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NEW BOUNDS FOR THE ČEBYŠEV FUNCTIONAL OF TWO FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract

Some new inequalities for the Čebyšev functional of two functions of self-adjoint linear operators in Hilbert spaces, under suitable assumptions for the involved functions and operators, are given.

1 Introduction

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle .,. \rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [6, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, *i.e.* $f(A)$ is a *positive operator* on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$f(t) \geq g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geq g(A) \tag{P}$$

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in the operator order of $B(H)$, the Banach algebra of all bounded linear operators on H.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [6] and the references therein.

For other results see [7], [9], [10] and [11].

We say that the functions $f,g:[a,b]\longrightarrow\mathbb{R}$ are *synchronous (asynchronous)* on the interval $[a,b]$ if they satisfy the following condition:

$$(f(t)-f(s))(g(t)-g(s))\geq(\leq)\,0 \text{ for each } t,s\in[a,b].$$

It is obvious that, if f,g are monotonic and have the same monotonicity on the interval $[a,b]$, then they are synchronous on $[a,b]$ while if they have opposite monotonicity, they are asynchronous.

For some extensions of the discrete *Čebyšev inequality* for *synchronous (asynchronous)* sequences of vectors in an inner product space, see [4] and [5].

For a selfadjoint operator A on the Hilbert space H with $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and for $f,g:[m,M]\longrightarrow\mathbb{R}$ that are continuous functions *on* $[m,M]$, we can define the following *Čebyšev functional*

$$C(f,g;A;x):=\langle f(A)g(A)x,x\rangle-\langle f(A)x,x\rangle\cdot\langle g(A)x,x\rangle$$

where $x\in H$ with $\|x\|=1$.

The following result provides an inequality of Čebyšev type for functions of selfadjoint operators, see [2]:

Theorem 1 (Dragomir, 2008, [2])**.** *Let A be a selfadjoint operator with $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$. If $f,g:[m,M]\longrightarrow\mathbb{R}$ are continuous and synchronous (asynchronous) on $[m,M]$, then*

$$C(f,g;A;x)\geq(\leq)\,0, \tag{1.1}$$

for any $x\in H$ with $\|x\|=1$.

The following result of Grüss' type can be stated as well, see [3]:

Theorem 2 (Dragomir, 2008, [3])**.** *Let A be a selfadjoint operator on the Hilbert space $(H;\langle .,.\rangle)$ and assume that $Sp(A)\subseteq[m,M]$ for some scalars $m<M$. If f and g are continuous on $[m,M]$ and $\gamma:=\min_{t\in[m,M]}f(t)$ and $\Gamma:=\max_{t\in[m,M]}f(t)$ then*

$$|C(f,g;A;x)|\leq\frac{1}{2}\cdot(\Gamma-\gamma)\,[C(g,g;A;x)]^{1/2}\left(\leq\frac{1}{4}(\Gamma-\gamma)(\Delta-\delta)\right), \tag{1.2}$$

for each $x\in H$ with $\|x\|=1$, where $\delta:=\min_{t\in[m,M]}g(t)$ and $\Delta:=\max_{t\in[m,M]}g(t)$.

The main aim of this paper is to provide other inequalities for the Čebyšev functional. Applications for particular functions of interest are also given.

2 A Refinement and Some Related Results

The following result that improves (1.2) can be stated:

Theorem 3. *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous with $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$, then*

$$|C(f, g; A; x)| \le \frac{1}{2}(\Delta - \delta)\langle |f(A) - \langle f(A)x, x\rangle \cdot 1_H| x, x\rangle \le \frac{1}{2}(\Delta - \delta) C^{1/2}(f, f; A; x), \quad (2.1)$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Since $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$, we have

$$\left| g(t) - \frac{\Delta + \delta}{2} \right| \le \frac{1}{2}(\Delta - \delta), \quad (2.2)$$

for any $t \in [m, M]$.

If we multiply the inequality (2.2) with $|f(t) - \langle f(A)x, x\rangle|$ we get

$$\left| f(t) g(t) - \langle f(A)x, x\rangle g(t) - \frac{\Delta + \delta}{2} f(t) + \frac{\Delta + \delta}{2}\langle f(A)x, x\rangle \right| \le \frac{1}{2}(\Delta - \delta)|f(t) - \langle f(A)x, x\rangle|, \quad (2.3)$$

for any $t \in [m, M]$ and for any $x \in H$ with $\|x\| = 1$.

Now, if we apply the property (P) for the inequality (2.3) and a selfadjoint operator B with $Sp(B) \subset [m, M]$, then we get the following inequality of interest in itself:

$$\begin{aligned} &|\langle f(B) g(B) y, y\rangle - \langle f(A)x, x\rangle \langle g(B) y, y\rangle \\ &-\frac{\Delta + \delta}{2}\langle f(B) y, y\rangle + \frac{\Delta + \delta}{2}\langle f(A)x, x\rangle \Big| \\ &\le \frac{1}{2}(\Delta - \delta)\langle |f(B) - \langle f(A)x, x\rangle \cdot 1_H| y, y\rangle, \end{aligned} \quad (2.4)$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

If we choose in (2.4) $y = x$ and $B = A$, then we deduce the first inequality in (2.1).

Further on, by the fact that for any bounded linear operator T we have

$$\langle |T| x, x\rangle = \left\langle (T^*T)^{1/2} x, x\right\rangle \le \langle (T^*T)x, x\rangle^{1/2} = \|Tx\|$$

for any $x \in H$ with $\|x\| = 1$, then we can state that

$$\begin{aligned}\langle |f(A) - \langle f(A)x, x\rangle \cdot 1_H| x, x\rangle &\le \|f(A)x - \langle f(A)x, x\rangle \cdot x\| \\ &= \left[\|f(A)x\|^2 - \langle f(A)x, x\rangle^2\right]^{1/2} \\ &= C^{1/2}(f, f; A; x),\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$, and the second part of (2.1) is also proved. ∎

Let U be a selfadjoint operator on the Hilbert space $(H, \langle ., .\rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be its *spectral family*. Then for any continuous function $f : [m, M] \to \mathbb{R}$, it is well known that we have the following representation in terms of the Riemann-Stieltjes integral:

$$\langle f(U)x, x\rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, x\rangle), \tag{2.5}$$

for any $x \in H$ with $\|x\| = 1$. The function $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* on the interval $[m, M]$ and

$$g_x(m-0) = 0 \text{ and } g_x(M) = 1 \tag{2.6}$$

for any $x \in H$ with $\|x\| = 1$.

The following result is of interest:

Theorem 4. *Let A and B be selfadjoint operators with $Sp(A), Sp(B) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is of $r - L-$Hölder type, i.e., for a given $r \in (0, 1]$ and $L > 0$ we have*

$$|f(s) - f(t)| \le L|s-t|^r \text{ for any } s, t \in [m, M],$$

then we have the following inequality:

$$|f(s) - \langle f(A)x, x\rangle| \le L\left[\frac{1}{2}(M-m) + \left|s - \frac{m+M}{2}\right|\right]^r, \tag{2.7}$$

for any $s \in [m, M]$ and any $x \in H$ with $\|x\| = 1$.

Moreover, we have

$$\begin{aligned}|\langle f(B)y, y\rangle - \langle f(A)x, x\rangle| &\le \langle |f(B) - \langle f(A)x, x\rangle \cdot 1_H| y, y\rangle \\ &\le L\left[\frac{1}{2}(M-m) + \left\langle \left|B - \frac{m+M}{2} \cdot 1_H\right| y, y\right\rangle\right]^r,\end{aligned} \tag{2.8}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Proof. We use the following Ostrowski type inequality for the Riemann-Stieltjes integral obtained by the author in [1]:

$$\left| f(s)[u(b)-u(a)] - \int_a^b f(t)\,du(t) \right| \le L\left[\frac{1}{2}(b-a) + \left| s - \frac{a+b}{2} \right|\right]^r V_a^b(u) \quad (2.9)$$

for any $s \in [a,b]$, provided that f is of $r-L-$Hölder type on $[a,b]$, u is of bounded variation on $[a,b]$ and $V_a^b(u)$ denotes the total variation of u on $[a,b]$.

Now, applying this inequality for $u(\lambda) = g_x(\lambda) := \langle E_\lambda x, x\rangle$ where $x \in H$ with $\|x\| = 1$ we get

$$\left| f(s) - \int_{m-0}^M f(\lambda)\,d(\langle E_\lambda x, x\rangle) \right| \le L\left[\frac{1}{2}(M-m) + \left| s - \frac{m+M}{2} \right|\right]^r V_{m-0}^M(g_x) \quad (2.10)$$

which, by (2.5) and (2.6) is equivalent with (2.7).

By applying the property (P) for the inequality (2.7) and the operator B we have

$$\begin{aligned}\langle |f(B) - \langle f(A)x,x\rangle \cdot 1_H|\, y, y\rangle &\le L\left\langle \left[\frac{1}{2}(M-m) + \left| B - \frac{m+M}{2}\cdot 1_H \right|\right]^r y, y \right\rangle \\ &\le L\left\langle \left[\frac{1}{2}(M-m) + \left| B - \frac{m+M}{2} \right| \cdot 1_H\right] y, y \right\rangle^r \\ &= L\left[\frac{1}{2}(M-m) + \left\langle \left| B - \frac{m+M}{2}\cdot 1_H \right| y, y \right\rangle\right]^r\end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$, which proves the second inequality in (2.8).

Further, by the Jensen inequality for convex functions of selfadjoint operators (see for instance [6, p. 5]) applied for the modulus, we can state that

$$|\langle h(A)x, x\rangle| \le \langle |h(A)|\, x, x\rangle \quad (\mathrm{M})$$

for any $x \in H$ with $\|x\| = 1$, where h is a continuous function on $[m, M]$.

Now, if we apply the inequality (M), then we have

$$|\langle [f(B) - \langle f(A)x,x\rangle \cdot 1_H]\, y, y\rangle| \le \langle |f(B) - \langle f(A)x,x\rangle \cdot 1_H|\, y, y\rangle$$

which shows the first part of (2.8), and the proof is complete. ∎

Remark 1. *With the above assumptions for f, A and B we have the following particular inequalities of interest:*

$$\left| f\left(\frac{m+M}{2}\right) - \langle f(A)x, x\rangle \right| \le \frac{1}{2^r} L (M-m)^r \quad (2.11)$$

and

$$|f(\langle Ax,x\rangle)-\langle f(A)x,x\rangle|\le L\left[\frac{1}{2}(M-m)+\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|\right]^r, \quad (2.12)$$

for any $x\in H$ with $\|x\|=1$.

We also have the inequalities:

$$\begin{aligned}|\langle f(A)y,y\rangle-\langle f(A)x,x\rangle| &\le \langle |f(A)-\langle f(A)x,x\rangle\cdot 1_H|\,y,y\rangle\\ &\le L\left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|y,y\right\rangle\right]^r, \quad (2.13)\end{aligned}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$,

$$\begin{aligned}|\langle[f(B)-f(A)]x,x\rangle| &\le \langle |f(B)-\langle f(A)x,x\rangle\cdot 1_H|\,x,x\rangle\\ &\le L\left[\frac{1}{2}(M-m)+\left\langle\left|B-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]^r \quad (2.14)\end{aligned}$$

and, more particularly,

$$\begin{aligned}&\langle |f(A)-\langle f(A)x,x\rangle\cdot 1_H|\,x,x\rangle\\ &\le L\left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]^r, \quad (2.15)\end{aligned}$$

for any $x\in H$ with $\|x\|=1$.

We also have the norm inequality

$$\|f(B)-f(A)\|\le L\left[\frac{1}{2}(M-m)+\left\|B-\frac{m+M}{2}\cdot 1_H\right\|\right]^r. \quad (2.16)$$

The following corollary of the above Theorem 4 can be useful for applications:

Corollary 1. *Let A and B be selfadjoint operators with $Sp(A), Sp(B)\subseteq[m,M]$ for some real numbers $m<M$. If $f:[m,M]\longrightarrow\mathbb{R}$ is absolutely continuous then we have the Ostrowski type inequality for selfadjoint operators:*

$$\begin{aligned}&|f(s)-\langle f(A)x,x\rangle|\\ &\le\begin{cases}\left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]\|f'\|_{\infty,[m,M]} & \text{if } f'\in L_\infty[m,M];\\ \left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]^{1/q}\|f'\|_{p,[m,M]} & \text{if } f'\in L_p[m,M],\\ & p,q>1,\ \frac{1}{p}+\frac{1}{q}=1,\end{cases} \quad (2.17)\end{aligned}$$

for any $s\in[m,M]$ and any $x\in H$ with $\|x\|=1$, where $\|\cdot\|_{p,[m,M]}$ are the Lebesgue norms, i.e.,

$$\|h\|_{\infty,[m,M]}:=ess\sup_{t\in[m,M]}\|h(t)\|$$

and

$$\|h\|_{p,[m,M]} := \left(\int_m^M |h(t)|^p\right)^{1/p}, p \geq 1.$$

Moreover, we have

$$\begin{aligned}
&|\langle f(B)y,y\rangle - \langle f(A)x,x\rangle| \leq \langle |f(B) - \langle f(A)x,x\rangle \cdot 1_H|\, y,y\rangle \\
&\leq \begin{cases} \left[\frac{1}{2}(M-m) + \left\langle \left|B - \frac{m+M}{2} \cdot 1_H\right| y,y\right\rangle\right] \|f'\|_{\infty,[m,M]} & \text{if } f' \in L_\infty[m,M]; \\ \left[\frac{1}{2}(M-m) + \left\langle \left|B - \frac{m+M}{2} \cdot 1_H\right| y,y\right\rangle\right]^{1/q} \|f'\|_{p,[m,M]} & \begin{array}{l}\text{if } f' \in L_p[m,M], \\ p,q > 1, \frac{1}{p}+\frac{1}{q}=1,\end{array} \end{cases}
\end{aligned} \tag{2.18}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1.$

Now, on utilising Theorem 3 we can provide the following bound for the Čebyšev functional that may be more useful in applications:

Corollary 2. *Let* A *be a selfadjoint operator with* $Sp(A) \subseteq [m,M]$ *for some real numbers* $m < M$. *If* $g : [m,M] \longrightarrow \mathbb{R}$ *is continuous with* $\delta := \min_{t\in[m,M]} g(t)$ *and* $\Delta := \max_{t\in[m,M]} g(t)$, *then for any* $f : [m,M] \longrightarrow \mathbb{R}$ *of* $r-L-$*Hölder type we have the inequality:*

$$|C(f,g;A;x)| \leq \frac{1}{2}(\Delta - \delta) L \left[\frac{1}{2}(M-m) + \left\langle \left|A - \frac{m+M}{2} \cdot 1_H\right| x,x\right\rangle\right]^r, \tag{2.19}$$

for any $x \in H$ *with* $\|x\| = 1.$

Remark 2. *With the assumptions from Corollary 2 for* g *and* A *and if* f *is absolutely continuos on* $[m,M]$, *then we have the inequalities:*

$$\begin{aligned}
&|C(f,g;A;x)| \leq \frac{1}{2}(\Delta-\delta) \\
&\times \begin{cases} \left[\frac{1}{2}(M-m) + \left\langle \left|A - \frac{m+M}{2} \cdot 1_H\right| x,x\right\rangle\right] \|f'\|_{\infty,[m,M]} & \text{if } f' \in L_\infty[m,M]; \\ \left[\frac{1}{2}(M-m) + \left\langle \left|A - \frac{m+M}{2} \cdot 1_H\right| x,x\right\rangle\right]^{1/q} \|f'\|_{p,[m,M]} & \begin{array}{l}\text{if } f' \in L_\infty[m,M], \\ p,q > 1, \frac{1}{p}+\frac{1}{q}=1\end{array} \end{cases}
\end{aligned} \tag{2.20}$$

for any $x \in H$ *with* $\|x\| = 1.$

3 Some Reverses of Jensen's Inequality

It is clear that all the above inequalities can be applied for various particular instances of functions f and g. However, in the following we only consider the inequalities

$$|f(\langle Ax,x\rangle) - \langle f(A)x,x\rangle| \leq L\left[\frac{1}{2}(M-m) + \left|\langle Ax,x\rangle - \frac{m+M}{2}\right|\right]^r \tag{3.1}$$

for any $x \in H$ with $\|x\| = 1$, where the function $f : [m, M] \to \mathbb{R}$ is of $r - L-$Hölder type, and

$$|f(\langle Ax, x\rangle) - \langle f(A)x, x\rangle| \le \begin{cases} \left[\frac{1}{2}(M-m) + \left|\langle Ax, x\rangle - \frac{m+M}{2}\right|\right] \|f'\|_{\infty,[m,M]}, & \text{if } f' \in L_\infty[m, M] \\ \left[\frac{1}{2}(M-m) + \left|\langle Ax, x\rangle - \frac{m+M}{2}\right|\right]^q \|f'\|_{p,[m,M]}, & \text{if } f' \in L_p[m, M]; \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1 \end{cases} \tag{3.2}$$

for any $x \in H$ with $\|x\| = 1$, where the function $f : [m, M] \to \mathbb{R}$ is absolutely continuous on $[m, M]$, which are related to the *Jensen's inequality* for convex functions.

1. Now, if we consider the concave function $f : [m, M] \subset [0, \infty) \to \mathbb{R}$, $f(t) = t^r$ with $r \in (0, 1)$ and take into account that it is of $r - L-$Hölder type with the constant $L = 1$, then from (3.1) we derive the following reverse for the *Hölder-McCarthy inequality [8]*

$$0 \le \langle A^r x, x\rangle - \langle Ax, x\rangle^r \le \left[\frac{1}{2}(M-m) + \left|\langle Ax, x\rangle - \frac{m+M}{2}\right|\right]^r \tag{3.3}$$

for any $x \in H$ with $\|x\| = 1$.

2. Now, if we consider the functions $f : [m, M] \subset (0, \infty) \to \mathbb{R}$ with $f(t) = t^s$ and $s \in (-\infty, 0) \cup (0, \infty)$, then they are absolutely continuous and

$$\|f'\|_{\infty,[m,M]} = \begin{cases} sM^{s-1} & \text{for } s \in [1, \infty), \\ |s|\, m^{s-1} & \text{for } s \in (-\infty, 0) \cup (0, 1). \end{cases}$$

If $p \ge 1$, then

$$\begin{aligned} \|f'\|_{p,[m,M]} &= |s| \left(\int_m^M t^{p(s-1)} dt\right)^{1/p} \\ &= |s| \times \begin{cases} \left(\frac{M^{p(s-1)+1} - m^{p(s-1)+1}}{p(s-1)+1}\right)^{1/p} & \text{if } s \ne 1 - \frac{1}{p} \\ \left[\ln\left(\frac{M}{m}\right)\right]^{1/p} & \text{if } s = 1 - \frac{1}{p}. \end{cases} \end{aligned}$$

On making use of the first inequality from (3.2) we deduce for a given $s \in (-\infty, 0) \cup (0, \infty)$ that

$$\begin{aligned} &|\langle Ax, x\rangle^s - \langle A^s x, x\rangle| \\ &\le \left[\frac{1}{2}(M-m) + \left|\langle Ax, x\rangle - \frac{m+M}{2}\right|\right] \\ &\quad \times \begin{cases} sM^{s-1} & \text{for } s \in [1, \infty), \\ |s|\, m^{s-1} & \text{for } s \in (-\infty, 0) \cup (0, 1). \end{cases} \end{aligned} \tag{3.4}$$

for any $x \in H$ with $\|x\| = 1.$

The second part of (3.2) will produce the following reverse of the *Hölder-McCarthy inequality* as well:

$$\begin{aligned} &|\langle Ax, x\rangle^s - \langle A^s x, x\rangle| \\ &\leq |s| \left[\frac{1}{2}(M-m) + \left|\langle Ax, x\rangle - \frac{m+M}{2}\right|\right]^q \\ &\times \begin{cases} \left(\frac{M^{p(s-1)+1} - m^{p(s-1)+1}}{p(s-1)+1}\right)^{1/p} & \text{if } s \neq 1 - \frac{1}{p} \\ \\ \left[\ln\left(\frac{M}{m}\right)\right]^{1/p} & \text{if } s = 1 - \frac{1}{p} \end{cases} \end{aligned} \tag{3.5}$$

for any $x \in H$ with $\|x\| = 1$, where $s \in (-\infty, 0) \cup (0, \infty)$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1.$

3. Now, if we consider the function $f(t) = \ln t$ defined on the interval $[m, M] \subset (0, \infty)$, then f is also absolutely continuous and

$$\|f'\|_{p,[m,M]} = \begin{cases} m^{-1} & \text{for } p = \infty, \\ \\ \left(\frac{M^{p-1} - m^{p-1}}{(p-1) M^{p-1} m^{p-1}}\right)^{1/p} & \text{for } p > 1, \\ \\ \ln\left(\frac{M}{m}\right) & \text{for } p = 1. \end{cases}$$

Making use of the first inequality in (3.2) we deduce

$$0 \leq \ln(\langle Ax, x\rangle) - \langle \ln(A) x, x\rangle \leq \left[\frac{1}{2}(M-m) + \left|\langle Ax, x\rangle - \frac{m+M}{2}\right|\right] m^{-1} \tag{3.6}$$

and

$$\begin{aligned} &0 \leq \ln(\langle Ax, x\rangle) - \langle \ln(A) x, x\rangle \\ &\leq \left[\frac{1}{2}(M-m) + \left|\langle Ax, x\rangle - \frac{m+M}{2}\right|\right]^q \left(\frac{M^{p-1} - m^{p-1}}{(p-1) M^{p-1} m^{p-1}}\right)^{1/p} \end{aligned} \tag{3.7}$$

for any $x \in H$ with $\|x\| = 1$, where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1.$

4 Some Particular Grüss' Type Inequalities

In this last section we provide some particular cases that can be obtained via the Grüss' type inequalities established before. For this purpose we select only two examples as follows.

Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $g : [m, M] \longrightarrow \mathbb{R}$ is continuous with $\delta := \min_{t \in [m,M]} g(t)$ and $\Delta :=$

$\max_{t\in[m,M]} g(t)$, then for any $f:[m,M]\longrightarrow\mathbb{R}$ of $r-L-$Hölder type we have the inequality:

$$\begin{aligned}&|\langle f(A)g(A)x,x\rangle-\langle f(A)x,x\rangle\cdot\langle g(A)x,x\rangle|\\&\qquad\le\frac{1}{2}(\Delta-\delta)L\left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]^r,\end{aligned}\tag{4.1}$$

for any $x\in H$ with $\|x\|=1$.

Moreover, if f is absolutely continuos on $[m,M]$, then we have the inequalities:

$$\begin{aligned}&|\langle f(A)g(A)x,x\rangle-\langle f(A)x,x\rangle\cdot\langle g(A)x,x\rangle|\le\frac{1}{2}(\Delta-\delta)\\&\times\begin{cases}\left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]\|f'\|_{\infty,[m,M]} & \text{if } f'\in L_\infty[m,M];\\ \\ \left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]^{1/q}\|f'\|_{p,[m,M]} & \begin{array}{l}\text{if } f'\in L_p[m,M],\\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1\end{array}\end{cases}\end{aligned}\tag{4.2}$$

for any $x\in H$ with $\|x\|=1$.

1. If we consider the concave function $f:[m,M]\subset[0,\infty)\to\mathbb{R}$, $f(t)=t^r$ with $r\in(0,1)$ and take into account that it is of $r-L-$Hölder type with the constant $L=1$, then from (4.1) we derive the following result:

$$\begin{aligned}&|\langle A^r g(A)x,x\rangle-\langle A^r x,x\rangle\cdot\langle g(A)x,x\rangle|\\&\qquad\le\frac{1}{2}(\Delta-\delta)\left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]^r,\end{aligned}\tag{4.3}$$

for any $x\in H$ with $\|x\|=1$, where $g:[m,M]\longrightarrow\mathbb{R}$ is continuous with $\delta:=\min_{t\in[m,M]} g(t)$ and $\Delta:=\max_{t\in[m,M]} g(t)$.

Now, consider the function $g:[m,M]\subset(0,\infty)\to\mathbb{R}$, $g(t)=t^p$ with $p\in(-\infty,0)\cup(0,\infty)$. Obviously,

$$\Delta-\delta=\begin{cases}M^p-m^p & \text{if } p>0,\\ \\ \frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}} & \text{if } p<0,\end{cases}$$

and by (4.3) we get for any $x\in H$ with $\|x\|=1$ that

$$\begin{aligned}0&\le\left\langle A^{r+p}x,x\right\rangle-\langle A^r x,x\rangle\cdot\langle A^p x,x\rangle\\&\qquad\le\frac{1}{2}(M^p-m^p)\left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]^r,\end{aligned}\tag{4.4}$$

when $p>0$ and

$$\begin{aligned}0&\le\langle A^r x,x\rangle\cdot\langle A^p x,x\rangle-\left\langle A^{r+p}x,x\right\rangle\\&\qquad\le\frac{1}{2}\cdot\frac{M^{-p}-m^{-p}}{M^{-p}m^{-p}}\left[\frac{1}{2}(M-m)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle\right]^r,\end{aligned}\tag{4.5}$$

when $p < 0$.

If $g : [m, M] \subset (0, \infty) \to \mathbb{R}$, $g(t) = \ln t$, then by (4.3) we also get the inequality for logarithm:

$$\begin{aligned} 0 &\le \langle A^r \ln Ax, x\rangle - \langle A^r x, x\rangle \cdot \langle \ln Ax, x\rangle \\ &\le \ln\sqrt{\frac{M}{m}} \cdot \left[\frac{1}{2}(M-m) + \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| x, x\right\rangle\right]^r, \end{aligned} \tag{4.6}$$

for any $x \in H$ with $\|x\| = 1$.

2. Now consider the functions $f, g : [m, M] \subset (0, \infty) \to \mathbb{R}$, with $f(t) = t^s$ and $g(t) = t^w$ with $s, w \in (-\infty, 0) \cup (0, \infty)$. We have

$$\|f'\|_{\infty,[m,M]} = \begin{cases} sM^{s-1} & \text{for } s \in [1, \infty), \\ |s|\, m^{s-1} & \text{for } s \in (-\infty, 0) \cup (0, 1). \end{cases}$$

and, for $p \ge 1$,

$$\|f'\|_{p,[m,M]} = |s| \times \begin{cases} \left(\frac{M^{p(s-1)+1} - m^{p(s-1)+1}}{p(s-1)+1}\right)^{1/p} & \text{if } s \ne 1 - \frac{1}{p} \\ \left[\ln\left(\frac{M}{m}\right)\right]^{1/p} & \text{if } s = 1 - \frac{1}{p}. \end{cases}$$

If $w > 0$, then by the first inequality in (4.2) we have

$$\begin{aligned} &\left|\left\langle A^{s+w} x, x\right\rangle - \langle A^s x, x\rangle \cdot \langle A^w x, x\rangle\right| \\ &\quad \le \frac{1}{2}(M^w - m^w)\left[\frac{1}{2}(M-m) + \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| x, x\right\rangle\right] \\ &\qquad \times \begin{cases} sM^{s-1} & \text{for } s \in [1, \infty), \\ |s|\, m^{s-1} & \text{for } s \in (-\infty, 0) \cup (0, 1), \end{cases} \end{aligned} \tag{4.7}$$

for any $x \in H$ with $\|x\| = 1$.

If $w < 0$, then by the same inequality we also have

$$\begin{aligned} &\left|\left\langle A^{s+w} x, x\right\rangle - \langle A^s x, x\rangle \cdot \langle A^w x, x\rangle\right| \\ &\quad \le \frac{1}{2} \cdot \frac{M^{-w} - m^{-w}}{M^{-w} m^{-w}}\left[\frac{1}{2}(M-m) + \left\langle \left| A - \frac{m+M}{2} \cdot 1_H \right| x, x\right\rangle\right] \\ &\qquad \times \begin{cases} sM^{s-1} & \text{for } s \in [1, \infty), \\ |s|\, m^{s-1} & \text{for } s \in (-\infty, 0) \cup (0, 1), \end{cases} \end{aligned} \tag{4.8}$$

for any $x \in H$ with $\|x\| = 1$.

Finally, if we assume that $p > 1$ and $w > 0$, then by the second inequality in (4.2) we have

$$\begin{aligned} &\left|\left\langle A^{s+w}x,x\right\rangle - \left\langle A^{s}x,x\right\rangle \cdot \left\langle A^{w}x,x\right\rangle\right| \\ &\quad\le \frac{1}{2}\left|s\right|\left(M^{w}-m^{w}\right)\left[\frac{1}{2}\left(M-m\right)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_{H}\right|x,x\right\rangle\right]^{1/q} \\ &\qquad\times\begin{cases}\left(\frac{M^{p(s-1)+1}-m^{p(s-1)+1}}{p(s-1)+1}\right)^{1/p} & \text{if } s\neq 1-\frac{1}{p} \\ \left[\ln\left(\frac{M}{m}\right)\right]^{1/p} & \text{if } s=1-\frac{1}{p},\end{cases} \end{aligned} \tag{4.9}$$

while for $w < 0$, we also have

$$\begin{aligned} &\left|\left\langle A^{s+w}x,x\right\rangle - \left\langle A^{s}x,x\right\rangle \cdot \left\langle A^{w}x,x\right\rangle\right| \\ &\quad\le \frac{1}{2}\left|s\right|\cdot\frac{M^{-w}-m^{-w}}{M^{-w}m^{-w}}\left[\frac{1}{2}\left(M-m\right)+\left\langle\left|A-\frac{m+M}{2}\cdot 1_{H}\right|x,x\right\rangle\right]^{1/q} \\ &\qquad\times\begin{cases}\left(\frac{M^{p(s-1)+1}-m^{p(s-1)+1}}{p(s-1)+1}\right)^{1/p} & \text{if } s\neq 1-\frac{1}{p} \\ \left[\ln\left(\frac{M}{m}\right)\right]^{1/p} & \text{if } s=1-\frac{1}{p},\end{cases} \end{aligned} \tag{4.10}$$

where $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $x \in H$ with $\|x\| = 1$.

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