Then the Vapnik–Chervonenkis dimension of $\bar{A}_N$ is upper-bounded as

$$V(\bar{\mathcal{A}}_N) \le 4N(d+1)\ln(N(d+1)).$$

*Proof:* We have $\bar{\mathcal{A}}_1 = \mathcal{A}$, and

$$\bar{\mathcal{A}}_N = \{A \cap B: A \in \mathcal{A},\, B \in \bar{\mathcal{A}}_{N-1}\}, \qquad \text{for } N \ge 2.$$

A well-known property of shatter coefficients (see, e.g., [20]) implies that for $N \ge 2$ and $k \ge 1$

$$\mathbb{S}_{\bar{\mathcal{A}}_N}(k) \le \mathbb{S}_{\mathcal{A}}(k)\mathbb{S}_{\bar{\mathcal{A}}_{N-1}}(k).$$

Thus, $\mathbb{S}_{\bar{\mathcal{A}}_N}(k) \le \mathbb{S}_{\mathcal{A}}(k)^N$ by induction. Define $\mathcal{D} = \{A^c: A \in \mathcal{A}\}$. It follows immediately from the definition that $\mathbb{S}_{\mathcal{A}}(k) = \mathbb{S}_{\mathcal{D}}(k)$. Hence, we obtain

$$\mathbb{S}_{\bar{\mathcal{A}}_N}(k) \le \mathbb{S}_{\mathcal{D}}(k)^N. \tag{14}$$

Since $\mathcal{D}$ is the collection of all closed balls in $\mathbb{R}^d$, we have $V(\mathcal{D}) = d + 1$ by a result of Dudley [23]. Next, we use a well-known consequence of Sauer's lemma which states that for any class of sets $\mathcal{B}$ and all integers $k \ge V(\mathcal{B})$

$$\mathbb{S}_{\mathcal{B}}(k) \le \left(\frac{ke}{V(\mathcal{B})}\right)^{V(\mathcal{B})}$$

(see, e.g., [20, Corollary 4.1]). This and (14) imply that for all $k \ge d+1$

$$\mathbb{S}_{\bar{\mathcal{A}}_N}(k) \le \left(\frac{ke}{V(\mathcal{D})}\right)^{NV(\mathcal{D})} = \left(\frac{ke}{d+1}\right)^{N(d+1)}. \tag{15}$$

An upper bound to $V(\bar{\mathcal{A}}_N)$ can now be obtained by finding a $k$ for which the right-hand side is less than $2^k$. It is easy to check that if $d \ge 2$, then $k = 4N(d+1)\ln(N(d+1))$ satisfies this requirement. Since for $d = 1$ we obviously have $V(\bar{\mathcal{A}}_N) \le 2N$, we obtain that for all $N,\, d \ge 1$,

$$V(\bar{\mathcal{A}}_N) \le 4N(d+1)\ln(N(d+1)). \qquad \square$$

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# New Bounds for the Marcum $Q$-Function


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***Abstract*—New bounds are proposed for the Marcum $Q$-function, which is defined by an integral expression where the 0th-order modified Bessel function appears. The proposed bounds are derived by suitable approximations of the 0th-order modified Bessel function in the integration region of the Marcum $Q$-function. They prove to be very tight and outperform bounds previously proposed in the literature. In particular, the proposed bounds are noticeably good for large values of the parameters of the Marcum $Q$-function, where previously introduced bounds fail and where exact computation of the function becomes critical due to numerical problems.**



***Index Terms*—Marcum $Q$-function, modified Bessel function of the first kind, upper and lower bounds.**


## I. Introduction

Calculation of the generalized Marcum $Q$-function of order $M$, usually referred to as $Q_M(a,\, b)$, and particularly the popular case $(M = 1)$ indicated as Marcum $Q$-function $Q(a,\, b)$, is important in many problems of signal detection [1], [2]. Immediate examples are the computation of error probability in transmission over fading channels or detection probability for code acquisition in a direct-sequence code-division multiple-access (DS-CDMA) system. Several algorithms have


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been devised in order to numerically evaluate this function [3]–[5], but it is sometimes useful to have simple bounds, in order to gain immediate physical insight and avoid tedious and difficult numerical computations.

Recently, new bounds have been proposed, based on suitable representations of the Marcum $Q$-function and generalized Marcum $Q$-function [6]–[8], [2]. However, one difficulty lies in the fact that the tightness of the bounds is strongly dependent on the values of the arguments. In particular, both upper and lower bounds are usually tight for $b \geq a$, but they are generally loose for $b < a$. Moreover, the tightness for $b \geq a$ may not be sufficient for some applications.

In this correspondence, we propose new bounds for the Marcum $Q$-function, which solve all of the aforementioned problems. Namely, they are tight also for $b < a$, and they improve significantly on the tightness for $b \geq a$. These satisfying results are obtained with the use of *ad hoc* bounds for the $0$th-order modified Bessel function valid only in the integration region of the Marcum $Q$-function. The resulting bounds require the computation of the $0$th-order modified Bessel function only.

The correspondence is structured as follows. In Section II, we recall the Marcum $Q$-function. In Section III, we propose new bounds for the Marcum $Q$-function, and numerical results are presented in Section IV. A few applications are mentioned in Section V and conclusions are drawn in Section VI.

## II. Marcum $Q$-Function

The generalized Marcum $Q$-function of order $M$ is defined by the integral

$$Q_M(a,\, b)=\int_b^{\infty} x\left(\frac{x}{a}\right)^{M-1}\exp\left(-\frac{x^2+a^2}{2}\right)I_{M-1}(ax)\,dx \quad (1)$$

where $I_{M-1}(\cdot)$ is the $(M-1)$th-order modified Bessel function of the first kind. The parameters $a$ and $b$ are positive real. It can be shown that [9]

$$Q_M(a,\, b) = Q_1(a,\, b) + e^{(a^2+b^2)/2}\sum_{k=1}^{M-1}\left(\frac{b}{a}\right)^k I_k(ab) \quad (2)$$

where

$$Q_1(a,\, b) = Q(a,\, b) = \int_b^{\infty} x\exp\left(-\frac{x^2+a^2}{2}\right)I_0(ax)\,dx \quad (3)$$

is usually referred to as *the* Marcum $Q$-function. By observing the preceding equations, it is evident that it is possible to focus our attention on finding bounds for the Marcum $Q$-function. In fact, we will bound the Marcum $Q$-function in terms of the $0$th-order modified Bessel function. The generalized Marcum $Q$-function of order $M$ is then automatically bounded in terms of modified Bessel functions, from $0$th to $(M-1)$th order. The spirit of the proposed bounds for the Marcum $Q$-function is that of simply considering a suitable integral function (the $0$th-order modified Bessel function), thus avoiding further integration.

In [6], [7], the authors claim that numerical problems can be encountered in evaluating (1), due to the semi-infinite integration region, and then they propose alternative integral expressions for the generalized Marcum $Q$-function with integration over a finite interval. In both cases, they consider suitable integral expressions for the $0$th-order modified Bessel function which appears inside the integral of the Marcum $Q$-function. In this correspondence, we take a different approach. Instead of transforming the integration region, we work directly on the classic expression (3) and we introduce bounds for the $0$th-order modified Bessel function which are extremely tight over the integration region.

## III. Bounds for the Marcum $Q$-Function

The starting point of the proposed bounding procedure is an observation about the integral (1) defining the Marcum $Q$-function. In fact, we are integrating in the semi-infinite region $[b,\, \infty)$. The integrand function can be seen as a Rice probability density function (pdf) with its parameter $\delta = 1$ (see [9]). This function is unimodal, and it is possible to show that its mode can be approximated by the parameter $a$ of the Marcum $Q$-function [10]. Assuming $b > a$, it is evident that the integrand is monotonic decreasing. In this case, it is possible to find a very tight upper bound for the integrand adding the constraint that the bounding function assumes the same value of the integrand function in $b$. On the other hand, if $b < a$, then the integrand is a complicated function which is first increasing and then decreasing. In this case, it is extremely difficult to find a bounding function that is also easy to integrate and tight. Therefore, it is expedient to evaluate $1 - Q(a,\, b)$, which essentially translates into integrating the Rice pdf in $[0,\, b]$. Again, theintegrand is now monotonic, but increasing. With some effort, it is possible to find tight bounding functions for this integrand.

In the following, since $Q(a,\, b) \in [0,\, 1]$, it is taken for granted that in the case of the proposed upper bounds one should consider $\min\{1,\, \text{upper bound}\}$, while in the case of the lower bounds one should consider $\max\{0,\, \text{lower bound}\}$. In fact, if the bounds are out of the interval $[0,\, 1]$, they become meaningless.

### A. First Case: $b \geq a$

*1) Upper Bound for $b \geq a$:* A known upper bound for the modified Bessel function is the following [9]:

$$I_0(ax) \leq \exp(ax), \qquad \forall\, x \geq 0. \quad (4)$$

Based on this inequality, it is easy to derive the following upper bound for the Marcum $Q$-function:

$$\begin{aligned} Q(a,\, b) &\leq \int_b^{\infty} x\exp\left(-\frac{x^2+a^2}{2}\right)e^{ax}\,dx \\ &= \int_b^{\infty} x\exp\left[-\frac{(x-a)^2}{2}\right]dx \\ &= \int_{b-a}^{\infty}(y+a)\exp\left(-\frac{y^2}{2}\right)dy \\ &= \int_{b-a}^{\infty} y\exp\left(-\frac{y^2}{2}\right)dy \\ &\quad + a\sqrt{2\pi}\int_{b-a}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)dy \\ &= \exp\left[-\frac{(b-a)^2}{2}\right] + a\sqrt{\frac{\pi}{2}}\,\mathrm{erfc}\left(\frac{b-a}{\sqrt{2}}\right). \end{aligned} \quad (5)$$

The key idea is the following. The upper bound can be significantly improved by observing that we are interested in bounding $I_0(ax)$ only in the interval $[b,\, \infty)$. Therefore, we may find a tighter approximation in this interval. We make the following observation. Since

$$\left(\frac{e^x}{I_0(x)}\right)' = e^x\,\frac{I_0(x)-I_1(x)}{I_0^2(x)} > 0, \qquad \forall\, x \geq 0$$

it follows that

$$I_0(x) \leq e^x\,\frac{I_0(b)}{\exp(b)}, \qquad \forall\, x \geq b. \quad (6)$$

In Fig. 1, the functions $e^x$, $I_0(x)$, and $e^x\,\frac{I_0(b)}{e^b}$ (for $b = 5$) are compared. Note that the upper bound $e^x\,\frac{I_0(b)}{e^b}$ for $I_0(x)$ is extremely tight,

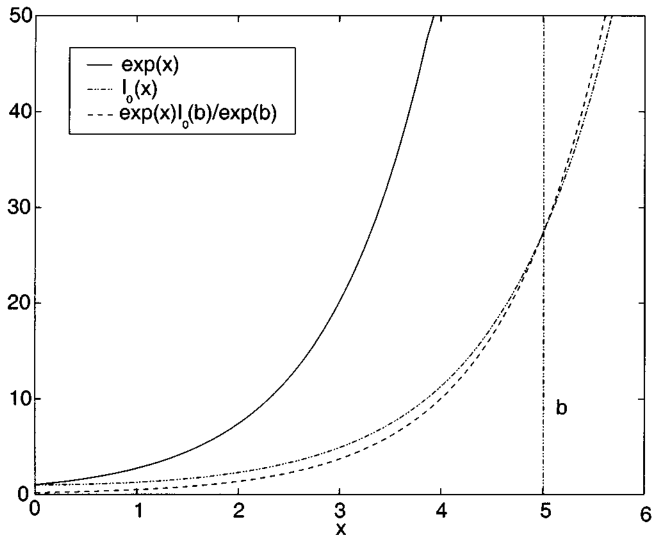


Fig. 1. Comparison between $I_0(x)$ and $e^x \frac{I_0(b)}{e^b}$, for $b = 5$.

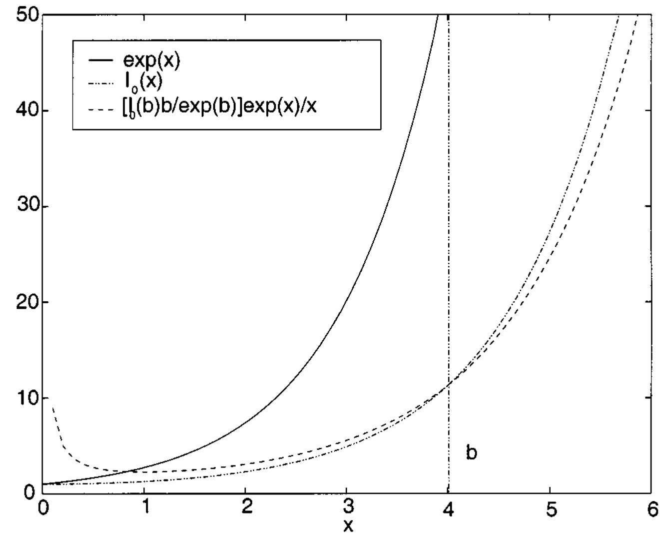


Fig. 2. Comparison between $I_0(x)$ and $\frac{e^x}{x} \frac{bI_0(b)}{e^b}$, for $b = 4$.

TABLE I
UPPER BOUNDS IN THE REGION $b \geq a$

| | |
|---|---|
| UB1 | $\frac{I_0(ab)}{\exp(ab)} \left\{\exp\left[-\frac{(b-a)^2}{2}\right] + a\sqrt{\frac{\pi}{2}}\mathrm{erfc}\left(\frac{b-a}{\sqrt{2}}\right)\right\}$ |
| UB1S | $\frac{b}{b-a}\exp\left[-\frac{(b-a)^2}{2}\right]$ |
| UB1C | $\exp\left(-\frac{a^2+b^2}{2}\right) I_0(ab) + a\sqrt{\frac{\pi}{8}}\mathrm{erfc}\left(\frac{b-a}{\sqrt{2}}\right)$ |
| UB1MG | $\exp\left[-\frac{(b-a)^2}{2}\right]$ |

TABLE II
LOWER BOUNDS IN THE REGION $b \geq a$

| | |
|---|---|
| LB1 | $\frac{I_0(ab)\,b}{\exp(ab)}\sqrt{\frac{\pi}{2}}\mathrm{erfc}\left(\frac{b-a}{\sqrt{2}}\right)$ |
| LB1S | $\frac{b}{b+a}\exp\left[-\frac{(b+a)^2}{2}\right]$ |
| LB1C | $\exp\left(-\frac{a^2+b^2}{2}\right) I_0(ab)$ |
| LB1MG | $\exp\left[-\frac{(b+a)^2}{2}\right]$ |

but holds exclusively for $x > b$, which luckily is the interval of interest. Hence, we can derive the following improved upper bound by multiplying the right-hand side of (5) by $\frac{I_0(ab)}{\exp(ab)}$, obtaining

$$Q(a, b) \leq \frac{I_0(ab)}{\exp(ab)} \left\{\exp\left[-\frac{(b-a)^2}{2}\right] + a\sqrt{\frac{\pi}{2}}\,\mathrm{erfc}\left(\frac{b-a}{\sqrt{2}}\right)\right\}. \quad (7)$$

This upper bound is referred to as UB1. In Table I, the expression of the new upper bound is shown, together with bounds that previously appeared in the literature. More precisely, the bound indicated as UB1S is proposed in [6], the bound UB1C is proposed in [7], and UB1MG is proposed in [8].

*2) Lower Bound for $b \geq a$:* In order to find a lower bound for the Marcum $Q$-function, the following inequality on the modified Bessel function $I_0(x)$ can be verified:

$$I_0(x) \geq \frac{I_0(b)b}{e^b}\,\frac{e^x}{x}, \qquad \forall\, x \geq b. \quad (8)$$

The functions involved in (8) in the lower bound are shown in Fig. 2 for $b = 4$. Again, the bound is tight and valid exclusively for $x > b$.

Based on this bound, we may derive the following lower bound on the Marcum $Q$-function:

$$\begin{aligned} Q(a, b) &\geq \int_b^\infty x\exp\left(-\frac{x^2+a^2}{2}\right)\frac{I_0(ab)ab}{e^{ab}}\,\frac{e^{ax}}{ax}\,dx \\ &= \frac{I_0(ab)b}{e^{ab}}\int_b^\infty \exp\left[-\frac{(x-a)^2}{2}\right]dx \\ &= \frac{I_0(ab)b}{e^{ab}}\sqrt{2\pi}\int_b^\infty \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{(x-a)^2}{2}\right]dx \\ &= \sqrt{\frac{\pi}{2}}\,\frac{I_0(ab)b}{e^{ab}}\,\mathrm{erfc}\left(\frac{b-a}{\sqrt{2}}\right). \end{aligned} \quad (9)$$

This lower bound will be referred to as LB1. In Table II, the new lower bound is shown and compared to bounds proposed in [6]–[8] and indicated as LB1S, LB1C, and LB1MG, respectively.

### *B. Second Case: $b < a$*

*1) Upper Bound for $b < a$:* In this case, as previously mentioned, we are integrating the Rice pdf over a region where the slope of the integrand is first positive and then negative. From the considerations previously made, to derive upper and lower bounds we should not use the inequalities (6) and (8) that we used in the preceding paragraph, because they yield extremely loose bounds. Instead, it is convenient to express the Marcum $Q$-function as follows:

$$Q(a,b) = 1 - \int_0^b x \exp\left(-\frac{x^2+a^2}{2}\right) I_0(ax)\,dx. \tag{10}$$

In this case, to find an *upper* bound for the Marcum $Q$-function we need to find a *lower* bound for the Bessel function in the interval $[0, b]$. It is now possible to reuse inequality (6) (which is indeed a lower bound for $x < b$), suitably rewritten as

$$I_0(ax) \ge \frac{I_0(ab)}{e^{ab}} e^{ax}, \qquad x \in [0, b]. \tag{11}$$

By substituting (11) into (10), the following upper bound on the Marcum $Q$-function can be derived:

$$\begin{aligned} Q(a,b) &= 1 - \int_0^b x \exp\left(-\frac{x^2+a^2}{2}\right) I_0(ax)\,dx \\ &\le 1 - \frac{I_0(ab)}{e^{ab}} \int_0^b x \exp\left[-\frac{(x-a)^2}{2}\right] dx \\ &= 1 - \frac{I_0(ab)}{e^{ab}} \left\{ \exp\left(-\frac{a^2}{2}\right) - \exp\left[-\frac{(b-a)^2}{2}\right] \right. \\ &\quad \left. + a\sqrt{\frac{\pi}{2}} \left[ \operatorname{erfc}\left(-\frac{a}{\sqrt{2}}\right) - \operatorname{erfc}\left(\frac{b-a}{\sqrt{2}}\right) \right] \right\}. \end{aligned} \tag{12}$$

This upper bound is referred to as UB2. It is important to remark that, to the best of the authors' knowledge, no upper bound can be found in the literature for $b < a$.

*2) Lower Bound for $b < a$:* Starting again from (10), to find a *lower* bound for the Marcum $Q$-function we need to find an *upper* bound for the Bessel function in the interval $[0, b]$. To this end, we make the following consideration. Letting $x_b = \frac{\log I_0(ab)}{a}$, we "bend" and "stretch" the exponential curve such that we bring the point $(x_b, I_0(ab))$ in the point $(b, I_0(ab))$, i.e., we consider the function

$$\exp\left(\frac{x_b}{b}\,ax\right) = \exp\left[\frac{\log I_0(ab)}{b} x\right].$$

Defining $\zeta \triangleq \frac{\log I_0(ab)}{b}$, it is possible to prove that

$$I_0(ax) \le \exp(\zeta x), \qquad \forall\, x \in [0, b]. \tag{13}$$

In fact, since

$$\frac{d^2}{dx^2} I_0(ax) = a^2 \frac{I_0(ax) + I_2(ax)}{2} > 0, \qquad \forall\, x \ge 0$$

and

$$\frac{d^2}{dx^2} \exp(\zeta x) = \zeta^2 \exp(\zeta x) > 0$$

it follows that $I_0(ax)$ and $\exp(\zeta x)$ are both concave. Since

$$I_0(0) = \exp(0) = 1$$

$$I_0(ab) = \exp(\zeta b) = \exp[\log I_0(ab)]$$

$$\left.\frac{dI_0(ax)}{dx}\right|_{x=0} = I_1(0) = 0$$

and

$$\left.\frac{d\,\exp(\zeta x)}{dx}\right|_{x=0} = \zeta > 0$$

we conclude that $I_0(ax) < \exp(\zeta x)$, for $x \in (0, b)$.

TABLE III
LOWER BOUNDS IN THE REGION $b < a$

| | |
|---|---|
| LB2 | $1 - \exp\left(-\frac{a^2-\zeta^2}{2}\right) \left\{ \exp\left(-\frac{\zeta^2}{2}\right) - \exp\left[-\frac{(b-\zeta)^2}{2}\right] + \zeta\sqrt{\frac{\pi}{2}} \left[\operatorname{erfc}\left(-\frac{\zeta}{\sqrt{2}}\right) - \operatorname{erfc}\left(\frac{b-\zeta}{\sqrt{2}}\right)\right] \right\}$, $\zeta \triangleq \frac{\log I_0(ab)}{b}$ |
| LB2S | $1 - \frac{a}{a-b} \exp\left[-\frac{(a-b)^2}{2}\right]$ |
| LB2aS | $1 - \frac{1}{2}\left\{ \exp\left[-\frac{(b-a)^2}{2}\right] - \exp\left[-\frac{(b+a)^2}{2}\right] \right\}$ |
| LB2C | $\exp\left(-\frac{b^2+a^2}{2}\right) I_0(ab)$ |

By substituting (13) into (10), it is possible to derive the following lower bound for the Marcum $Q$-function, which is referred to as LB2:

$$\begin{aligned} Q(a,b) &= 1 - \int_0^b x \exp\left(-\frac{x^2+a^2}{2}\right) I_0(ax)\,dx \\ &\ge 1 - \int_0^b x \exp\left(-\frac{x^2+a^2}{2}\right) \exp(\zeta x)\,dx \\ &= 1 - \exp\left(-\frac{a^2-\zeta^2}{2}\right) \cdot \int_0^b x \exp\left[-\frac{(x-\zeta)^2}{2}\right] dx \\ &= 1 - \exp\left(-\frac{a^2-\zeta^2}{2}\right) \\ &\quad \cdot \left\{ \exp\left(-\frac{\zeta^2}{2}\right) - \exp\left[-\frac{(b-\zeta)^2}{2}\right] \right. \\ &\quad \left. + \zeta\sqrt{\frac{\pi}{2}} \left[ \operatorname{erfc}\left(-\frac{\zeta}{\sqrt{2}}\right) - \operatorname{erfc}\left(\frac{b-\zeta}{\sqrt{2}}\right) \right] \right\}. \end{aligned} \tag{14}$$

It is interesting to note the formal similarity between the upper bound (12) and the lower bound (14). The latter bound is reported in Table III, together with the bounds LB2S, LB2aS, and LB2C proposed in [6], [8], and [7], respectively.

## IV. NUMERICAL RESULTS

In this section, we compare the Marcum $Q$-function with the obtained bounds and with some bounds previously proposed in the literature [6]–[8] and considered in Section III. It is worth remarking that simplified bounds can be derived by using approximations for the error function and the Bessel function [11]. Examples of simplified bounds are shown in [10].

Numerical results are presented as follows. We consider the two cases $b \ge a$ and $b < a$. For each of these cases, we present the upper bounds and the lower bounds. We consider several fixed values of the parameter $a$ and, for each case, we consider the behavior of the Marcum $Q$-function and the considered bounds as a function of the remaining parameter $b$. An extensive analysis of the performance of the proposed bounds (especially for larger values of the parameters $a$ and $b$) can be found in [10].

TABLE IV
UPPER BOUNDS COMPARISON: $a = 1$ AND $b \geq a$

| $b$ | $Q(a,b)$ | UB1 | UB1 $\varepsilon\%$ | UB1S | UB1S $\varepsilon\%$ | UB1C | UB1C $\varepsilon\%$ | UB1MG | UB1MG $\varepsilon\%$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.73 | 1.04 | 43.20 | – | – | 1.09 | 49.05 | 1.00 | 36.44 |
| 2 | 0.26 | 0.30 | 15.16 | – | – | 0.38 | 43.47 | 0.60 | 125.46 |
| 3 | 4.37E-02 | 4.67E-02 | 6.92 | 0.20 | 364.36 | 6.13E-02 | 40.45 | 0.13 | 209.57 |
| 4 | 2.88E-03 | 3.00E-03 | 3.82 | 1.48E-02 | 412.60 | 3.99E-03 | 38.13 | 1.11E-02 | 284.45 |
| 5 | 7.43E-05 | 7.61E-05 | 2.39 | 4.19E-04 | 463.90 | 1.01E-04 | 36.17 | 3.35E-04 | 351.12 |
| 6 | 7.28E-07 | 7.40E-07 | 1.63 | 4.47E-06 | 513.49 | 9.80E-07 | 34.48 | 3.72E-06 | 411.24 |
| 7 | 2.68E-09 | 2.72E-09 | 1.17 | 1.77E-08 | 560.55 | 3.57E-09 | 33.01 | 1.52E-08 | 466.18 |
| 8 | 3.71E-12 | 3.74E-12 | 0.88 | 2.61E-11 | 605.09 | 4.88E-12 | 31.71 | 2.28E-11 | 516.96 |
| 9 | 1.90E-15 | 1.91E-15 | 0.69 | 1.42E-14 | 647.35 | 2.48E-15 | 30.55 | 1.26E-14 | 564.31 |
| 10 | 3.63E-19 | 3.65E-19 | 0.55 | 2.86E-18 | 687.56 | 4.70E-19 | 29.51 | 2.57E-18 | 608.81 |
| 11 | 2.56E-23 | 2.58E-23 | 0.45 | 2.12E-22 | 725.97 | 3.30E-23 | 28.58 | 1.92E-22 | 650.88 |
| 12 | 6.71E-28 | 6.74E-28 | 0.38 | 5.79E-27 | 762.76 | 8.57E-28 | 27.73 | 5.31E-27 | 690.87 |
| 13 | 6.48E-33 | 6.51E-33 | 0.32 | 5.82E-32 | 798.12 | 8.23E-33 | 26.96 | 5.38E-32 | 729.03 |
| 14 | 2.31E-38 | 2.32E-38 | 0.27 | 2.15E-37 | 832.19 | 2.92E-38 | 26.25 | 2.00E-37 | 765.61 |

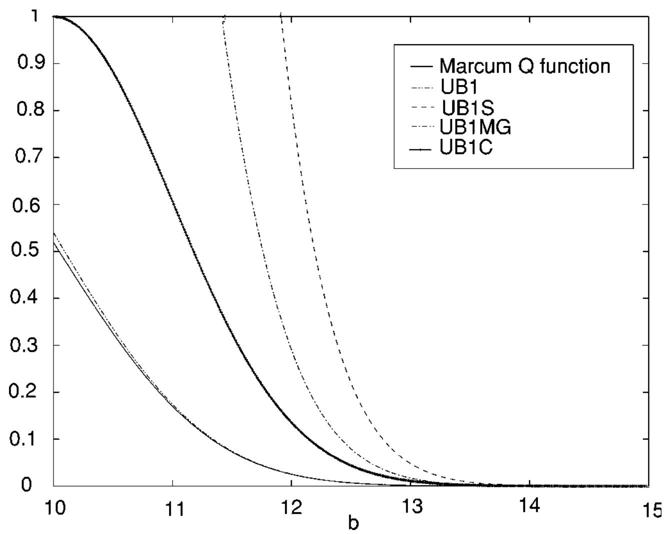


Fig. 3. Upper bounds comparison (linear scale): $a = 10$ and $b \geq a$.

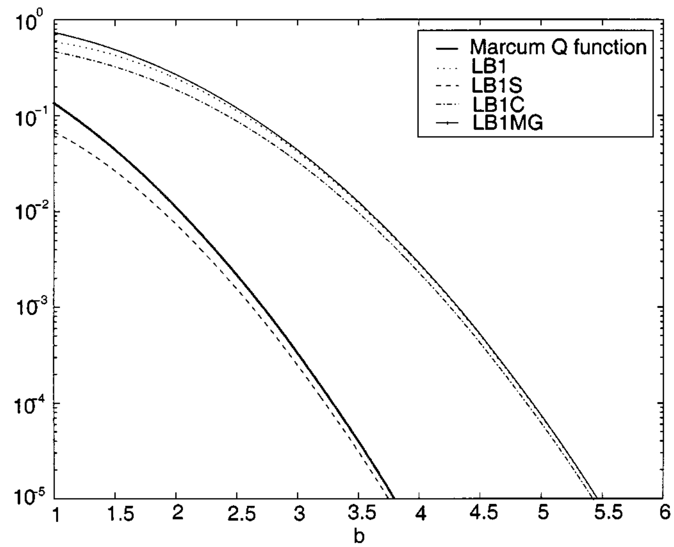


Fig. 4. Lower bounds comparison (logarithmic scale): $a = 1$ and $b \geq a$.

### A. *First Case:* $b \geq a$

*1) Upper Bounds for* $b \geq a$*:* We compare the proposed bound UB1 to bounds previously proposed in the literature. In Table IV, we present a comparison among the considered bounds for $a = 1$ and several values of $b$. For each pair $(a, b)$, we present the exact value $Q(a, b)$ of the Marcum $Q$-function, while for each of the considered bounds we show the exact value and the relative error, with respect to the Marcum $Q$-function, expressed as $100 \times \frac{\text{bound} - Q(a,b)}{Q(a,b)}$ and indicated as $\varepsilon\%$. The proposed bound UB1 is the best one. The improvement with respect to the second best bound (UB1C), in terms of the relative error $\varepsilon\%$, can be as large as two orders of magnitude for large $b$. As can be seen, the bound UB1S is greater than 1 for $b < 2$. The bound is in this case meaningless, hence the corresponding entries in Table IV are filled with $-$. In Fig. 3, the case for $a = 10$ is considered and the bounds are compared in linear scale. The improvement of the proposed bound over the others for large $b$ is now between three and four orders of magnitude [10].

*2) Lower Bounds for* $b \geq a$*:* In Fig. 4, we consider the lower bounds for $a = 1$ in linear scale. As one can see, the proposed bound LB1 performs very well. The only bound previously proposed which has a similar performance is the bound indicated as LB1C [7]. For increasing values of $a$, the performance improvement of the proposed bound is even more pronounced [10].

### B. *Second Case:* $b < a$

*1) Upper Bounds for* $b < a$*:* Since no upper bound was found in the literature for the case $b < a$, we consider the proposed bound UB2 and compare it to the Marcum $Q$-function only. The two curves are shown in Fig. 5. As can be seen, the bound UB2 becomes less tight for increasing values of $b$. Similar behavior is observed for almost all values of the parameter $a$.

*2) Lower Bounds for* $b < a$*:* In Fig. 6, we compare several lower bounds for $a = 1$ in linear scale. As is evident, the proposed bound LB2 is very tight, while the simple bound LB2S is loose. The maximum relative error with the proposed bound is around 1.4%, and it is remarkably less than with the other bounds [10]. In this case, the only lower bound comparable to the proposed bound LB2 is the bound LB2aS. As shown in Fig. 6, the bound LB2aS has a "strange" behavior, since for increasing values of $b$ a sort of floor appears. This situation

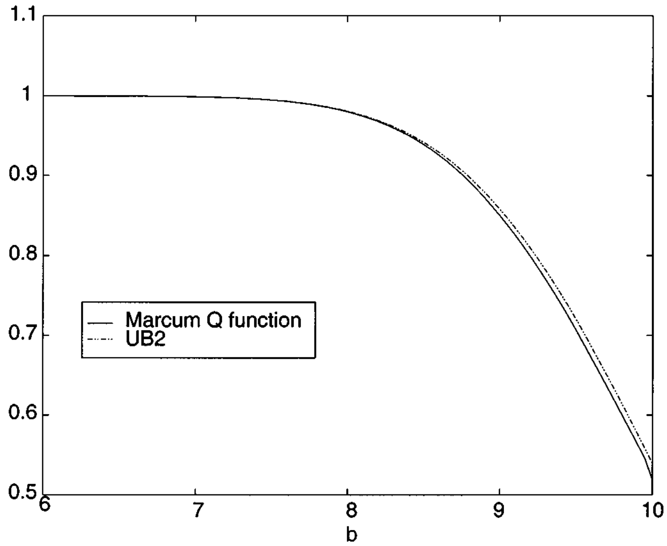


Fig. 5. Upper bound UB1 versus the Marcum $Q$-function: $a = 10$ and $b < a$.

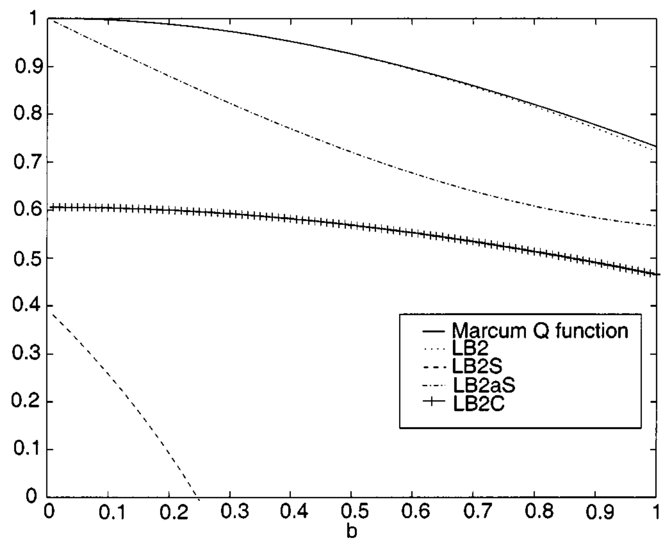


Fig. 6. Lower bounds comparison: $a = 1$ and $b < a$.

is even more pronounced for increasing values of $a$, where the bound LB2 is still the tightest one [10].

## V. Applications

It can be shown that the cumulative distribution function of a noncentral chi-square random variable with $2M$ degrees of freedom can be expressed in terms of the generalized Marcum $Q$-function of order $M$ [12].

When characterizing the performance of differentially coherent and noncoherent digital communications, the generic form of the expression for the error probability typically involves the Marcum $Q$-function, the arguments of which are proportional to the square root of the instantaneous signal-to-noise ratio of the received signal [8], [2]. Hence, the proposed bounds may be successfully employed to bound this error probability.

## VI. Conclusion and Discussion

In this correspondence, we presented new bounds for the Marcum $Q$-function. The proposed bounds, especially when compared to bounds previously introduced in the literature, have shown to be extremely tight. In particular, the following specific remarks can be made.

- For large values of the parameters $a$ and $b$, the computation of the Marcum $Q$-function according to any definition becomes critical, because of numerical problems in the integration region. The proposed bounds are valid for large values of $a$ and $b$ without suffering any numerical problem.

- Looking at the tightness tables relative to all possible combinations of the parameters $a$ and $b$ [10], the following remark can be made. Considering a value of the Marcum $Q$-function around $10^{-5}$, for increasing $a$ the proposed bounds improve, while the bounds previously proposed in the literature worsen. Moreover, in the case $b \geq a$, the performance, i.e., the tightness, improves quickly for increasing $b$.

- As far as we know, no upper bound for the Marcum $Q$-function has been ever introduced for $b < a$. Hence, the proposed upper bounds may be very useful in the computation of a Rice cumulative density function for very small values of this function, i.e., for very low probabilities.

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