}{ }^{6} c_{1}}-\frac{12\left(c_{2} c_{1}+576 c_{0}{ }^{6} c_{1}\left(\ln c_{0}\right)^{2}-24 c_{0}{ }^{3} \ln c_{0}\right)}{c_{0}{ }^{7} r^{2}} \\
& +\frac{6 c_{1}\left(7 c_{2}+4032 c_{0}{ }^{6}\left(\ln c_{0}\right)^{2}-168 c_{0}{ }^{3} c_{3} \ln c_{0}-8 c_{0}{ }^{3} c_{3}+384 c_{0}{ }^{6} \ln c_{0}\right)}{c_{0}{ }^{8} r^{4}}+\mathcal{O}\left(r^{-6}\right), \\
& R(r \rightarrow 0) \approx-24+\frac{48 c_{0}{ }^{2} r^{4}}{c_{1}{ }^{2}}+\mathcal{O}\left(r^{6}\right), \\
& r(R) \approx \pm 2 \sqrt{\frac{3 c_{1}\left(c_{2} c_{1}+576 c_{0}{ }^{6} c_{1}\left(\ln c_{0}\right)^{2}-24 c_{0}{ }^{3} \ln c_{0}\right)}{c_{0}\left(144 c_{0}{ }^{3} c_{4} \ln c_{0}-R c_{0}{ }^{6} c_{1}-6 c_{2} c_{1}-3456 c_{0}{ }^{6} c_{1}\left(\ln c_{0}\right)^{2}\right)}}, \quad r \rightarrow \infty . \tag{32}
\end{align*}
$$

The form of $f(r)$ of the BH solution (17) has the following form:

$$
\begin{align*}
& f(r)=\frac{1}{\left(c_{0} r^{2}+c_{1}\right) c_{0}{ }^{5}\left(-c_{0} r^{2}+c_{1}\right)^{7} c_{1}}\left\{4 c_{1}{ }^{9} c_{2}-96 c_{0}{ }^{11} r^{16} \ln \left(c_{0} r^{2}+c_{1}\right)+c_{5} c_{0}{ }^{5} c_{1}{ }^{9}+12 c_{0}{ }^{7} r^{14} c_{2} c_{1}{ }^{2}-24 c_{1}{ }^{8} r^{2} c_{0} c_{2}\right. \\
& +56 c_{1}{ }^{7} r^{4} c_{0}{ }^{2} c_{2}-56 c_{1}{ }^{6} r^{6} c_{0}{ }^{3} c_{2}+20 c_{1}{ }^{4} c_{0}{ }^{5} r^{10} c_{2}-108 c_{1}{ }^{3} c_{0}{ }^{6} r^{12} c_{2}+192 c_{0}{ }^{11} r^{16} \ln r-14 c_{5} c_{0}{ }^{8} c_{1}{ }^{6} r^{6}+96 c_{0}{ }^{10} r^{14} c_{1} \\
& +864 c_{1}{ }^{3} c_{0}{ }^{8} r^{10} \ln \left(c_{0} r^{2}+c_{1}\right)+1248 c_{1}{ }^{2} c_{0}{ }^{9} r^{12} \ln \left(c_{0} r^{2}+c_{1}\right)+288 c_{0}{ }^{10} r^{14} c_{1} \ln \left(c_{0} r^{2}+c_{1}\right)-4608 \ln \left(c_{0} r^{2}+c_{1}\right) c_{0}{ }^{13} c_{1}{ }^{2} r^{14} \\
& -20736 r^{10} c_{0}{ }^{11}\left(\ln \left(c_{0} r^{2}+c_{1}\right)\right)^{2} c_{1}{ }^{4}-29952 r^{12} c_{0}{ }^{12}\left(\ln \left(c_{0} r^{2}+c_{1}\right)\right)^{2} c_{1}{ }^{3}-6912 r^{14} c_{0}{ }^{13}\left(\ln \left(c_{0} r^{2}+c_{1}\right)\right)^{2} c_{1}{ }^{2} \\
& +2304 r^{16} c_{0}{ }^{14}\left(\ln \left(c_{0} r^{2}+c_{1}\right)\right)^{2} c_{1}+13824 \ln \left(c_{0} r^{2}+c_{1}\right) c_{0}{ }^{10} c_{1}{ }^{5} r^{8}-1536 \ln \left(c_{0} r^{2}+c_{1}\right) c_{0}{ }^{9} c_{1}{ }^{6} r^{6}+480 c_{1}{ }^{8} r^{2} c_{0}{ }^{7} \\
& +52224 \ln \left(c_{0} r^{2}+c_{1}\right) c_{0}{ }^{11} c_{1}{ }^{4} r^{10}+13824 \ln \left(c_{0} r^{2}+c_{1}\right) c_{0}{ }^{12} c_{1}{ }^{3} r^{12}-1280 c_{1}{ }^{7} r^{4} c_{0}{ }^{8}+2880 c_{1}{ }^{6} r^{6} c_{0}{ }^{9}-21696 c_{1}{ }^{5} c_{0}{ }^{10} r^{8} \\
& -119808 r^{12} c_{0}{ }^{12}(\ln r)^{2} c_{1}{ }^{3}-27648 r^{14} c_{0}{ }^{13}(\ln r)^{2} c_{1}{ }^{2}+9216 r^{16} c_{0}{ }^{14}(\ln r)^{2} c_{1}+3072 c_{1}{ }^{6} c_{0}{ }^{9} r^{6} \ln r-27648 c_{1}{ }^{5} c_{0}{ }^{10} r^{8} \ln r \\
& -104448 c_{1}{ }^{4} c_{0}{ }^{11} r^{10} \ln r-27648 c_{1}{ }^{3} c_{0}{ }^{12} r^{12} \ln r+9216 c_{1}{ }^{2} c_{0}{ }^{13} r^{14} \ln r-288 c_{1}{ }^{4} c_{0}{ }^{7} r^{8}-7200 c_{1}{ }^{4} c_{0}{ }^{11} r^{10}+2304 c_{1}{ }^{3} c_{0}{ }^{12} r^{12} \\
& +\ln \left(c_{0} r^{2}+c_{1}\right)\left[27648 c_{0}{ }^{13} c_{1}{ }^{2} \ln r r^{14}-9216 c_{0}{ }^{14} c_{1} r^{16} \ln r+119808 \ln c_{0}{ }^{12} c_{1}{ }^{3} r^{12} \ln r+82944 \ln c_{0}{ }^{11} c_{1}{ }^{4} r^{10} \ln r\right] \\
& -64 c_{1}{ }^{9} c_{0}{ }^{6}+14 c_{5} c_{0}{ }^{7} c_{1}{ }^{7} r^{4}-6 c_{5} c_{0}{ }^{6} c_{1}{ }^{8} r^{2}+14 c_{5} c_{0}{ }^{10} c_{1}{ }^{4} r^{10}-14 c_{5} c_{0}{ }^{11} c_{1}{ }^{3} r^{12}+6 c_{5} c_{0}{ }^{12} c_{1}{ }^{2} r^{14}-82944 r^{10} c_{0}{ }^{11}(\ln r)^{2} c_{1}{ }^{4} \\
& -c_{5} c_{0}{ }^{13} c_{1} r^{16}-1728 c_{0}{ }^{8} \ln r c_{1}{ }^{3} r^{10}-2496 c_{0}{ }^{9} \ln r c_{1}{ }^{2} r^{12}-576 c_{0}{ }^{10} c_{1} r^{14} \ln r+32 c_{1}{ }^{5} r^{6} c_{0}{ }^{6}-1088 c_{1}{ }^{3} c_{0}{ }^{8} r^{10}-288 c_{1}{ }^{2} c_{0}{ }^{9} r^{12} \\
& \left.+96 \ln \left(\frac{2}{3}\right) c_{0}{ }^{3}\left[14 c_{1}{ }^{5} r^{6} c_{0}{ }^{3}-c_{1}{ }^{8}-6 c_{0}{ }^{7} r^{14} c_{1}-14 c_{0}{ }^{2} c_{1}{ }^{6} r^{4}+6 c_{0} c_{1}{ }^{7} r^{2}-14 c_{1}{ }^{3} c_{0}{ }^{5} r^{10}+c_{0}{ }^{8} r^{16}+14 c_{1}{ }^{2} c_{0}{ }^{6} r^{12}\right]\right\} \text {, } \\
& f(r \rightarrow \infty) \approx \frac{1}{c_{0}{ }^{7} c_{1} r^{4}}\left\{96 c_{0}{ }^{5} r^{4} \ln \left(\frac{3}{2}\right)-96 c_{0}{ }^{5} r^{4} \ln c_{0}+2304 c_{0}{ }^{8} r^{4} c_{1}\left(\ln c_{0}\right)^{2}-c_{0}{ }^{7} r^{4} c_{5} c_{1}-288 c_{1} r^{2} c_{0}{ }^{4} \ln c_{0}\right. \\
& -c_{5} c_{0}{ }^{13} c_{1} r^{16}-1728 c_{0}{ }^{8} \ln r c_{1}{ }^{3} r^{10}-2496 c_{0}{ }^{9} \ln r c_{1}{ }^{2} r^{12}-576 c_{0}{ }^{10} c_{1} r^{14} \ln r+32 c_{1}{ }^{5} r^{6} c_{0}{ }^{6}-1088 c_{1}{ }^{3} c_{0}{ }^{8} r^{10}-288 c_{1}{ }^{2} c_{0}{ }^{9} r^{12} \\
& \left.+6912 c_{1}^{2} r^{2} c_{0}{ }^{7}\left(\ln c_{0}\right)^{2}+12 c_{1}^{2} r^{2} c_{0} c_{2}+864 c_{1}{ }^{2} c_{0}{ }^{3} \ln c_{0}-20736 c_{1}{ }^{3} c_{0}{ }^{6}\left(\ln c_{0}\right)^{2}-36 c_{1}{ }^{3} c_{2}-2304 c_{1}{ }^{3} c_{0}{ }^{6} \ln c_{0}+48 c_{0}{ }^{3} c_{1}{ }^{2}\right\} \\
& -c_{5} c_{0}{ }^{13} c_{1} r^{16}-1728 c_{0}{ }^{8} \ln r c_{1}{ }^{3} r^{10}-2496 c_{0}{ }^{9} \ln r c_{1}{ }^{2} r^{12}-576 c_{0}{ }^{10} c_{1} r^{14} \ln r+32 c_{1}{ }^{5} r^{6} c_{0}{ }^{6}-1088 c_{1}{ }^{3} c_{0}{ }^{8} r^{10}-288 c_{1}{ }^{2} c_{0}{ }^{9} r^{12} \\
& +\mathcal{O}\left(r^{-6}\right) \text {, } \\
& f(r \rightarrow 0) \approx \frac{4 c_{2} c_{1}{ }^{2}+96 c_{1} c_{0}{ }^{3} \ln \left(\frac{3}{2}\right)+c_{5} c_{0}{ }^{5} c_{1}{ }^{2}-64 c_{0}{ }^{6} c_{1}{ }^{2}+96 c_{1} c_{0}{ }^{7} r^{2}+192 r^{4} c_{0}{ }^{8}}{c_{0}{ }^{5} c_{1}{ }^{2}}+\mathcal{O}\left(r^{6}\right) . \tag{34}
\end{align*}
$$

To avoid the tachyonic instability, we check the Dolgov-Kawasaki stability criterion [52-55] which states that the second derivative of the gravitational model $f_{R R}$ must be always positive. Using the chain rule we get

$$
\begin{equation*}
f_{R}=\frac{d f(R)}{d R}=\frac{d f(r)}{d r} \frac{d r}{d R}=c_{0}+\frac{c_{1}}{r^{2}} \tag{35}
\end{equation*}
$$

$f_{R R}=\left(c_{0} r^{2}+c_{1}\right)^{3}\left(c_{0} r^{2}-c_{1}\right)^{8} c_{1}\left\{12 r^{6}\left[\ln \left(c_{0} r^{2}+c_{1}\right)\left[24 c_{0}{ }^{7} r^{16}-101376 c_{0}{ }^{7} c_{1}{ }^{4} r^{10} \ln r-34560 c_{0}{ }^{6} c_{1}{ }^{5} r^{8} \ln r\right]-8 c_{0}{ }^{2} c_{1}{ }^{9}\right.\right.$ $-c_{0}{ }^{4} c_{2} r^{16} c_{1}-888 r^{12} c_{1}{ }^{3} c_{0}{ }^{8}+6480 r^{10} c_{1}{ }^{4} c_{0}{ }^{7}+15848 r^{8} c_{1}{ }^{5} c_{0}{ }^{6}+656 r^{8} c_{1}{ }^{4} c_{0}{ }^{3}-32 c_{0}{ }^{6} r^{14} c_{1}+7712 c_{0}{ }^{5} c_{1}{ }^{6} r^{6}+280 c_{0}{ }^{5} r^{12} c_{1}{ }^{2}$ $-488 c_{0}{ }^{4} c_{1}{ }^{7} r^{4}+816 c_{0}{ }^{4} r^{10} c_{1}{ }^{3}+12 c_{0}{ }^{3} c_{2} r^{14} c_{1}{ }^{2}+42 c_{0}{ }^{2} c_{2} r^{12} c_{1}{ }^{3}+44 c_{0} c_{2} r^{10} c_{1}{ }^{4}+15 c_{2} r^{8} c_{1}{ }^{5}+16 c_{0}{ }^{3} r^{2} c_{1}{ }^{8}+80 c_{0}{ }^{2} c_{1}{ }^{5} r^{6}$ $+2304 r^{8} c_{0}{ }^{6} c_{1}(\ln r)^{2}\left[44 c_{0} c_{1}{ }^{3} r^{2}+15 c_{1}{ }^{4}-r^{8} c_{0}{ }^{4}+12 c_{0}{ }^{3} c_{1} r^{6}+42 c_{0}{ }^{2} c_{1}{ }^{2} r^{4}\right]+48 r^{4} c_{0}{ }^{3} c_{1} \ln r\left[44 c_{0} c_{1}{ }^{2} r^{6}-16 c_{0} c_{1}{ }^{6}+15 c_{1}{ }^{3} r^{4}\right.$ $\left.-64 c_{0}{ }^{6} c_{1} r^{10}+560 c_{0}{ }^{5} c_{1}{ }^{2} r^{8}+1632 c_{0}{ }^{4} c_{1}{ }^{3} r^{6}+12 c_{0}{ }^{3} r^{10}+160 c_{0}{ }^{2} c_{1}{ }^{5} r^{2}+42 c_{0}{ }^{2} c_{1} r^{8}+1312 c_{0}{ }^{3} c_{1}{ }^{4} r^{4}\right]-8 c_{0} c_{1}{ }^{6} r^{4}$ $+24 r^{4} c_{1} c_{0}{ }^{3} \ln \left(c_{0} r^{2}+c_{1}\right)\left[64 r^{10} c_{1} c_{0}{ }^{6}-560 r^{8} c_{1}{ }^{2} c_{0}{ }^{5}-1632 r^{6} c_{1}{ }^{3} c_{0}{ }^{4}-1312 r^{4} c_{1}{ }^{4} c_{0}{ }^{3}-12 c_{0}{ }^{3} r^{8} r^{2}-160 c_{0}{ }^{2} c_{1}{ }^{5}-42 c_{0}{ }^{2} r^{8} c_{1}\right.$ $\left.+16 c_{0} c_{1}{ }^{6}-44 c_{0} r^{6} c_{1}{ }^{2}-15 r^{4} c_{1}{ }^{3}-1152 c_{0}{ }^{6} c_{1} r^{10} \ln r\right]+576 c_{1} c_{0}{ }^{6} r^{8}\left(\ln \left(c_{0} r^{2}+c_{1}\right)\right)^{2}\left[12 r^{6} c_{1} c_{0}{ }^{3}+42 r^{4} c_{1}{ }^{2} c_{0}{ }^{2}+44 r^{2} c_{1}{ }^{3} c_{0}\right.$ $\left.\left.\left.+15 c_{1}{ }^{4}-r^{8} c_{0}{ }^{4}\right]-48 c_{0}{ }^{7} r^{16} \ln r-96768 \ln \left(c_{0} r^{2}+c_{1}\right) c_{0}{ }^{8} c_{1}{ }^{3} r^{12} \ln r+2304 \ln \left(c_{0} r^{2}+c_{1}\right) r^{16} \ln r c_{1} c_{0}{ }^{10}\right]\right\}^{-1}$.


Figure 2. Systematic plots of; (a) the Ricci scalar given by Eq. (32); (b) the analytic function f(r) given (34); (c) the derivative function $f_{R}$ given by Eq. (35), and (d) the second derivative $f_{R R}$ given by Eq. (36). All the figures are plotted using the following values of the constants, $c_{0}=10^{2}, c_{1}=10, c_{2}=-10^{5}, c_{3}=1$ and $c_{4}=1$. These values satisfy the constrains given by Eq. (24).

The behavior of the Ricci scalar, $f(r), f_{R}$ and $f_{R R}$ are given in Figure 2. Following the same procedure of the non-rotating we get the behavior of the Ricci scalar, $f(r), f_{R}$ and $f_{R R}$, in the rotating case, as: As figure 2 (a)-2 (d) shows that the Ricci scalar has a positive value, $f(r)$ has a costive value then a non-defined value then a negative value; the first derivative of $f(R)$ has a positive value as well as the second derivative of $f(R)$.

## V. THERMODYNAMICS OF THE BHS

Now, we are ready to study the physical properties the BHs (12) and (17) from the thermodynamics viewpoint. To do this, we are going to write the basic definitions of the thermodynamical quantities that we will use.


Figure 3. Systematic plots of; (a) the metric potentials given by Eq. (12); (b) the horizons of the temporal component of $g_{t t}(r)$ given (37); (c) the Hawking temperature given by Eq. (39), and (d) the second derivative $f_{R R}$ given by Eq. (36). All the figures are plotted using the following values of the constants, $M=1, c_{0}=0.01, c_{1}=0.001, c_{2}=-10^{5}, c_{3}=1$ and $c_{4}=1$. These values satisfy the constrains given by Eq. (24).

## A. Thermodynamics of the BH (12)

The asymptote form of the temporal component of the BH solution (12) has the form:

$$
\begin{equation*}
g_{t t} \approx \frac{24 r^{2} c_{0}^{3} \ln c_{0}}{c_{1}}-r^{2} c_{2}-3 c_{0}^{2}-\frac{6 c_{0} c_{1}}{r^{2}} \tag{37}
\end{equation*}
$$

Equation (37) has four real roots given as:

$$
\begin{align*}
& r_{ \pm}=\frac{\sqrt{\left( \pm \sqrt{9 c_{0}^{4}+576 c_{0}^{4} \ln c_{0}-24 c_{0} c_{2} c_{1}}-3 c_{0}^{2}\right) c_{1}}}{\sqrt{2\left(24 c_{0}^{3} \ln c_{0}-c_{2} c_{1}\right)}} \\
& r_{(1,2)}=-\frac{\sqrt{\left( \pm \sqrt{9 c_{0}^{4}+576 c_{0}^{4} \ln c_{0}-24 c_{0} c_{2} c_{1}}-3 c_{0}^{2}\right) c_{1}}}{\sqrt{2\left(24 c_{0}^{3} \ln c_{0}-c_{2} c_{1}\right)}} \tag{38}
\end{align*}
$$

where $r_{( \pm)}$are the inner and outer horizons of the spacetime. In Fig. 3 (a) we plot the metric potentials of $g_{t t}$ and $g_{r r}$ showing their behavior. In Fig. 3 (b) we show the horizons given by Eq. (38) showing that for the specific values of the constant $c_{2}$ we can have two horizons, inner and out, or the two horizons coincide constitute a degenerate horizon or we can enter a region where there is no horizon, appearance naked singularity.

The temperature of Hawking is given by [91-96]

$$
\begin{equation*}
T\left(r_{+}\right)=\frac{g_{t t}^{\prime}}{4 \pi}=\frac{24 r^{6} \ln \left(c_{0}+\frac{c_{1}}{r^{2}}\right) c_{0}^{3}\left[r^{2} c_{0}+c_{1}\right]-r^{8} c_{2} c_{1}, c_{0}-r^{6} c_{2} c_{1}^{2}-24 r^{6} c_{0}^{3} c_{1}-6 r^{4} c_{0}^{2} c_{1}^{2}-4 r^{2} c_{0} c_{1}^{3}+2 c_{1}^{4}}{\pi\left(c_{0} r^{2}+c_{1}\right) c_{1} r^{5}} . \tag{39}
\end{equation*}
$$

In Fig. 3 (c) we show the behavior of the Hawking temperature of the BH solution (12) showing that the temperature is always positive. The semi classical Bekenstein-Hawking entropy of the horizons is defined as

$$
\begin{equation*}
S\left(r_{+}\right)=\frac{\mathcal{A}}{4 G} f_{R}\left(r_{+}\right)=4 \pi^{2} r_{1} f_{R}\left(r_{+}\right)=4 \pi^{2} r_{+}\left(c_{0}+\frac{c_{1}}{r_{+}{ }^{2}}\right) \tag{40}
\end{equation*}
$$

with $\mathcal{A}=2 \pi r_{+}$being the area of the event horizons and the gravitational constant $G$ equals $G=\frac{1}{8 \pi}$. The behavior of the entropy is depicted in Fig. 3 (d) showing that the BH solution given by Eq. (12) has always positive entropy. Finally, the heat capacity is figured out as [97, 98]

$$
\begin{align*}
& C\left(r_{+}\right)=T\left(r_{+}\right)\left(\frac{S^{\prime}\left(r_{+}\right)}{T^{\prime}\left(r_{+}\right)}\right)=\left\{\left(24 r^{8} \ln \left(c_{0} r^{2}+c_{1}\right) c_{0}{ }^{4}+24 r^{6} \ln \left(c_{0} r^{2}+c_{1}\right) c_{0}{ }^{3} c_{1}-r^{8} c_{2} c_{1} c_{0}-r^{6} c_{2} c_{1}{ }^{2}-24 r^{6} c_{0}{ }^{3} c_{1}\right.\right. \\
& \left.\left.-48 r^{8} \ln r c_{0}{ }^{4}-48 r^{6} \ln r c_{0}{ }^{3} c_{1}-6 r^{4} c_{0}{ }^{2} c_{1}{ }^{2}-4 r^{2} c_{0} c_{1}{ }^{3}+2 c_{1}{ }^{4}\right) \pi^{2}\left(r^{4} c_{0}{ }^{2}-c_{1}{ }^{2}\right)\right\}\left\{4 r \left(24 r^{10} \ln \left(c_{0} r^{2}+c_{1}\right) c_{0}{ }^{5}\right.\right. \\
& -24 r^{8} c_{0}{ }^{4} c_{1}-r^{6} c_{2} c_{1}{ }^{3}-54 r^{6} c_{0}{ }^{3} c_{1}{ }^{2}-48 r^{10} \ln r c_{0}^{5}+26 r^{4} c_{0}{ }^{2} c_{1}{ }^{3}-2 r^{2} c_{0} c_{1}^{4}+48 r^{8} \ln \left(c_{0} r^{2}+c_{1}\right) c_{0}{ }^{4} c_{1} \\
& \left.\left.+24 r^{6} \ln \left(c_{0} r^{2}+c_{1}\right) c_{0}{ }^{3} c_{1}{ }^{2}-r^{10} c_{2} c_{1} c_{0}{ }^{2}-2 r^{8} c_{2} c_{1}{ }^{2} c_{0}-96 r^{8} \ln r c_{0}{ }^{4} c_{1}-48 r^{6} \ln r c_{0}{ }^{3} c_{1}{ }^{2}-10 c_{1}{ }^{5}\right)\right\}^{-1}, \tag{41}
\end{align*}
$$

where $S^{\prime}\left(r_{1}\right)$ and $T^{\prime}\left(r_{1}\right)$ are the derivative of entropy and Hawking temperature with respect to the outer horizon respectively. We depict the behavior of the heat capacity given by Eq. (41) in Fig. 3 (e) which shows that we have a stable model because the heat capacity is always positive.

## B. Thermodynamics of the BH (17)

Now we are going the same steps used in the non-rotating case and get the asymptote form of the temporal component of the BH solution (17) to has the following form:

$$
\begin{equation*}
g_{t t} \approx \frac{24 r^{2} c_{0}^{3} \ln c_{0}}{c_{1}}-r^{2} c_{2}+r^{2} c_{4}^{2}-48 r^{2} c_{0}^{3} c_{4} \ln c_{0}+3 c_{0}^{2}-6 c_{1} c_{0}^{2} c_{4}+12 \frac{c_{1}^{2} c_{4} c_{0}}{r^{2}}-6 \frac{c_{1} c_{0}}{r^{2}} \tag{42}
\end{equation*}
$$

Equation (42) has four real roots given as:

$$
\begin{align*}
& r_{(1,2)}= \pm \frac{\sqrt[4]{c_{0} c_{1}^{2}} \sqrt{3 c_{0}^{3 / 2}\left[2 c_{4} c_{1}-1\right]+\sqrt{3} \sqrt{2 c_{1} c_{4}-1} \sqrt{6 c_{0}^{3} c_{4} c_{1}-8 c_{4}^{2} c_{1}-3 c_{0}^{3}+192 c_{0}^{3} \ln c_{0}\left[2 c_{4} c_{1}-1\right]+8 c_{2} c_{1}}}}{\sqrt{2\left(24 c_{0}^{3} \ln c_{0}-c_{2} c_{1}+c_{4}^{2} c_{1}-48 c_{0}^{3} c_{4} c_{1} \ln c_{0}\right)}} \\
& r_{(5,6)}= \pm \frac{\sqrt[4]{c_{0} c_{1}^{2}} \sqrt{3 c_{0}^{3 / 2}\left[2 c_{4} c_{1}-1\right]-\sqrt{3} \sqrt{2 c_{1} c_{4}-1} \sqrt{6 c_{0}{ }^{3} c_{4} c_{1}-8 c_{4}^{2} c_{1}-3 c_{0}^{3}+192 c_{0}^{3} \ln c_{0}\left[2 c_{4} c_{1}-1\right]+8 c_{2} c_{1}}}}{\sqrt{2\left(24 c_{0}^{3} \ln c_{0}-c_{2} c_{1}+c_{4}^{2} c_{1}-48 c_{0}^{3} c_{4} c_{1} \ln c_{0}\right)}} \tag{43}
\end{align*}
$$

As Eq. (43) shows that there will real roots if $c_{0}>0, c_{1}>\frac{1}{c_{4}}$ and $6 c_{0}{ }^{3} c_{4} c_{1}-8 c_{4}{ }^{2} c_{1}-3 c_{0}{ }^{3}+192 c_{0}{ }^{3} \ln c_{0}\left[2 c_{4} c_{1}-1\right]+8 c_{2} c_{1}>0$. In this case we will deal with one horizon which we call it $r_{h}$ that given by

$$
\begin{equation*}
r_{h}=\frac{\sqrt[4]{c_{0} c_{1}^{2}} \sqrt{3 c_{0}^{3 / 2}\left[2 c_{4} c_{1}-1\right]+\sqrt{3} \sqrt{2 c_{1} c_{4}-1} \sqrt{6 c_{0}^{3} c_{4} c_{1}-8 c_{4}{ }^{2} c_{1}-3 c_{0}^{3}+192 c_{0}^{3} \ln c_{0}\left[2 c_{4} c_{1}-1\right]+8 c_{2} c_{1}}}}{\sqrt{2\left(24 c_{0}{ }^{3} \ln c_{0}-c_{2} c_{1}+c_{4}{ }^{2} c_{1}-48 c_{0}{ }^{3} c_{4} c_{1} \ln c_{0}\right)}} . \tag{44}
\end{equation*}
$$



Figure 4. Systematic plots of; (a) the metric potentials given by Eq. (17); (b) the horizons of the temporal component of $g_{t t}(r)$ given (43); (c) the Hawking temperature given by Eq. (45), and (d) the heat capacity given by Eq. (41). All the figures are plotted using the following values of the constants, $M=1, c_{0}=0.01, c_{1}=0.001, c_{2}=-10^{5}, c_{3}=1$ and $c_{4}=1$. These values satisfy the constrains given by Eq. (24).

In Fig. 4 (a) we plot the metric potentials of $g_{t t}, g_{r r}$, and $g_{t \phi}$ showing their behavior. In Fig. 4 (b) we show the horizons given by Eq. (44) showing that for specific values of the constant $c_{1}$ we can have one horizon a degenerate horizon or we can enter a region where there is no horizon, appearance naked singularity.

The temperature of Hawking of the BH solution (17) is given by [91-96]:

$$
\begin{align*}
& T\left(r_{h}\right)=\frac{1}{2 \pi\left(c_{0} r^{2}+c_{1}\right) r^{5} c_{1}}\left\{24 r^{6} c_{0}{ }^{3} \ln \left(c_{0} r^{2}+c_{1}\right) c_{1}-48 r^{6} c_{0}{ }^{3} \ln r c_{1}+48 r^{6} c_{0}{ }^{3} c_{1}{ }^{2} c_{4}+12 r^{4} c_{1}{ }^{3} c_{0}{ }^{2} c_{4}+8 r^{2} c_{0} c_{1}{ }^{4} c_{4}\right. \\
& -r^{8} c_{1} c_{2} c_{0}+r^{8} c_{1} c_{4}{ }^{2} c_{0}-4 c_{1}^{5} c_{4}+24 r^{8} c_{0}^{4} \ln \left(c_{0} r^{2}+c_{1}\right)-48 r^{8} c_{0}{ }^{4} \ln r-24 r^{6} c_{0}{ }^{3} c_{1}-6 r^{4} c_{0}{ }^{2} c_{1}{ }^{2}-4 r^{2} c_{0} c_{1}^{3}-r^{6} c_{2} c_{1}^{2} \\
& \left.+r^{6} c_{4}{ }^{2} c_{1}^{2}-48 r^{8} c_{0}{ }^{4} \ln \left(c_{0} r^{2}+c_{1}\right) c_{1} c_{4}-48 r^{6} c_{0}^{3} \ln \left(c_{0} r^{2}+c_{1}\right) c_{1}{ }^{2} c_{4}+96 r^{8} c_{0}^{4} \ln r c_{1} c_{4}+96 r^{6} c_{0}^{3} \ln r c_{1}^{2} c_{4}+2 c_{1}^{4}\right\} \tag{45}
\end{align*}
$$

In Fig. 4 (c) we show the behavior of the Hawking temperature of the BH solution (17) showing that the temperature
is always positive. Finally, the heat capacity of the BH solution (17) is figured out as [97, 98]

$$
\begin{align*}
& C\left(r_{h}\right)=T\left(r_{h}\right)\left(\frac{S^{\prime}\left(r_{h}\right)}{T^{\prime}\left(r_{h}\right)}\right)=\left[4 \left(2 c_{1}^{4}+24 r^{6} c_{0}{ }^{3} \ln \left(c_{0} r^{2}+c_{1}\right) c_{1}-48 r^{6} c_{0}{ }^{3} \ln r c_{1}+48 r^{6} c_{0}{ }^{3} c_{1}{ }^{2} c_{4}+12 r^{4} c_{1}{ }^{3} c_{0}{ }^{2} c_{4}\right.\right. \\
& +8 r^{2} c_{0} c_{1}{ }^{4} c_{4}-r^{8} c_{1} c_{2} c_{0}+r^{8} c_{1} c_{4}{ }^{2} c_{0}-48 r^{8} c_{0}{ }^{4} \ln \left(c_{0} r^{2}+c_{1}\right) c_{1} c_{4}-48 r^{6} c_{0}{ }^{3} \ln \left(c_{0} r^{2}+c_{1}\right) c_{1}{ }^{2} c_{4}+96 r^{8} c_{0}{ }^{4} \ln r c_{1} c_{4} \\
& +96 r^{6} c_{0}{ }^{3} \ln r c_{1}{ }^{2} c_{4}-4 c_{1}{ }^{5} c_{4}+24 r^{8} c_{0}{ }^{4} \ln \left(c_{0} r^{2}+c_{1}\right)-48 r^{8} c_{0}{ }^{4} \ln r-24 r^{6} c_{0}{ }^{3} c_{1}-6 r^{4} c_{0}{ }^{2} c_{1}{ }^{2}-4 r^{2} c_{0} c_{1}{ }^{3}-r^{6} c_{2} c_{1}{ }^{2} \\
& \left.\left.+r^{6} c_{4}{ }^{2} c_{1}{ }^{2}\right) \pi^{2}\left(c_{0}{ }^{2} r^{4}-c_{1}{ }^{2}\right)\right]\left\{r \left(108 r^{6} c_{1}{ }^{3} c_{0}{ }^{3} c_{4}+48 r^{8} c_{0}{ }^{4} c_{1}{ }^{2} c_{4}-r^{10} c_{1} c_{2} c_{0}{ }^{2}-52 r^{4} c_{1}{ }^{4} c_{0}{ }^{2} c_{4}-2 r^{8} c_{1}{ }^{2} c_{2} c_{0}+r^{10} c_{1} c_{4}{ }^{2} c_{0}{ }^{2}\right.\right. \\
& +2 r^{8} c_{1}{ }^{2} c_{4}{ }^{2} c_{0}+4 r^{2} c_{0} c_{1}^{5} c_{4}-48 r^{10} c_{0}{ }^{5} \ln \left(c_{0} r^{2}+c_{1}\right) c_{1} c_{4}-96 r^{8} c_{0}{ }^{4} \ln \left(c_{0} r^{2}+c_{1}\right) c_{1}{ }^{2} c_{4}-48 r^{6} c_{0}{ }^{3} \ln \left(c_{0} r^{2}+c_{1}\right) c_{1}{ }^{3} c_{4} \\
& +96 r^{10} c_{0}{ }^{5} \ln r c_{1} c_{4}+192 r^{8} c_{0}{ }^{4} \ln r c_{1}{ }^{2} c_{4}+96 r^{6} c_{0}{ }^{3} \ln r c_{1}{ }^{3} c_{4}+20 c_{1}{ }^{6} c_{4}+26 r^{4} c_{1}{ }^{3} c_{0}{ }^{2}-2 c_{0} r^{2} c_{1}{ }^{4}-54 r^{6} c_{0}{ }^{3} c_{1}{ }^{2} \\
& +48 r^{8} c_{0}{ }^{4} \ln \left(c_{0} r^{2}+c_{1}\right) c_{1}+24 c_{0}{ }^{3} \ln \left(c_{0} r^{2}+c_{1}\right) r^{6} c_{1}{ }^{2}-96 r^{8} c_{0}{ }^{4} \ln r c_{1}-48 c_{0}{ }^{3} \ln r r^{6} c_{1}{ }^{2}-10 c_{1}{ }^{5}-24 r^{8} c_{0}{ }^{4} c_{1}+r^{6} c_{4}{ }^{2} c_{1}{ }^{3} \\
& \left.\left.-r^{6} c_{2} c_{1}{ }^{3}+24 r^{10} c_{0}{ }^{5} \ln \left(c_{0} r^{2}+c_{1}\right)-48 r^{10} c_{0}{ }^{5} \ln r\right)\right\}^{-1}, \tag{46}
\end{align*}
$$

where $S^{\prime}\left(r_{h}\right)$ and $T^{\prime}\left(r_{h}\right)$ are the derivative of entropy and Hawking temperature concerning the event horizon respectively. We depict the behavior of the heat capacity given by Eq. (46) in Fig. 4 (d) which shows that we have a stable model because the heat capacity is always positive.

## VI. DISCUSSION AND CONCLUSIONS

In this work, we have explored two new classes of three dimensional BH solutions in $f(R)$ gravity. By varying the action we derived the field equations. Taking the trace of the field equations, we have rewritten the field equations in the form of Eq. (5). We applied the form of the field equation of $f(R)$, written in terms of $f_{R}$, to 3D spacetime that has three unknown functions, one of them is responsible for making the metric to have a rotating form, i.e., $b_{2}(r)$. We classified the resulting field equations into four cases: (i) $f_{R}=$ constant and $b_{2}(r)=0$, (ii) $f_{R}=$ constant and $b_{2}(r) \neq 0$, (iii) $f_{R} \neq$ constant and $b_{2}(r)=0$, and finally, (iv) $f_{R} \neq$ constant and $b_{2}(r) \neq 0$. We focus on the last two cases because the first two coincide with the rotating/non-rotating three dimensional solutions of GR.

Assuming the form of $f_{R}(r)=c_{0}+\frac{c_{1}}{r^{2}}$ we are able to solve the field equation with/without an unknown rotating function. Our solutions cannot coincide with the 3D solutions of GR because the constant $c_{1}$ is not equal to zero. Although our field equation do not involve the cosmological constant term, however the study of the asymptote of these BH solutions showed that they behave as $\mathrm{AdS} / \mathrm{dS}$. In fact, the constant $c_{0}$ plays the role of the cosmological constant which indeed means that this constant cannot be vanished. We also showed that the invariants of these BHs have a true singularity at the origin and a strong one as compared to the 3 D solutions of GR. The source of the strong singularity comes from the non-trivial forms of the Ricci scalar of the two BHs. We calculated the form of $f(R)$ of the BHs and showed that both of them behave as a polynomial one. Moreover, we calculated the second derivative of $f(R)$, i.e., $f_{R R}$ of the two BHs and showed analytically and graphically that they have positive values, which means that the Dolgov-Kawasaki stability criterion is satisfied [69] and this ensures that our BHs avoid tachyonic instabilities [52, 54, 55, 99].

We also investigated the causal structure of the solutions and showed that they possess several horizons. We found out that these solutions show stable thermodynamic behavior in all regions since the Hawking temperature and heat capacity are always positive and free from singular points, which indeed ensure stable BHs thermodynamics configuration where no phase transitions occur.

Finally, we would like to mention that many issues remain for further investigations. One may consider the geodesic structure and Penrose diagrams of these spacetimes. Besides, the techniques presented in [100, 101] could be applied for describing the issue of the structure and nature of the horizons and singularities of the obtained solutions. It would be also interesting to explore thermodynamic properties of the BH solutions, as well as possible holographic applications.
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[^1]:    ${ }^{1}$ We assume the form of the first derivative of $f(R)$ to depend on the radial coordinate since our present study using spherically symmetric ansate.

[^2]:    ${ }^{2}$ It is important to stress that the form of $F(r)$ cannot assume zero value in this study because when $F(r)=0$ yields that $f(R)=$ constant which is out the scope of this study.

[^3]:    ${ }^{3}$ There are many forms that one can assume for $F(r)$ but in this study we restrict ourselves to the form that can give reasonable physical results.
    ${ }^{4}$ These assumptions of the constants $C, C_{1}$ and $C_{2}$ give a relations between the two unknown functions $b(r)$ and $b_{1}(r)$ to take the form $b 1(r)=\frac{b(r)}{\left(c_{0}-\frac{c_{1}}{r^{2}}\right)^{6}}$.

