

NEW BOUNDS FOR THE OSTROWSKI-LIKE TYPE INEQUALITIES

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ABSTRACT. We improve some inequalities of Ostrowski-like type and further generalize them.

1. Introduction

In 1938, Ostrowski [8] proved the following interesting integral inequality which has received considerable attention from many researchers.

Theorem 1 (See [8]). *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) whose derivative function $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , i.e., $\|f'\|_\infty = \sup_{t \in (a, b)} |f'(t)| < \infty$. Then*

$$(1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right) (b-a) \|f'\|_\infty$$

for all $x \in [a, b]$.

This inequality gives an upper bound for the approximation of the integral average $\frac{1}{b-a} \int_a^b f(t) dt$ by the value $f(x)$ at point $x \in [a, b]$. The first generalization of Ostrowski inequality was given by G. V. Milovanović and J. E. Pečarić in [7]. However, note that estimate (1) can be applied only if f' is bounded. In the first part of this paper, we will improve (1) by assuming $f' \in L^p(a, b)$ for some $1 \leq p < \infty$. More precisely, we obtain the following theorem.

Theorem 2. *Assume that $1 \leq p$. Let $I \subset \mathbb{R}$ be an open interval such that $[a, b] \subset I$ and let $f : I \rightarrow \mathbb{R}$ be a differentiable function such that $f' \in L^p(a, b)$. Then we have*

$$(2) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq A(x, q) \|f'\|_p$$

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for all $x \in [a, b]$ where

$$A(x, q) = \left(\frac{1}{b-a} \left(\frac{1}{q+1} \left(\frac{b-a}{2} \right)^{q+1} \right)^{\frac{1}{q}} + \left| x - \frac{a+b}{2} \right|^{\frac{1}{q}} \right)$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 1. $\lim_{q \rightarrow +\infty} A(x, q) = \frac{3}{2}$ for each $x \in [a, b]$.

Example 1. Let us consider the integral

$$\int_0^1 \sqrt[3]{\sin(t^2)} dt.$$

Then we have

$$f(t) = \sqrt[3]{\sin(t^2)} \quad \text{and} \quad f'(t) = \frac{2t \cos(t^2)}{3 \sqrt[3]{\sin^2(t^2)}}$$

such that $f'(t) \rightarrow \infty$ as $t \rightarrow 0$. On the other hand, we have

$$\int_0^1 |f'(t)|^2 dt \leq \frac{4}{9} \max_{0 \leq t \leq 1} \left| \frac{t^2 \cos(t^2)}{\sin(t^2)} \right| \int_0^1 \frac{dt}{\sqrt[3]{\sin(t^2)}} \leq \frac{16}{9},$$

i.e., $\|f'\|_{L^2} \leq \frac{4}{3}$. It follows that

$$\left| \sqrt[3]{\sin(x^2)} - \int_0^1 \sqrt[3]{\sin(t^2)} dt \right| \leq \frac{4}{3} \left(\frac{1}{\sqrt{24}} + \sqrt{x - \frac{1}{2}} \right)$$

for all $x \in [0, 1]$.

In recent years, a number of authors have written about generalizations of Ostrowski inequality. For example, this topic is considered in [1, 3, 4, 6, 11, 5]. In this way, some new types of inequalities are formed, such as inequalities of Ostrowski-Griiss type, inequalities of Ostrowski-Chebyshev type, etc. The first inequality of Ostrowski-Grüss type was given by Dragomir and Wang in [4]. It was generalized and improved by Matić, Pečarić, and Ujević in [6]. Cheng gave a sharp version of the mentioned inequality in [3]. Recently in [11], Ujević proved the following result which gives much better results than estimations based on [3].

Theorem 3 (See [11, Theorem 4]). *Let $f : I \rightarrow \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a twice continuously differentiable mapping in the interior $\overset{\circ}{I}$ of I with $f'' \in L^2(a, b)$ and let $a, b \in \overset{\circ}{I}$, $a < b$. Then we have*

$$(3) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \left(x - \frac{a+b}{2} \right) \frac{f(b) - f(a)}{b-a} \right| \leq \frac{(b-a)^{\frac{3}{2}}}{2\pi\sqrt{3}} \|f''\|_2$$

for all $x \in [a, b]$.

If we assume f is such that f'' is of class L^p for some $1 \leq p < \infty$, then we obtain:

Theorem 4. *Let $f : I \rightarrow \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a twice continuously differentiable mapping in the interior $\overset{\circ}{I}$ of I with $f'' \in L^p(a, b)$, $1 \leq p < \infty$, we have*

$$(4) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \left(x - \frac{a+b}{2}\right) \frac{f(b) - f(a)}{b-a} \right| \leq B(q) \|f''\|_p$$

for all $x \in [a, b]$ where

$$B(q) = \left[\frac{3}{2} \left(\frac{(b-a)^{q+1}}{q+1} \right)^{\frac{1}{q}} + \frac{1}{2(b-a)} \left(\frac{(b-a)^{2q+1}}{2q+1} \right)^{\frac{1}{q}} \right]$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 2. $\lim_{q \rightarrow +\infty} B(q) = 2(b-a)$.

2. Proofs

Before proving our main theorem, we need an essential lemma below. It is well-known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

Lemma 5 (See [2]). *Let $f : [a, b] \rightarrow \mathbb{R}$ and let r be a positive integer. If f is such that $f^{(r-1)}$ is absolutely continuous on $[a, b]$, $x_0 \in (a, b)$, then for all $x \in (a, b)$ we have*

$$f(x) = T_{r-1}(f, x_0, x) + R_{r-1}(f, x_0, x),$$

where $T_{r-1}(f, x_0, \cdot)$ is a Taylor's polynomial of degree $r-1$, that is,

$$T_{r-1}(f, x_0, x) = \sum_{k=0}^{r-1} \frac{f^{(k)}(x_0) (x-x_0)^k}{k!}$$

and the remainder can be given by

$$(5) \quad R_{r-1}(f, x_0, x) = \int_{x_0}^x \frac{(x-t)^{r-1} f^{(r)}(t)}{(r-1)!} dt.$$

By a simple calculation, the remainder in (5) can be rewritten as

$$R_{r-1}(f, x_0, x) = \int_0^{x-x_0} \frac{(x-x_0-t)^{r-1} f^{(r)}(x_0+t)}{(r-1)!} dt$$

which helps us to deduce a similar representation of f as following

$$(6) \quad f(x+u) = \sum_{k=0}^{r-1} \frac{u^k}{k!} f^{(k)}(x) + \int_0^u \frac{(u-t)^{r-1}}{(r-1)!} f^{(r)}(x+t) dt.$$

Proof of Theorem 2. Denote

$$F(x) = \int_a^x f(t) dt.$$

By Fundamental Theorem of Calculus

$$I(f) = F(b) - F(a).$$

Applying Lemma 5 gives

$$F(b) = F\left(\frac{a+b}{2}\right) + \frac{b-a}{2} F'\left(\frac{a+b}{2}\right) + \int_{\frac{a+b}{2}}^b (b-t) F''(t) dt$$

which implies that

$$F(b) - F\left(\frac{a+b}{2}\right) = \frac{b-a}{2} f\left(\frac{a+b}{2}\right) + \int_{\frac{a+b}{2}}^b (b-t) f'(t) dt.$$

We see that

$$F(a) = F\left(\frac{a+b}{2}\right) + \frac{a-b}{2} F'\left(\frac{a+b}{2}\right) + \int_{\frac{a+b}{2}}^a (a-t) F''(t) dt$$

which yields

$$F(a) - F\left(\frac{a+b}{2}\right) = \frac{a-b}{2} f\left(\frac{a+b}{2}\right) + \int_a^{\frac{a+b}{2}} (t-a) f'(t) dt.$$

Therefore,

$$F(b) - F(a) = (b-a) f\left(\frac{a+b}{2}\right) + \int_{\frac{a+b}{2}}^b (b-t) f'(t) dt - \int_a^{\frac{a+b}{2}} (t-a) f'(t) dt.$$

By changing $t = a + b - x$, we get

$$\int_{\frac{a+b}{2}}^b (b-t) f'(t) dt = \int_a^{\frac{a+b}{2}} (t-a) f'(a+b-t) dt$$

which helps us to deduce that

$$\int_a^b f(t) dt = (b-a) f\left(\frac{a+b}{2}\right) + \int_a^{\frac{a+b}{2}} (t-a) (f'(a+b-t) - f'(t)) dt.$$

On the other hand,

$$f(x) - f\left(\frac{a+b}{2}\right) = \int_{\frac{a+b}{2}}^x f'(t) dt.$$

Then

$$\begin{aligned} & f(x) - \frac{1}{b-a} \int_a^b f(t) dt \\ &= \int_{\frac{a+b}{2}}^x f'(t) dt - \frac{1}{b-a} \int_a^{\frac{a+b}{2}} (t-a) (f'(a+b-t) - f'(t)) dt. \end{aligned}$$

Next we consider the case $1 < p < \infty$. We first have the following estimates

$$\begin{aligned}
& \left| \int_a^{\frac{a+b}{2}} (t-a) (f'(a+b-t) - f'(t)) dt \right| \\
& \leq \left| \int_a^{\frac{a+b}{2}} (t-a) f'(a+b-t) dt \right| + \left| \int_a^{\frac{a+b}{2}} (t-a) f'(t) dt \right| \\
& \leq \left(\int_a^{\frac{a+b}{2}} |f'(a+b-t)|^p dt \right)^{\frac{1}{p}} \left(\int_a^{\frac{a+b}{2}} |t-a|^q dt \right)^{\frac{1}{q}} \\
& \quad + \left(\int_a^{\frac{a+b}{2}} |f'(t)|^p dt \right)^{\frac{1}{p}} \left(\int_a^{\frac{a+b}{2}} |t-a|^q dt \right)^{\frac{1}{q}} \\
& = \left(\frac{1}{q+1} \left(\frac{b-a}{2} \right)^{q+1} \right)^{\frac{1}{q}} \|f'\|_p.
\end{aligned}$$

Clearly,

$$\begin{aligned}
\left| \int_{\frac{a+b}{2}}^x f'(t) dt \right| & \leq \left| \int_{\frac{a+b}{2}}^x |f'(t)|^p dt \right|^{\frac{1}{p}} \left| \int_{\frac{a+b}{2}}^x 1^q dt \right|^{\frac{1}{q}} \\
& \leq \left| \int_a^b |f'(t)|^p dt \right|^{\frac{1}{p}} \left| \int_{\frac{a+b}{2}}^x 1^q dt \right|^{\frac{1}{q}} \\
& = \|f'\|_p \left| x - \frac{a+b}{2} \right|^{\frac{1}{q}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
& \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\
& \leq \left(\frac{1}{b-a} \left(\frac{1}{q+1} \left(\frac{b-a}{2} \right)^{q+1} \right)^{\frac{1}{q}} + \left| x - \frac{a+b}{2} \right|^{\frac{1}{q}} \right) \|f'\|_p.
\end{aligned}$$

If $p = 1$, then

$$\begin{aligned}
& \left| \int_a^{\frac{a+b}{2}} (t-a) (f'(a+b-t) - f'(t)) dt \right| \\
& \leq \frac{b-a}{2} \int_a^{\frac{a+b}{2}} (|f'(a+b-t)| + |f'(t)|) dt \\
& = \frac{b-a}{2} \|f'\|_1
\end{aligned}$$

and

$$\left| \int_{\frac{a+b}{2}}^x f'(t) dt \right| \leq \|f'\|_1$$

which helps us to claim that

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{3}{2} \|f'\|_1. \quad \square$$

Corollary 1. *If we put $x = \frac{a+b}{2}$, then under the assumptions of Theorem 1 and $1 \leq p < \infty$, we have*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{b-a} \left(\frac{1}{q+1} \left(\frac{b-a}{2}\right)^{q+1} \right)^{\frac{1}{q}} \|f'\|_p.$$

Note that

$$\frac{1}{b-a} \left(\frac{1}{q+1} \left(\frac{b-a}{2}\right)^{q+1} \right)^{\frac{1}{q}} = \frac{1}{2} \left(\frac{1}{q+1} \right)^{\frac{1}{q}} \left(\frac{b-a}{2}\right)^{\frac{1}{q}}.$$

Proof of Theorem 4. Clearly, by Lemma 5 one has

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(t) dt &= \frac{1}{b-a} (F(b) - F(a)) \\ &= \frac{1}{b-a} \left((b-a)F'(a) + \frac{(b-a)^2}{2} F''(a) + \int_a^b \frac{(b-t)^2}{2} F'''(t) dt \right) \\ &= f(a) + \frac{b-a}{2} f'(a) + \frac{1}{b-a} \int_a^b \frac{(b-x)^2}{2} f''(t) dt. \end{aligned}$$

Similarly,

$$f(x) = f(a) + (x-a) f'(a) + \int_a^b (b-t) f''(t) dt$$

and

$$\begin{aligned} \frac{f(b) - f(a)}{b-a} &= \frac{1}{b-a} \left((b-a) f'(a) + \int_a^b (b-t) f''(t) dt \right) \\ &= f'(a) + \frac{1}{b-a} \int_a^b (b-t) f''(t) dt. \end{aligned}$$

Therefore,

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \left(x - \frac{a+b}{2}\right) \frac{f(b) - f(a)}{b-a} \right| \\ &= \left| \int_a^b (b-x) f''(x) dt - \frac{1}{b-a} \int_a^b \frac{(b-t)^2}{2} f''(t) dt - \frac{x - \frac{a+b}{2}}{b-a} \int_a^b (b-t) f''(t) dt \right|. \end{aligned}$$

If $1 < p < \infty$, then by the Hölder inequality, one has

$$\left| \int_a^b (b-t) f''(t) dt \right| \leq \|f''\|_p \left(\int_a^b (b-t)^q dt \right)^{\frac{1}{q}} = \left(\frac{(b-a)^{q+1}}{q+1} \right)^{\frac{1}{q}} \|f''\|_p,$$

and

$$\begin{aligned} \frac{1}{b-a} \left| \int_a^b \frac{(b-t)^2}{2} f''(t) dt \right| &\leq \frac{1}{2(b-a)} \|f''\|_p \left(\int_a^b (b-t)^{2q} dt \right)^{\frac{1}{q}} \\ &= \frac{1}{2(b-a)} \left(\frac{(b-a)^{2q+1}}{2q+1} \right)^{\frac{1}{q}} \|f''\|_p, \end{aligned}$$

and

$$\begin{aligned} \left| \frac{x - \frac{a+b}{2}}{b-a} \int_a^b (b-t) f''(t) dt \right| &\leq \frac{1}{2} \left| \int_a^b (b-t) f''(t) dt \right| \\ &\leq \frac{1}{2} \left(\frac{(b-a)^{q+1}}{q+1} \right)^{\frac{1}{q}} \|f''\|_p. \end{aligned}$$

Thus,

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \left(x - \frac{a+b}{2} \right) \frac{f(b) - f(a)}{b-a} \right| \\ &\leq \left[\frac{3}{2} \left(\frac{(b-a)^{q+1}}{q+1} \right)^{\frac{1}{q}} + \frac{1}{2(b-a)} \left(\frac{(b-a)^{2q+1}}{2q+1} \right)^{\frac{1}{q}} \right] \|f''\|_p. \end{aligned}$$

If $1 = p$, then again by the Hölder inequality, one has

$$\left| \int_a^b (b-t) f''(t) dt \right| \leq (b-a) \int_a^b |f''(t)| dt = (b-a) \|f''\|_1,$$

and

$$\frac{1}{b-a} \left| \int_a^b \frac{(b-t)^2}{2} f''(t) dt \right| \leq \frac{1}{b-a} \frac{(b-a)^2}{2} \int_a^b |f''(t)| dt = \frac{b-a}{2} \|f''\|_1,$$

and

$$\left| \frac{x - \frac{a+b}{2}}{b-a} \int_a^b (b-t) f''(t) dt \right| \leq \frac{1}{2} \left| \int_a^b (b-t) f''(t) dt \right| \leq \frac{1}{2} (b-a) \|f''\|_1.$$

Hence,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \left(x - \frac{a+b}{2} \right) \frac{f(b) - f(a)}{b-a} \right| \leq 2(b-a) \|f''\|_1.$$

Therefore,

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \left(x - \frac{a+b}{2}\right) \frac{f(b) - f(a)}{b-a} \right| \\ & \leq \left[\frac{3}{2} \left(\frac{(b-a)^{q+1}}{q+1} \right)^{\frac{1}{q}} + \frac{1}{2(b-a)} \left(\frac{(b-a)^{2q+1}}{2q+1} \right)^{\frac{1}{q}} \right] \|f''\|_p. \quad \square \end{aligned}$$

Corollary 2. *If we put $x = \frac{a+b}{2}$, then under the assumptions of Theorem 3 and $1 \leq p < \infty$, we have*

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \left[\frac{3}{2} \left(\frac{(b-a)^{q+1}}{q+1} \right)^{\frac{1}{q}} + \frac{1}{2(b-a)} \left(\frac{(b-a)^{2q+1}}{2q+1} \right)^{\frac{1}{q}} \right] \|f''\|_p. \end{aligned}$$

3. Applications in numerical integral

Let $\Gamma = \{x_0 = a < x_1 < \dots < x_n = b\}$ be a given subdivision of the interval $[a, b]$ such that $h = x_{i+1} - x_i = \frac{b-a}{n}$. Then we obtain the following theorem by using Corollary 1.

Theorem 6. *Under the assumptions of Theorem 2 and $1 \leq p < \infty$, we have*

$$\left| \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_{i-1} + x_i}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{2n} \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left(\frac{b-a}{2}\right)^{\frac{1}{q}} \|f'\|_p.$$

Proof. We have

$$\left| f\left(\frac{x_{i-1} + x_i}{2}\right) - \frac{n}{b-a} \int_{x_{i-1}}^{x_i} f(t) dt \right| \leq \frac{1}{2} \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left(\frac{b-a}{2n}\right)^{\frac{1}{q}} \|f'\|_{p, [x_{i-1}, x_i]}$$

where

$$\|f'\|_{p, [x_{i-1}, x_i]} = \left(\int_{x_{i-1}}^{x_i} |f'(t)|^p dt \right)^{\frac{1}{p}}.$$

Then,

$$\begin{aligned} & \left| \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_{i-1} + x_i}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{2n} \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left(\frac{b-a}{2n}\right)^{\frac{1}{q}} \sum_{i=1}^n \|f'\|_{p, [x_{i-1}, x_i]}. \end{aligned}$$

Put

$$\alpha_i = \int_{x_{i-1}}^{x_i} |f'(t)|^p dt.$$

Then

$$\sum_{i=1}^n \|f'\|_{p, [x_{i-1}, x_i]} = \sum_{i=1}^n \alpha_i^{\frac{1}{p}} \leq n^{1-\frac{1}{p}} \left(\sum_{i=1}^n \alpha_i \right)^{\frac{1}{p}} = n^{1-\frac{1}{p}} \|f'\|_p.$$

Therefore,

$$\begin{aligned} & \left| \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_{i-1} + x_i}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{2n} \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left(\frac{b-a}{2n}\right)^{\frac{1}{q}} n^{1-\frac{1}{p}} \|f'\|_p \\ & = \frac{1}{2n} \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left(\frac{b-a}{2}\right)^{\frac{1}{q}} \|f'\|_p. \quad \square \end{aligned}$$

If we use Corollary 2, we then obtain the following theorem whose proof will be omitted.

Theorem 7. *Under the assumptions of Theorem 4 and $1 \leq p < \infty$, we have*

$$\begin{aligned} & \left| \frac{1}{n} \sum_{i=1}^n f\left(\frac{x_{i-1} + x_i}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{n^2} \left[\frac{3}{2} \left(\frac{(b-a)^{q+1}}{q+1}\right)^{\frac{1}{q}} + \frac{1}{2(b-a)} \left(\frac{(b-a)^{2q+1}}{2q+1}\right)^{\frac{1}{q}} \right] \|f''\|_p. \end{aligned}$$

References

- [1] G. A. Anastassiou, *Ostrowski type inequalities*, Proc. Amer. Math. Soc. **123** (1995), no. 12, 3775–3781.
- [2] G. A. Anastassiou and S. S. Dragomir, *On some estimates of the remainder in Taylor's formula*, J. Math. Anal. Appl. **263** (2001), no. 1, 246–263.
- [3] X. L. Cheng, *Improvement of some Ostrowski-Grüss type inequalities*, Comput. Math. Appl. **42** (2001), no. 1-2, 109–114.
- [4] S. S. Dragomir and S. Wang, *An inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules*, Comput. Math. Appl. **33** (1997), no. 11, 15–20.
- [5] V. N. Huy and Q. A. Ngõ, *New inequalities of Ostrowski-like type involving n knots and the L^p -norm of the m -th derivative*, Appl. Math. Lett. **22** (2009), no. 9, 1345–1350.
- [6] M. Matic, J. E. Pečarić, and N. Ujević, *Improvement and further generalization of inequalities of Ostrowski-Grüss type*, Comput. Math. Appl. **39** (2000), no. 3-4, 161–175.
- [7] G. V. Milovanović and J. E. Pečarić, *On generalization of the inequality of A. Ostrowski and some related applications*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 544-576 (1976), 155–158.
- [8] A. M. Ostrowski, *Über die absolutabweichung einer differentiebaren funktion von ihrem integralmittelwert*, Comment. Math. Helv. **10** (1938), 226–227.
- [9] N. Ujević, *Error inequalities for a quadrature formula of open type*, Rev. Colombiana Mat. **37** (2003), no. 2, 93–105.

- [10] ———, *Error inequalities for a quadrature formula and applications*, *Comput. Math. Appl.* **48** (2004), no. 10-11, 1531–1540.
- [11] ———, *New bounds for the first inequality of Ostrowski-Grüss type and applications*, *Comput. Math. Appl.* **46** (2003), no. 2-3, 421–427.

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