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New Bounds on Crossing Numbers*

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Abstract. The *crossing number*, $\mathrm{cr}(G)$, of a graph G is the least number of crossing points in any drawing of G in the plane. Denote by $\kappa(n, e)$ the minimum of $\mathrm{cr}(G)$ taken over all graphs with n vertices and at least e edges. We prove a conjecture of Erdős and Guy by showing that $\kappa(n, e)n^2/e^3$ tends to a positive constant as $n \to \infty$ and $n \ll e \ll n^2$. Similar results hold for graph drawings on any other surface of fixed genus.

We prove better bounds for graphs satisfying some monotone properties. In particular, we show that if G is a graph with n vertices and $e \geq 4n$ edges, which does not contain a cycle of length *four* (resp. *six*), then its crossing number is at least ce^4/n^3 (resp. ce^5/n^4), where $c > 0$ is a suitable constant. These results cannot be improved, apart from the value of the constant. This settles a question of Simonovits.

1. Introduction

Let G be a simple undirected graph with $n(G)$ nodes (vertices) and $e(G)$ edges. A *drawing* of G in the *plane* is a mapping f that assigns to each vertex of G a distinct point in the plane and to each edge uv a continuous arc connecting $f(u)$ and $f(v)$, not passing through the image of any other vertex. For simplicity, the arc assigned to uv is also called an *edge*, and if this leads to no confusion, it is also denoted by uv. We assume

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that no three edges have an interior point in common. The *crossing number*, $\mathrm{cr}(G)$, of G is the minimum number of crossing points in any drawing of G.

The determination of $\mathrm{cr}(G)$ is an NP-complete problem [GJ]. It was discovered by Leighton [L2] that the crossing number can be used to estimate the chip area required for the VLSI circuit layout of a graph. He proved the following general lower bound for $\mathrm{cr}(G)$, which was discovered independently by Ajtai et al. [ACNS]. The best known constant, $1/33.75$, in the theorem is due to Pach and Tóth.

Theorem A [ACNS], [L2], [PT]. *Let G be a graph with $n(G) = n$ nodes and $e(G) = e$ edges, $e \geq 7.5n$. Then we have*

$$\mathrm{cr}(G) \ \geq \ \frac{1}{33.75}\frac{e^3}{n^2}.$$

Theorem A can be used to deduce the best known upper bounds for the number of unit distances determined by n points in the plane [S3], for the number of different ways how a line can split a set of n points into two equal parts [D], and it has some other interesting corollaries [PS].

It is easy to see that the bound in Theorem A is tight, apart from the value of the constant. However, as was suggested by Simonovits [S1], it may be possible to strengthen the theorem for some special classes of graphs, e.g., for graphs not containing some fixed, so-called *forbidden* subgraph. In Sections 2 and 3 of this paper we verify this conjecture.

A graph property $\mathcal{P}$ is said to be *monotone* if

- whenever a graph G satisfies $\mathcal{P}$, then every subgraph of G also satisfies $\mathcal{P}$;
- whenever G_1 and G_2 satisfy $\mathcal{P}$, then their disjoint union also satisfies $\mathcal{P}$.

For any monotone property $\mathcal{P}$, let $\mathrm{ex}(n, \mathcal{P})$ denote the maximum number of edges that a graph of n vertices can have if it satisfies $\mathcal{P}$. In the special case when $\mathcal{P}$ is the property that the graph does not contain a subgraph isomorphic to a fixed forbidden subgraph H, we write $\mathrm{ex}(n, H)$ for $\mathrm{ex}(n, \mathcal{P})$.

Theorem 1. *Let $\mathcal{P}$ be a monotone graph property with $\mathrm{ex}(n, \mathcal{P}) = O(n^{1+\alpha})$ for some $\alpha > 0$. Then there exist two constants $c, c' > 0$ such that the crossing number of any graph G with property $\mathcal{P}$, which has n vertices and $e \geq cn \log^2 n$ edges, satisfies*

$$\mathrm{cr}(G) \geq c' \frac{e^{2+1/\alpha}}{n^{1+1/\alpha}}.$$

If $\mathrm{ex}(n, \mathcal{P}) = \Theta(n^{1+\alpha})$, then this bound is asymptotically tight, up to a constant factor.

In some interesting special cases when we know the precise order of magnitude of the function $\mathrm{ex}(n, \mathcal{P})$, we obtain some slightly stronger results. The *girth* of a graph is the length of its shortest cycle.

Theorem 2. *Let G be a graph with n vertices and $e \geq 4n$ edges, whose girth is larger than $2r$, for some $r > 0$ integer. Then the crossing number of G satisfies*

$$\mathrm{cr}(G) \geq c_r \frac{e^{r+2}}{n^{r+1}},$$

where $c_r > 0$ *is a suitable constant. For* $r = 2, 3,$ *and* 5*, these bounds are asymptotically tight, up to a constant factor.*

What happens if the girth of G is larger than $2r + 1$? Since one can destroy every odd cycle of a graph by deleting at most half of its edges, even in this case we cannot expect an asymptotically better lower bound for the crossing number of G than the bound given in Theorem 2.

Theorem 3. *Let* G *be a graph with* n *vertices and* $e \geq 4n$ *edges, which does not contain a complete bipartite subgraph* $K_{r,s}$ *with* r *and* s *vertices in its classes,* $s \geq r$*. Then the crossing number of* G *satisfies*

$$\operatorname{cr}(G) \geq c_{r,s} \frac{e^{3+1/(r-1)}}{n^{2+1/(r-1)}},$$

where $c_{r,s} > 0$ *is a suitable constant. These bounds are tight up to a constant factor if* $r = 2, 3,$ *or if* r *is arbitrary and* $s > (r-1)!$.

The *bisection width*, $b(G)$, of a graph G is defined as the minimum number of edges whose removal splits the graph into two roughly equal subgraphs. More precisely, $b(G)$ is the minimum number of edges running between V_1 and V_2, over all partitions of the vertex set of G into two parts $V_1 \cup V_2$ such that $|V_1|, |V_2| \geq n(G)/3$.

Leighton [L1] observed that there is an intimate relationship between the bisection width and the crossing number of a graph, which is based on the Lipton–Tarjan separator theorem for planar graphs [LT]. The proofs of Theorems 1–3 are based on repeated application of the following version of this relationship.

Theorem B [PSS]. *Let* G *be a graph of* n *vertices, whose degrees are* $d_1, d_2, \ldots, d_n$*. Then*

$$b(G) \leq 10\sqrt{\operatorname{cr}(G)} + 2\sqrt{\sum_{i=1}^{n} d_i^2}.$$

Let $\kappa(n, e)$ denote the minimum crossing number of a graph G with n vertices and at least e edges. That is,

$$\kappa(n, e) = \min_{\substack{n(G) = n \\ e(G) \geq e}} \operatorname{cr}(G).$$

It follows from Theorem A that, for $e \geq 4n$, $\kappa(n, e)n^2/e^3$ is bounded from below and from above by two positive constants. Erdős and Guy [EG] conjectured that if $e \gg n$, then $\lim \kappa(n, e)n^2/e^3$ exists. (We use the notation $f(n) \gg g(n)$ to express that $\lim_{n\to\infty} f(n)/g(n) = \infty$.) In Section 4, we settle this problem.

Theorem 4. *If $n \ll e \ll n^2$, then*

$$\lim_{n\to\infty} \kappa(n,e)\frac{n^2}{e^3} = C > 0$$

exists.

We call the constant $C > 0$ in Theorem 4 the *midrange crossing constant*. It is necessary to limit the range of e from below and from above. (See Remark 4.4 at the end of Section 4.)

All of the above problems can be reformulated for graph drawings on other surfaces. Let S_g denote a torus with g holes, i.e., a compact oriented surface of *genus* g with no boundary. Define $\mathrm{cr}_g(G)$, the crossing number of G on S_g, as the minimum number of crossing points in any drawing of G on S_g. Let

$$\kappa_g(n,e) = \min_{\substack{n(G)=n\\ e(G)\geq e}} \mathrm{cr}_g(G).$$

With this notation, $\mathrm{cr}_0(G)$ is the planar crossing number and $\kappa_0(n,e) = \kappa(n,e)$.

In Section 5 we prove that there is a midrange crossing constant for graph drawings on any surface S_g of fixed genus $g \geq 0$.

Theorem 5. *For every $g \geq 0$, if $n \ll e \ll n^2$, then the limit*

$$\lim_{n\to\infty} \kappa_g(n,e)\frac{n^2}{e^3}$$

exists and is equal to the constant $C > 0$ in Theorem 4.

To prove this result, we have to generalize Theorem B.

Theorem 6. *Let G be a graph with n vertices, whose degrees are $d_1, d_2, \ldots, d_n$. Then*

$$b(G) \leq 300(1+g^{3/4})\sqrt{\mathrm{cr}_g(G) + \sum_{i=1}^{n} d_i^2}.$$

For more problems and results on crossing numbers, see [RT] and [WB].

2. Crossing Numbers and Monotone Properties—Proof of Theorem 1

Let $\mathcal{P}$ be a monotone graph property with $\mathrm{ex}(n,\mathcal{P}) \leq An^{1+\alpha}$, for some $A, \alpha > 0$. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, where $|V(G)| = n(G) = n$ and $|E(G)| = e(G) = e$. Suppose that G satisfies property $\mathcal{P}$ and $e \geq cn\log^2 n$. To prove Theorem 1, we assume that

$$\mathrm{cr}(G) < c'\frac{e^{2+1/\alpha}}{n^{1+1/\alpha}},$$

and, if c and c' are suitable constant, we will obtain a contradiction.

We break G into smaller components, according to the following procedure.

DECOMPOSITION ALGORITHM

Step 0. **Let** $G^0 = G$, $G_1^0 = G$, $M_0 = 1$, $m_0 = 1$.

Suppose that we have already executed Step i, and that the resulting graph, G^i, consists of M_i components, $G_1^i, G_2^i, \ldots, G_{M_i}^i$, each of at most $(2/3)^i n$ vertices. Assume, without loss of generality, that the first m_i components of G^i have at least $(2/3)^{i+1}n$ vertices and the remaining $M_i - m_i$ have fewer. Then

$$(2/3)^{i+1}n(G) \le n(G_j^i) \le (2/3)^i n(G) \qquad (j = 1, 2, \ldots, m_i).$$

Thus, we have that $m_i \le (3/2)^{i+1}$.

Step $i+1$. **If**

$$\left(\frac{2}{3}\right)^i < \frac{1}{(2A)^{1/\alpha}} \cdot \frac{e^{1/\alpha}}{n^{1+1/\alpha}}, \tag{1}$$

then STOP. Inequality (1) is called the *stopping rule*.

Else, for $j = 1, 2, \ldots, m_i$, delete $b(G_j^i)$ edges from G_j^i such that G_j^i falls into two components, each of at most $(2/3)n(G_j^i)$ vertices. Let G^{i+1} denote the resulting graph on the original set of n vertices. Clearly, each component of G^{i+1} has at most $(2/3)^{i+1}n$ vertices.

Suppose that the DECOMPOSITION ALGORITHM terminates in Step $k+1$. If $k > 0$, then

$$\left(\frac{2}{3}\right)^k < \frac{1}{(2A)^{1/\alpha}} \cdot \frac{e^{1/\alpha}}{n^{1+1/\alpha}} \le \left(\frac{2}{3}\right)^{k-1}.$$

First, we give an upper bound on the total number of edges deleted from G.

Using that, for any nonnegative reals $a_1, a_2, \ldots, a_m$,

$$\sum_{j=1}^{m} \sqrt{a_j} \le \sqrt{m \sum_{j=1}^{m} a_j}, \tag{2}$$

we obtain that, for any $0 \le i < k$,

$$\sum_{j=1}^{m_i} \sqrt{\mathrm{cr}(G_j^i)} \le \sqrt{m_i \sum_{j=1}^{m_i} \mathrm{cr}(G_j^i)} \le \sqrt{\left(\frac{3}{2}\right)^{i+1}} \sqrt{\mathrm{cr}(G)} < \sqrt{\left(\frac{3}{2}\right)^{i+1}} \sqrt{\frac{c' e^{2+1/\alpha}}{n^{1+1/\alpha}}}.$$

Denoting by $d(v, G_j^i)$ the degree of vertex v in G_j^i, we have

$$\begin{aligned}
\sum_{j=1}^{m_i} \sqrt{\sum_{v\in V(G_j^i)} d^2(v, G_j^i)} &\le \sqrt{\left(\frac{3}{2}\right)^{i+1}} \sqrt{\sum_{v\in V(G^i)} d^2(v, G^i)} \\
&\le \sqrt{\left(\frac{3}{2}\right)^{i+1}} \sqrt{\max_{v\in V(G^i)} d(v, G^i) \sum_{v\in V(G^i)} d(v, G^i)} \\
&\le \sqrt{\left(\frac{3}{2}\right)^{i+1}} \sqrt{\left(\frac{2}{3}\right)^{i} n(2e)} = \sqrt{3en}.
\end{aligned}$$

In view of Theorem B in the Introduction, the total number of edges deleted during the procedure is

$$\begin{aligned}
\sum_{i=0}^{k-1} \sum_{j=1}^{m_i} b(G_j^i) &\le 10 \sum_{i=0}^{k-1} \sum_{j=1}^{m_i} \sqrt{\mathrm{cr}(G_j^i)} + 2 \sum_{i=0}^{k-1} \sum_{j=1}^{m_i} \sqrt{\sum_{v\in V(G_j^i)} d^2(v, G_j^i)} \\
&< 10\sqrt{c'} \sqrt{\frac{e^{2+1/\alpha}}{n^{1+1/\alpha}}} \sum_{i=0}^{k-1} \sqrt{\left(\frac{3}{2}\right)^{i}} + 2k\sqrt{3en} \\
&\le 250\sqrt{c'} \sqrt{\frac{e^{2+1/\alpha}}{n^{1+1/\alpha}}} \sqrt{(2A)^{1/\alpha} \frac{n^{1+1/\alpha}}{e^{1/\alpha}}} + 2k\sqrt{3en} \le \frac{e}{2},
\end{aligned}$$

provided that c' is sufficiently small and c is sufficiently large.

Therefore, the number of edges of the graph G^k obtained in the final Step of the algorithm satisfies

$$e(G^k) \ge \frac{e}{2}.$$

(Note that this inequality trivially holds if the algorithm terminates in the very first Step, i.e., when $k = 0$.)

Next we give a lower bound on $e(G^k)$. The number of vertices of each connected component of G^k satisfies

$$n(G_j^k) \le \left(\frac{2}{3}\right)^{k} n < \frac{1}{(2A)^{1/\alpha}} \cdot \frac{e^{1/\alpha}}{n^{1+1/\alpha}} n = \left(\frac{e}{2An}\right)^{1/\alpha} \qquad (j = 1, 2, \ldots, M_k).$$

Since each G_j^k has property $\mathcal{P}$, it follows that

$$e(G_j^k) \le An^{1+\alpha}(G_j^k) < An(G_j^k) \cdot \frac{e}{2An}.$$

Therefore, for the total number of edges of G_k, we have

$$e(G^k) = \sum_{j=1}^{M_k} e(G_j^k) < A\frac{e}{2An}\sum_{j=1}^{M_k} n(G_j^k) = \frac{e}{2},$$

the desired contradiction. This proves the bound of Theorem 1.

It remains to show that the bound is tight up to a constant factor. Suppose that $\mathrm{ex}(n, \mathcal{P}) \geq A'n^{1+\alpha}$. For every e ($cn < e \leq An^{1+\alpha}$), we construct a graph G of at most n vertices and at least e edges, which has property $\mathcal{P}$ and crossing number

$$\mathrm{cr}(G) \leq c''\frac{e^{2+1/\alpha}}{n^{1+1/\alpha}},$$

for a suitable constant $c'' = c''(A', \alpha)$.

Let

$$k = \left\lceil \frac{2e}{A'n} \right\rceil^{1/\alpha},$$

and let G_k denote a graph of k vertices and at least $A'k^{1+\alpha}$ edges, which has property $\mathcal{P}$. Clearly,

$$\mathrm{cr}(G_k) \leq e^2(G_k) \leq (Ak^{1+\alpha})^2 = A^2k^{2+2\alpha}.$$

Let G be the union of $\lfloor n/k \rfloor$ disjoint copies of G_k. Then $n(G) = \lfloor n/k \rfloor k \leq n$,

$$e(G) = \left\lfloor \frac{n}{k} \right\rfloor e(G_k) \geq \frac{n}{2k}A'kk^{\alpha} \geq e,$$

$$\mathrm{cr}(G) = \left\lfloor \frac{n}{k} \right\rfloor \mathrm{cr}(G_k) \leq \frac{n}{k}A^2k^{2+2\alpha} \leq A^2n\left(2\left(\frac{2e}{A'n}\right)^{1/\alpha}\right)^{1+2\alpha} = \frac{2^{3+2\alpha+1/\alpha}A^2}{(A')^{2+1/\alpha}}\cdot\frac{e^{2+1/\alpha}}{n^{1+1/\alpha}},$$

as required. □

3. Forbidden Subgraphs—Proofs of Theorems 2 and 3

In Section 1 we established Theorem 1 under the assumption $e \geq cn\log^2 n$, where c is a suitable constant depending on property $\mathcal{P}$. It seems very likely that the same result is true for every $e \geq cn$. The appearance of the $\log^2 n$ factor was due to the fact that to estimate the total number of edges deleted during the DECOMPOSITION ALGORITHM, we applied Theorem B. We used a poor upper bound on the term $\sum d_i^2$, because some of the degrees d_i may be very large. However, in some interesting special cases, this difficulty can be avoided by a simple trick. We can split each vertex of high degree into vertices of "average degree," unless the new graph ceases to have property $\mathcal{P}$.

We illustrate this technique by proving the following result, which is the $r = s = 2$ special case of Theorem 3 and a slight modification of Theorem 2 for $r = 2$.

Theorem 3.1. *Let G be a $K_{2,2}$-free (C_4-free) graph with $n(G) = n$ vertices and $e(G) = e$ edges, $e \geq 1000n$. Then*

$$\mathrm{cr}(G) \geq \frac{1}{10^8}\frac{e^4}{n^3}.$$

This bound is tight up to a constant factor.

Proof. Let G be a graph with n vertices and $e \geq 1000n$ edges, which does not contain $K_{2,2}$ as a subgraph. Suppose, in order to obtain a contradiction, that

$$\mathrm{cr}(G) < \frac{1}{10^8}\frac{e^4}{n^3},$$

and G is drawn in the plane with $\mathrm{cr}(G)$ crossings.

First, we split every vertex of G whose degree exceeds $\overline{d} := 2e/n$ into vertices of degree at most $\overline{d}$, as follows. Let v be a vertex of G with degree $d(v, G) = d(v) = d > \overline{d}$, and let $vw_1, vw_2, \ldots, vw_d$ be the edges incident to v, listed in clockwise order. Replace v by $\lceil d/\overline{d}\rceil$ new vertices, $v_1, v_2, \ldots, v_{\lceil d/\overline{d}\rceil}$, placed in clockwise order on a very small circle around v. Without introducing any new crossings, connect w_j to v_i if and only if $\overline{d}(i-1) < j \leq \overline{d}i$ $(1 \leq j \leq d, 1 \leq i \leq \lceil d/\overline{d}\rceil)$. Repeat this procedure for every vertex whose degree exceeds $\overline{d}$, and denote the resulting graph by G'.

Obviously, G' is also $K_{2,2}$-free, $e(G') = e(G) = e$, and

$$\mathrm{cr}(G') \leq \mathrm{cr}(G) < \frac{1}{10^8}\frac{e^4(G)}{n^3(G)}.$$

Since all but at most n vertices of G' have degree $\overline{d}$, we have $n(G') < 2n(G) = 2n$.

Apply the DECOMPOSITION ALGORITHM described in the previous section to the graph G' with the difference that, instead of (1), use the following stopping rule: STOP in Step $i+1$ if

$$\left(\frac{2}{3}\right)^i < \frac{e^2(G')}{16n^3(G')}.$$

Suppose that the algorithm terminates in Step $k+1$. If $k > 0$, then

$$\left(\frac{2}{3}\right)^k < \frac{e^2(G')}{16n^3(G')} \leq \left(\frac{2}{3}\right)^{k-1}.$$

Just like in the proof of Theorem 1, for every $i < k$, we have that

$$\sum_{j=1}^{m_i}\sqrt{\mathrm{cr}(G_j^i)} \leq \sqrt{\left(\frac{3}{2}\right)^{i+1}}\sqrt{\mathrm{cr}(G)} < \frac{1}{10^4}\sqrt{\left(\frac{3}{2}\right)^{i+1}}\frac{e^2}{n^{3/2}}$$

and, using the fact that the maximum degree in G' is at most $\overline{d}$,

$$\begin{aligned}\sum_{j=1}^{m_i}\sqrt{\sum_{v\in V(G_j^i)} d^2(v, G_j^i)} &\leq \sqrt{\left(\frac{3}{2}\right)^{i+1}}\sqrt{\sum_{v\in V(G')} d^2(v, G')}\\ &\leq \sqrt{\left(\frac{3}{2}\right)^{i+1}}\sqrt{\overline{d}2e(G')} \leq 2\sqrt{\left(\frac{3}{2}\right)^{i+1}}\frac{e}{\sqrt{n}}.\end{aligned}$$

Hence, by Theorem B, the total number of edges deleted during the algorithm is

$$\begin{aligned}\sum_{i=0}^{k-1}\sum_{j=1}^{m_i} b(G_j^i) &\le 10\sum_{i=0}^{k-1}\sum_{j=1}^{m_i}\sqrt{\mathrm{cr}(G_j^i)} + 2\sum_{i=0}^{k-1}\sum_{j=1}^{m_i}\sqrt{\sum_{v\in V(G_j^i)} d^2(v, G_j^i)}\\ &< \frac{1}{1000}\frac{e^2}{n^{3/2}}\sum_{i=0}^{k-1}\sqrt{\left(\frac{3}{2}\right)^{i+1}} + 4\frac{e}{\sqrt{n}}\sum_{i=0}^{k-1}\sqrt{\left(\frac{3}{2}\right)^{i+1}}\\ &= \sqrt{\frac{3}{2}}\frac{\sqrt{(3/2)^k}-1}{\sqrt{3/2}-1}\left(\frac{e^2}{1000n^{3/2}} + \frac{4e}{\sqrt{n}}\right)\\ &< 100\frac{n^{3/2}}{e}\left(\frac{e^2}{1000n^{3/2}} + \frac{4e}{\sqrt{n}}\right) < \frac{e}{10} + 400n < \frac{e}{2}.\end{aligned}$$

Therefore, for the resulting graph,

$$e(G^k) \ge \frac{e}{2}.$$

On the other hand, each component of G^k has relatively few vertices:

$$n(G_j^k) < \left(\frac{2}{3}\right)^k n(G') < \frac{e^2}{16n^2(G')} = \frac{e^2}{16n^2(G^k)} \qquad (j = 1, 2, \ldots, M_k).$$

Claim C [R]. *Let* $\mathrm{ex}(n, K_{2,2})$ *denote the maximum number of edges that a* $K_{2,2}$*-free graph with n vertices can have. Then*

$$\mathrm{ex}(n, K_{2,2}) \le \frac{n\left(1+\sqrt{4n-3}\right)}{4} \le n^{3/2}.$$

Applying the claim to each G_k^j, we obtain

$$e(G_j^k) \le n^{3/2}(G_j^k) < n(G_j^k)\cdot\sqrt{\frac{e^2}{16n^2(G^k)}},$$

therefore,

$$e(G^k) = \sum_{j=1}^{M_k} e(G_j^k) < \frac{e}{4n(G^k)}\sum_{j=1}^{M_k} n(G_j^k) = \frac{e}{4},$$

the desired contradiction. The tightness of Theorem 3.1 immediately follows from the fact that Theorem 1 was tight. □

Theorems 2 and 3 can be proved similarly. It is enough to notice that splitting a vertex of high degree does not decrease the girth of a graph G and does not create a subgraph isomorphic to $K_{r,s}$. Instead of Claim C, now we need

Claim C′ [BS], [B2], [B1], [S2], [W]. *For a fixed positive integer r, let $\mathcal{G}_{2r}$ denote the property that the girth of a graph is larger than $2r$. Then the maximum number of edges of a graph with n vertices, which has property $\mathcal{G}_{2r}$, satisfies*

$$\mathrm{ex}(n, \mathcal{G}_{2r}) = O(n^{1+1/r}).$$

For $r = 2, 3$, and 5, this bound is tight.

Claim C″ [KST], [F], [ER], [B2], [ARS]. *For any integers $s \geq r \geq 2$, the maximum number of edges of a $K_{r,s}$-free graph of n vertices, satisfies*

$$\mathrm{ex}(n, K_{r,s}) = O(n^{2-1/r}).$$

This bound is tight for $s > (r-1)!$.

In case $r = 3$, we obtain the following slight generalization of Theorem 2.

Theorem 3.2. *Let G be a graph of n vertices and $e \geq 4n$ edges, which contains no cycle C_6 of length 6.*

Then, for a suitable constant $c'_6 > 0$, we have

$$\mathrm{cr}(G) \geq c'_6 \frac{e^5}{n^4}.$$

To establish Theorem 3.2, it is enough to modify the proof of Theorem 2 at one point. Before splitting the high-degree vertices of G and running the DECOMPOSITION ALGORITHM, we have to turn G into a bipartite graph, by deleting at most half of its edges. After that, splitting a vertex cannot create a C_6, and the rest of the above argument shows that the crossing number of the remaining graph still exceeds $c'_6 e^5/n^4$.

We do not see, however, how to obtain the analogous generalization of Theorem 2 for $r > 3$.

4. Midrange Crossing Constant in the Plane—Proof of Theorem 4

Lemma 4.1.

(i) *For any $a > 0$, the limit*

$$\gamma[a] = \lim_{n\to\infty} \frac{\kappa(n, na)}{n}$$

exists and is finite.

(ii) *$\gamma[a]$ is a convex continuous function.*

(iii) *For any $a \geq 4$, $1 > \delta > 0$,*

$$\gamma[a] - \gamma[a(1-\delta)] \leq \gamma[a(1+\delta)] - \gamma[a] \leq 10^3 \delta \gamma[a].$$

Proof. Clearly, any two graphs, G_1 and G_2, can be drawn in the plane so that the edges of G_1 do not intersect the edges of G_2. Therefore,

$$\kappa(n_1 + n_2, e_1 + e_2) \leqq \kappa(n_1, e_1) + \kappa(n_2, e_2). \tag{3}$$

In particular, the function $f_a(n) = \kappa(n, na)$ is subadditive and hence the limit

$$\gamma[a] = \lim_{n\to\infty} \frac{\kappa(n, na)}{n}$$

exists and is finite for every fixed $a > 0$. It also follows from (3) that, for any $a, b > 0$ and $1 > \alpha > 0$, if n and αn are both integers,

$$\kappa(n, (\alpha a + (1-\alpha)b)n) \leqq \kappa(\alpha n, \alpha a n) + \kappa((1-\alpha)n, (1-\alpha)bn),$$

so, for any $1 > \alpha > 0$ rational,

$$\gamma[\alpha a + (1-\alpha)b] \leq \alpha\gamma[a] + (1-\alpha)\gamma[b].$$

However, since the function $\gamma[a]$ is monotone increasing, it follows that, for *any* $1 > \alpha > 0$,

$$\gamma[\alpha a + (1-\alpha)b] \leq \alpha\gamma[a] + (1-\alpha)\gamma[b]. \tag{4}$$

That is, the function $\gamma[a]$ is *convex*. In particular, for every $1 > \delta > 0$, we have

$$\gamma[a] - \gamma[a(1-\delta)] \leq \gamma[a(1+\delta)] - \gamma[a].$$

It is known that, for any $a \geq 4$,

$$\frac{a^3 n}{100} \leq \kappa(n, an) \leq a^3 n \quad \Rightarrow \quad \frac{a^3}{100} \leq \gamma[a] \leq a^3 \tag{5}$$

(see, e.g., [PT]). Let $a \geq 4$, $1 > \delta > 0$. By (4),

$$\gamma[a(1+\delta)] \leq (1-\delta)\gamma[a] + \delta\gamma[2a].$$

Therefore, using (5),

$$\gamma[a(1+\delta)] - \gamma[a] \leq \delta\gamma[2a] \leq \delta 8a^3 < 10^3\delta\gamma[a]. \qquad \square$$

Set

$$C := \limsup_{a\to\infty} \frac{\gamma[a]}{a^3}.$$

By (5), we have that $C < 1$.

Lemma 4.2. *For any* $0 < \varepsilon < 1$, *there exists* $N = N(\varepsilon)$ *such that* $\kappa(n, e) > C(e^3/n^2)(1-\varepsilon)$, *whenever* $\min\{n, e/n, n^2/e\} > N$.

Proof. Let $A > 10^9/\varepsilon^3$ be a rational number satisfying

$$\frac{\gamma[A]}{A^3} > C\left(1 - \frac{\varepsilon}{10}\right). \tag{6}$$

Let $N = N(\varepsilon) \geq A$ such that, if $n > N$, $e = nA'$, and $|A - A'| \leq A\varepsilon$, then

$$\kappa(n, e) > \gamma[A']\left(1 - \frac{\varepsilon}{10}\right)n. \tag{7}$$

Let n and e be fixed, $\min\{n, e/n, n^2/e\} > N$ and let $G = (V, E)$ be a graph with $|V| = n$ vertices and $|E| = e$ edges, drawn in the plane with $\kappa(n, e)$ crossings. Set $p = An/e$. Let U be a randomly chosen subset of V with $\Pr[v \in U] = p$, independently for all $v \in V$. Let $\nu = |U|$, and let η (resp. ξ) be the number of edges (resp. crossings) in the (drawing of the) subgraph of G induced by the elements of U.

ν has mean pn and variance $p(1-p)n \leqq pn$, so, by the Chebyshev Inequality,

$$\Pr\left[|\nu - pn| > \frac{\varepsilon}{10^4}pn\right] < \frac{\varepsilon}{10}.$$

Write $\eta = \sum I_{uv}$, where the sum is taken over all edges $uv = vu \in E$, and I_{uv} denotes the indicator for the event $u, v \in U$. Obviously, $E[\eta] = \sum_{uv\in E} E[I_{uv}] = ep^2$. We decompose

$$\mathrm{Var}[\eta] = \sum_{uv\in E} \mathrm{Var}[I_{uv}] + \sum_{uv,uw\in E} \mathrm{Cov}[I_{uv}, I_{uw}],$$

as $\mathrm{Cov}[I_{uv}, I_{wz}] = 0$ when all four indices are distinct. As always with indicators, we have

$$\sum_{uv\in E} \mathrm{Var}[I_{uv}] \leqq \sum_{uv\in E} E[I_{uv}] = E[\eta] = ep^2.$$

Using the bound $\mathrm{Cov}[I_{uv}, I_{uw}] \leqq E[I_{uv}I_{uw}] = p^3$, we obtain

$$\mathrm{Var}[\eta] \leqq p^2 e + p^3 \sum_{v\in V} \binom{d(v)}{2},$$

where $d(v)$ is the degree of vertex v in G. However, $\sum_{v\in V} d(v) = 2e$ and all $d(v) < n$, so

$$\sum_{v\in V} \binom{d(v)}{2} \leqq \tfrac{1}{2} \sum_{v\in V} d^2(v) \leqq en.$$

Thus, we have

$$\mathrm{Var}[\eta] \leqq p^2 e + p^3 en \leqq 2p^3 en,$$

as $pn = An^2/e \geq 1$. Again, by the Chebyshev Inequality,

$$\Pr\left[|\eta - p^2 e| > \frac{\varepsilon}{10^4}p^2 e\right] < \frac{\varepsilon}{10}.$$

With probability at least $1 - \varepsilon/5$,

$$pn\left(1 - \frac{\varepsilon}{10^4}\right) < \nu < pn\left(1 + \frac{\varepsilon}{10^4}\right) \quad \text{and} \quad p^2 e\left(1 - \frac{\varepsilon}{10^4}\right) < \eta < p^2 e\left(1 + \frac{\varepsilon}{10^4}\right),$$

so with probability at least $1 - \varepsilon/5$,

$$A\left(1 - \frac{3\varepsilon}{10^4}\right) < \frac{\eta}{\nu} = A' < A\left(1 + \frac{3\varepsilon}{10^4}\right).$$

Therefore, in view of (7), with probability at least $1 - \varepsilon/5$, the subgraph of G induced by U has at least $pn(1 - \varepsilon/10)\gamma[A'](1 - \varepsilon/10)$ crossings. However, then we have

$$\begin{aligned} E[\xi] &\geq \left(1 - \frac{\varepsilon}{5}\right) pn\left(1 - \frac{\varepsilon}{10}\right)\gamma[A']\left(1 - \frac{\varepsilon}{10}\right) \\ &\geq \left(1 - \frac{\varepsilon}{5}\right) pn\left(1 - \frac{\varepsilon}{10}\right)\gamma[A]\left(1 - \frac{3\varepsilon}{10}\right)\left(1 - \frac{\varepsilon}{10}\right) \\ &\geq \left(1 - \frac{\varepsilon}{5}\right) pn\left(1 - \frac{\varepsilon}{10}\right) CA^3\left(1 - \frac{\varepsilon}{10}\right)\left(1 - \frac{3\varepsilon}{10}\right)\left(1 - \frac{\varepsilon}{10}\right) \\ &\geq (1 - \varepsilon)CA^3 pn, \end{aligned}$$

where the second and third inequalities follow from Lemma 4.1(iii) and from the choice of A, respectively.

On the other hand,

$$E[\xi] = p^4\kappa(n, e),$$

as every crossing lies in U with probability p^4. Thus

$$\kappa(n, e) \geq (1 - \varepsilon)\frac{pnCA^3}{p^4} = C\frac{e^3}{n^2}(1 - \varepsilon)$$

as desired. □

To complete the proof of Theorem 4, we have to establish the "counterpart" of Lemma 4.2.

Lemma 4.3. *For any* $1 > \varepsilon > 0$, *there exists* $M = M(\varepsilon)$ *such that* $\kappa(n, e) < C(e^3/n^2)(1 + \varepsilon)$, *whenever* $\min\{n, e/n, n^2/e\} > M$.

Proof. Let $A > 10^4/\varepsilon^2$ be a rational number satisfying

$$C\left(1 - \frac{\varepsilon}{10}\right) < \frac{\gamma[A]}{A^3} < C\left(1 + \frac{\varepsilon}{10}\right).$$

Let $M_1 = M_1(\varepsilon) \geq A$ such that, if $n > M_1$ and $e = nA$, then

$$CA^3 n\left(1 - \frac{\varepsilon}{5}\right) < \kappa(n, e) < CA^3 n\left(1 + \frac{\varepsilon}{5}\right).$$

Let $G_1 = G_1(n_1, e_1)$ be a graph with $n_1 > M_1$ vertices, $e_1 = An_1$ edges, and suppose that G_1 is drawn in the plane with $\kappa(n_1, e_1)$ crossings, where $CA^3n_1(1-\varepsilon/5) < \kappa(n_1, e_1) < CA^3n_1(1+\varepsilon/5)$. For each vertex v of G_1 with degree $d(v) > A^{3/2}$, we do the following. Let $d(v) = rA^{3/2} + s$, where $0 \le s < A^{3/2}$. Substitute v with $r+1$ vertices, each of degree $A^{3/2}$, except one which has degree s, each drawn very close to the original position of v. Clearly, this can be done without creating any additional crossing. We obtain a graph $G_2(n_2, e_2)$ such that

$$n_1 \le n_2 \le n_1\left(1 + \frac{2}{\sqrt{A}}\right) \le n_1\left(1 + \frac{\varepsilon}{10}\right),$$

$e_2 = e_1$, and G_2 is drawn in the plane with $\kappa(n_1, e_1)$ crossings.

Suppose that n and e are fixed, $\min\{n, e/n, n^2/e\} > M(\varepsilon) = 10M_1/\varepsilon$. Let

$$L = \frac{e/n}{e_2/n_2} \quad \text{and} \quad K = \frac{n^2/e}{n_2^2/e_2},$$

so that

$$n = KLn_2 \quad \text{and} \quad e = KL^2e_2.$$

Let

$$\tilde{L} = \left\lfloor L\left(1 + \frac{\varepsilon}{10}\right)\right\rfloor \quad \text{and} \quad \tilde{K} = \left\lfloor K\left(1 - \frac{\varepsilon}{10}\right)\right\rfloor$$

and let

$$\tilde{n} = \tilde{K}\tilde{L}n_2 \quad \text{and} \quad \tilde{e} = \tilde{K}\tilde{L}^2e_2.$$

Then $n(1-\varepsilon/5) < \tilde{n} < n$ and $e_2 < \tilde{e} \le e_2(1+\varepsilon/4)$, so we have $\kappa(n, e) < \kappa(\tilde{n}, \tilde{e})$.

Substitute each vertex of G_2 with $\tilde{L}$ very close vertices, and substitute each edge of G_2 with the corresponding $\tilde{L}^2$ edges, all running very close to the original edge. Make $\tilde{K}$ copies of this drawing, each separated from the others. This way we got a graph $\tilde{G}(\tilde{n}, \tilde{e})$ drawn in the plane. We estimate the number of crossings X in this drawing.

A crossing in the original drawing of G_2 corresponds to $\tilde{K}\tilde{L}^4$ crossings in the present drawing of $\tilde{G}$. For any two edges of G_2 with common endpoint, uv and uw, the edges arise from them have at most $\tilde{K}\tilde{L}^4$ crossings with each other. So

$$X \le \tilde{K}\tilde{L}^4\left(\kappa(n_1, e_1) + \sum_{v \in V(G_2)} \binom{d(v)}{2}\right).$$

However, $\sum_{v\in V(G_2)} d(v) = 2e_2$ and $d(v) \le A^{3/2}$, so

$$\sum_{v \in V(G_2)} \binom{d(v)}{2} < 3A^{5/2}n_2.$$

Therefore,

$$\begin{aligned} \kappa(n, e) &< \kappa(\tilde{n}, \tilde{e}) \le c < \tilde{K}\tilde{L}^4\kappa(n_1, e_1) + \tilde{K}\tilde{L}^4 3A^{5/2}n_2 < \tilde{K}\tilde{L}^4\kappa(n_1, e_1)\left(1 + \frac{\varepsilon}{10}\right) \\ &< \tilde{K}\tilde{L}^4CA^3n_1\left(1 + \frac{\varepsilon}{5}\right)\left(1 + \frac{\varepsilon}{10}\right) = \tilde{K}\tilde{L}^4C\frac{e_1^3}{n_1^2}\left(1 + \frac{\varepsilon}{5}\right)\left(1 + \frac{\varepsilon}{10}\right) \\ &< KL^4C\frac{e_2^3}{n_2^2}\left(1 + \frac{\varepsilon}{10}\right)^6\left(1 + \frac{\varepsilon}{5}\right)\left(1 + \frac{\varepsilon}{10}\right) < C(1+\varepsilon)\frac{e^3}{n^2}. \end{aligned}$$

□

Remark 4.4. It was shown in [PT] that $0.06 \geq C \geq 0.029$.

We cannot decide whether Theorem 4 remains true under the weaker condition that $C_1 n \leq e \leq C_2 n^2$ for suitable positive constants C_1 and C_2. If the answer were in the affirmative, then, clearly, $C_1 > 3$. We would also have that $C_2 < \frac{1}{2}$, because, by [G], for $e = \binom{n}{2}$, $\mathrm{cr}(K_n) > (\frac{1}{10} - \varepsilon)(e^3/n^2)$ for any $\varepsilon > 0$ if n is large enough.

5. Midrange Crossing Constants on Other Surfaces—Proof of Theorem 5

Lemma 5.1. *For any integer $g \geq 0$ and for any $1 > \varepsilon > 0$, there exists $N = N(g, \varepsilon)$ such that $\kappa_g(n, e) > C(e^3/n^2)(1 - \varepsilon)$, whenever $\min\{n, e/n, n^{3/2}/e\} > N$.*

Proof. For $g = 0$, the assertion follows from Lemma 4.2. Suppose that $g > 0$ is fixed and we have already proved the lemma for $g - 1$. For any $\varepsilon > 0$, let $N(g, \varepsilon) = (10^5/\varepsilon^2) g N(g - 1, \varepsilon/10)$. Suppose, in order to get a contradiction, that $\min\{n, e/n, n^{3/2}/e\} > N$, and let $G(n, e)$ be a graph drawn on S_g with $\mathrm{cr}_g(G) = \kappa_g(n, e) < C(e^3/n^2)(1 - \varepsilon)$ crossings.

As long as there is an edge with at least $4C(e^2/n^2)$ crossings, delete it. Let the resulting graph be $G_1(n_1, e_1)$. Suppose that we deleted e' edges. Then G_1 has $n_1 = n$ vertices, $e_1 = e - e'$ edges, and the number of crossings in the resulting drawing of G_1 is at most $\mathrm{cr}_g(G) - 4C(e^2/n^2)e'$. Therefore, $e' < e/4$, so $e \geq e_1 \geq 3e/4$. It is not hard to check that $\mathrm{cr}_g(G_1) < C(e_1^3/n_1^2)(1 - \varepsilon)$ and G_1 contains no edge with more than $4C(e^2/n^2) < 8C(e_1^2/n_1^2)$ crossings.

Consider all cycles of G_1, as they are drawn on S_g. If each cycle is *trivial*, i.e., each cycle is contractible to a point of S_g, then every connected component of G is contractible to a point. That is, in this case, our drawing of G on S_g is equivalent to a drawing of G_1 on the plane. Consequently, $\mathrm{cr}_{g-1}(G_1) \leq \mathrm{cr}_0(G_1) < C(e^3/n^2)(1 - \varepsilon)$ contradicting the induction hypothesis.

Suppose that there is a nontrivial (i.e., noncontractible) cycle $\mathcal{C}$ of G_1 with at most $(\varepsilon/80C)$, (n_1^2/e_1) edges. Clearly, $\mathcal{C}$ contains a nontrivial closed curve, $\mathcal{C}'$, which does not intersect itself. The total number of crossings along $\mathcal{C}'$ is at most

$$\frac{\varepsilon}{80C}\frac{n_1^2}{e_1}8C\frac{e_1^2}{n_1^2} = \frac{\varepsilon}{10}e_1.$$

Delete all edges that cross $\mathcal{C}'$. Cut S_g along $\mathcal{C}'$. Replace every vertex (resp. edge) $\mathcal{C}'$ by two vertices, one on each side of the cut. Every edge of G arriving at a vertex v of $\mathcal{C}'$ from a given side of the cut will be connected to the copy of v lying on the same side. Thus, we obtain a graph $G_2(n_2, e_2)$, drawn with fewer than $\mathrm{cr}_g(G_1)$ crossings. Attaching a half-sphere to each side of the cut, we obtain either a surface of genus $g - 1$ or two surfaces whose genuses are smaller than g. We discuss only the former case (the calculation in the latter one is very similar). Since we doubled at most

$$\frac{\varepsilon}{80C}\frac{n_1^2}{e_1} = \varepsilon n_1 \frac{n_1}{e_1}\frac{1}{80C} < \varepsilon n_1 \frac{1}{N} < n_1 \frac{\varepsilon}{10}$$

vertices and deleted at most $(\varepsilon/10)e$ edges, we have $n_2 \leq n_1(1+\varepsilon/10)$ and $e_2 \geq e_1(1-\varepsilon/0)$. In the resulting drawing there are fewer than $\mathrm{cr}_g(G_1)$ crossings, therefore

$$\mathrm{cr}_{g-1}(G_2) < \mathrm{cr}_g(G_1) < C\frac{e_1^3}{n_1^2}(1-\varepsilon) \leq C\frac{e_2^3}{n_2^2}(1-\varepsilon)\left(1-\frac{\varepsilon}{10}\right)^{-3}\left(1+\frac{\varepsilon}{10}\right)^2 \leq C\frac{e_2^3}{n_2^2}\left(1-\frac{\varepsilon}{10}\right),$$

contradicting the induction hypothesis.

Thus, we can assume that every nontrivial cycle of G_1 contains at least $(\varepsilon/80C)(n_1^2/e_1)$ edges. For each vertex v of G_1 with degree $d(v) > 10e_1/\varepsilon n_1$, we do the following. Let $d(v) = r(10e_/\varepsilon n_1) + s$, where $0 \leq s < 10e_1/\varepsilon n_1$. Without creating any new crossing, replace v by $r+1$ nearby vertices, each of degree $10e_1/\varepsilon n$, except one, whose degree is s. We obtain a graph $G_3(n_3, e_3)$ drawn on S_g with $n_1 \leq n_3 \leq n_1(1+\varepsilon/5)$, $e_3 = e_1$, and with the same number of crossings as G_1. Hence,

$$\mathrm{cr}_g(G_3) \leq \mathrm{cr}_g(G_1) \leq C\frac{e_1^3}{n_1^2}(1-\varepsilon) \leq C\frac{e_3^3}{n_3^2}(1-\varepsilon)\left(1+\frac{\varepsilon}{5}\right)^2 \leq C\frac{e_3^3}{n_3^2}\left(1-\frac{\varepsilon}{2}\right).$$

The maximum degree D in G_3 cannot exceed $10e_1/\varepsilon n_1 < 18e_3/\varepsilon n_3$, and the length of each nontrivial cycle is at least

$$\frac{\varepsilon}{80C}\frac{n_1^2}{e_1} \geq \frac{\varepsilon}{100C}\frac{n_3^2}{e_3}.$$

Apply to G_3 the Decomposition Algorithm described in Section 2 with the difference that, instead of (1), use the following stopping rule: stop in Step $i+1$ if

$$\left(\frac{2}{3}\right)^i < \frac{\varepsilon}{100C}\frac{n_3}{e_3}.$$

Suppose that the algorithm terminates in Step $k+1$. Then

$$\left(\frac{2}{3}\right)^k < \frac{\varepsilon}{100C}\frac{n_3}{e_3} \leq \left(\frac{2}{3}\right)^{k-1}.$$

First, we give an upper bound on the total number of edges deleted from G_3. Let $G^0 = G_1^0 = G_3$ and $m_0 = 1$. Using (2), we obtain that, for every $0 \leq i < k$,

$$\sum_{j=1}^{m_i}\sqrt{\mathrm{cr}_g(G_j^i)} \leq \sqrt{m_i\sum_{j=1}^{m_i}\mathrm{cr}_g(G_j^i)} \leq \sqrt{\left(\frac{3}{2}\right)^{i+1}}\sqrt{\mathrm{cr}_g(G_3)} \leq \sqrt{\left(\frac{3}{2}\right)^{i+1}}\sqrt{C\frac{e_3^3}{n_3^2}\left(1-\frac{\varepsilon}{2}\right)}.$$

Denoting by $d(v, G_j^i)$ the degree of vertex v in G_j^i, we have

$$\begin{aligned}\sum_{j=1}^{m_i}\sqrt{\sum_{v\in V(G_j^i)} d^2(v, G_j^i)} &\le \sqrt{\left(\frac{3}{2}\right)^{i+1}}\sqrt{\sum_{v\in V(G^i)} d^2(v, G^i)}\\ &\le \sqrt{\left(\frac{3}{2}\right)^{i+1}}\sqrt{\max_{v\in V(G^i)} d(v, G^i)\sum_{v\in V(G^i)} d(v, G^i)}\\ &\le \sqrt{\left(\frac{3}{2}\right)^{i+1}}\sqrt{\frac{18e_3^3}{\varepsilon n_3^2}(2e_3)} = 12\sqrt{\left(\frac{3}{2}\right)^{i+1}}\frac{e_3}{\sqrt{\varepsilon n_3}}.\end{aligned}$$

By Theorem 6 (proved in the next section), the total number of edges deleted during the algorithm is

$$\begin{aligned}\sum_{i=0}^{k-1}\sum_{j=1}^{m_i} b(G_j^i) &\le 300(1+g^{3/4})\sum_{i=0}^{k-1}\sum_{j=1}^{m_i}\sqrt{\mathrm{cr}_g(G_j^i) + \sum_{v\in V(G_j^i)} d^2(v, G_j^i)}\\ &\le 300(1+g^{3/4})\sum_{i=0}^{k-1}\sum_{j=1}^{m_i}\sqrt{\mathrm{cr}_g(G_j^i)}\\ &\quad + 300(1+g^{3/4})\sum_{i=0}^{k-1}\sum_{j=1}^{m_i}\sqrt{\sum_{v\in V(G_j^i)} d^2(v, G_j^i)}\\ &\le 300(1+g^{3/4})\sum_{i=0}^{k-1}\sqrt{\left(\frac{3}{2}\right)^{i+1}}\left(\sqrt{C\frac{e_3^3}{n_3^2}\left(1-\frac{\varepsilon}{2}\right)} + 6\frac{e_3}{\sqrt{\varepsilon n_3}}\right)\\ &\le 300(1+g^{3/4})\sqrt{\frac{3}{2}}\frac{\sqrt{(3/2)^k}-1}{\sqrt{3/2}-1}\left(\sqrt{C\frac{e_3^3}{n_3^2}\left(1-\frac{\varepsilon}{2}\right)} + 6\frac{e_3}{\sqrt{\varepsilon n_3}}\right)\\ &\le 2000(1+g^{3/4})\sqrt{\frac{C}{\varepsilon}}\sqrt{\frac{e}{n}}\left(\sqrt{C\frac{e_3^3}{n_3^2}\left(1-\frac{\varepsilon}{2}\right)} + 6\frac{e_3}{\sqrt{\varepsilon n_3}}\right) \le e_3\frac{\varepsilon}{10}.\end{aligned}$$

Therefore, the number of edges $e(G^k)$ of the graph G^k obtained in the final Step of the algorithm satisfies $e(G^k) \ge e_3(1-\varepsilon/10)$. Consider the drawing of G^k on S_g inherited from the drawing of G_3. Each connected component of G^k has fewer than $(\varepsilon/100C)(n_3^2/e_3)$ vertices, therefore, each cycle of G^k, as drawn on S_g, is contractible to a point. Consequently, this drawing is equivalent to a planar drawing of G^k. Hence,

$$\begin{aligned}\mathrm{cr}_{g-1}(G^k) \le \mathrm{cr}_0(G^k) \le \mathrm{cr}_g(G_3) \le C\frac{e_3^3}{n_3^2}\left(1-\frac{\varepsilon}{2}\right) &\le C\frac{e^3(G^k)}{n^2(G^k)}\left(1-\frac{\varepsilon}{2}\right)\left(1-\frac{\varepsilon}{10}\right)^{-3}\\ &< C\frac{e^3(G^k)}{n^2(G^k)}\left(1-\frac{\varepsilon}{10}\right),\end{aligned}$$

a contradiction. This concludes the proof of Lemma 5.1. □

Lemma 5.2. *For any integer $g \geq 0$ and for any $\varepsilon > 0$, there exists $N' = N'(g, \varepsilon)$ such that $\kappa_g(n, e) > C(e^3/n^2)(1 - \varepsilon)$, whenever $\min\{n, e/n, n^2/e\} > N'$.*

Proof. The proof is analogous to that of Lemma 4.2. □

Lemma 5.3. *For any integer $g \geq 0$ and for any $\varepsilon > 0$, there exists $M = M(g, \varepsilon)$ such that $\kappa_g(n, e) < C(e^3/n^2)(1 + \varepsilon)$, whenever $\min\{n, e/n, n^2/e\} > M$.*

Proof. Clearly, for any graph G and for any $g \geq 0$, we have $\mathrm{cr}_0(G) \geq \mathrm{cr}_g(G)$. Therefore, Lemma 5.3 is a direct consequence of Lemma 4.3. □

Theorem 5 now readily follows from Lemmas 5.2 and 5.3.

6. A Separator Theorem—Proof of Theorem 6

For the proof of Theorem 6, we need a slight variation of the notion of bisection width. The *weak bisection width*, $\overline{b}(G)$, of a graph G is defined as the minimum number of edges whose removal splits the graph into two components, each of size at least $|V(G)|/5$. That is,

$$\overline{b}(G) = \min_{|V_A|,|V_B| \geq n/5} |E(V_A, V_B)|,$$

where $E(V_A, V_B)$ denotes the number of edges between V_A and V_B, and the minimum is taken over all partitions $V(G) = V_A \cup V_B$ with $|V_A|, |V_B| \geq |V(G)|/5$.

Lemma 6.1. *For any graph G, we have*

$$\overline{b}(G) \leq b(G) \leq 2 \max_{H \subset G} \overline{b}(H).$$

Proof. The first inequality is obviously true. To prove the second one, let $|V(G)| = n$ and consider a partition $V(G) = V_A \cup V_B$ such that $n/5 \leq |V_A|, |V_B| \leq 4n/5$ and $|E(V_A, V_B)| = \overline{b}(G)$. Suppose that $|V_A| \leq |V_B|$. If $n/3 \leq |V_A|$, then $b(G) = \overline{b}(G)$ and we are done. So we can assume that $n/5 \leq |V_A| \leq n/3$ and $2n/3 \leq |V_B| \leq 4n/5$.

Let H be the subgraph of G induced by V_B. By definition, there is a partition $V_B = V'_B \cup V''_B$ such that $|V_B|/5 \leq |V'_B|, |V''_B| \leq 4|V_B|/5$, and $|E(V'_B, V''_B)| = \overline{b}(H)$. We can assume that $|V'_B| \leq |V''_B|$. Then

$$\frac{n}{3} \leq \frac{|V_B|}{2} \leq |V''_B| \leq \frac{4|V_B|}{5} \leq \frac{16n}{25} < \frac{2n}{3}.$$

Letting $V_1 = V_A \cup V'_B$ and $V_2 = V''_B$, we have $V(G) = V_1 \cup V_2$, $n/3 \leq |V_1|, |V_2| \leq 2n/3$,

$$|E(V_1, V_2)| \leq |E(V_A, V_B)| + |E(V'_B, V''_B)| \leq \overline{b}(G) + \overline{b}(H),$$

and the result follows. □

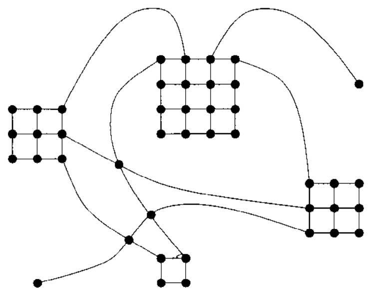

Fig. 1. The definition of H.

Theorem 6 is an immediate consequence of Lemma 6.1 and the following statement.

Theorem 6.2. *Let G be a graph with n vertices of degrees* $d_1, d_2, \ldots, d_n$. *Then*

$$\overline{b}(G) \leq 150(1 + g^{3/4})\sqrt{cr_g(G) + \sum_{i=1}^{n} d_i^2}.$$

Proof. Clearly, we can assume that G contains no isolated vertices, that is, $d_i > 0$ for all $1 \leq i \leq n$. Consider a drawing of G on S_g with exactly $\mathrm{cr}_g(G)$ crossings. Let $v_1, v_2, \ldots, v_n$ be the vertices of G with degrees $d_1, d_2, \ldots, d_n$, respectively. Introduce a new vertex at each crossing. Denote the set of these vertices by V_0. Replace each $v_i \in V(G)$ $(i = 1, 2 \ldots, n)$ by a set V_i of vertices forming a $d_i \times d_i$ piece of a square grid, in which each vertex is connected to its horizontal and vertical neighbors. Let each edge incident to v_i be hooked up to distinct vertices along one side of the boundary of V_i without creating any crossing. These d_i vertices will be called the *special boundary vertices* of V_i.

Thus, we obtain a graph H of $\sum_{i=0}^{n} |V_i| = \mathrm{cr}_g(G) + \sum_{i=1}^{n} d_i^2$ vertices and no crossing (see Fig. 1). For each $1 \leq i \leq n$, assign weight $1/d_i$ to each special boundary vertex of V_i. Assign weight 0 to all other vertices of H. For any subset ν of the vertex set of H, let $w(\nu)$ denote the total weight of the vertices belonging to ν. With this notation, $w(V_i) = 1$ for each $1 \leq i \leq n$. Consequently, $w(V(H)) = n$.

Since H is drawn on S_g without crossing, H does not contain K_α as a minor, where $\alpha = \lfloor 4 + 4\sqrt{g} \rfloor$ [RY]. Then, by a result of Alon et al. [AST1] (see also [AST2]), the vertices of H can be partitioned into three sets, A, B, and C, such that $w(A), w(B) \geq n/3$ and $|C| \leq 25(1 + g^{3/4})\sqrt{\mathrm{cr}_g(G) + \sum_{i=1}^{n} d_i^2}$, and there is no edge from A to B. Let $A_i = A \cap V_i$, $B_i = B \cap V_i$, $C_i = C \cap V_i$ $(i = 0, 1, \ldots, n)$.

For any $1 \leq i \leq n$, we say that V_i is of *type A* (resp. *type B*) if $w(A_i) \geq \frac{5}{6}$ (resp. $w(B_i) \geq \frac{5}{6}$), and it is of *type C*, otherwise.

Define a partition $V(G) = V_A \cup V_B$ of the vertex set of G, as follows. For any $1 \leq i \leq n$, let $v_i \in V_A$ (resp. $v_i \in V_B$) if V_i is of type A (resp. type B). The remaining vertices, $\{v_i \mid V_i \text{ is of type } C\}$ are assigned either to V_A or to V_B so as to minimize $||V_A| - |V_B||$.

Claim 1. $n/5 \le |V_A|, |V_B| \le 4n/5$

To prove the claim, define another partition $V(H) = \overline{A} \cup \overline{B} \cup \overline{C}$ such that $\overline{A} \cap V_i = A \cap V_i$ and $\overline{B} \cap V_i = B \cap V_i$, for $i = 0$ and for every V_i of type C. If V_i is of type A (resp. type B), then let $V_i = \overline{A}_i \subset \overline{A}$ (resp. $V_i = \overline{B}_i \subset \overline{B}$), finally, let $\overline{C} = V(H) - \overline{A} - \overline{B}$.

For any V_i of type A, $w(\overline{A}_i) - w(A_i) \le w(A_i)/5$. Similarly, for any V_i of type B, $w(\overline{B}_i) - w(B_i) \le w(B_i)/5$. Therefore,

$$|w(\overline{A}) - w(A)| \le \frac{\max\{w(A), w(B)\}}{5} \le \frac{2n}{15}.$$

Hence, $n/5 \le w(\overline{A}) \le 4n/5$ and, analogously, $n/5 \le w(\overline{B}) \le 4n/5$. In particular, $|w(\overline{A}) - w(\overline{B})| \le 3n/5$. Using the minimality of $||V_A| - |V_B||$, we obtain that $||V_A| - |V_B|| \le 3n/5$, which implies Claim 1.

Claim 2. *For any* $1 \le i \le n$,

(i) *if V_i is of type A (resp. of type B), then $w(B_i)d_i \le |C_i|$ (resp. $w(A_i)d_i \le |C_i|$);*
(ii) *if V_i is of type C, then $d_i/6 \le |C_i|$.*

In V_i, every connected component belonging to A_i is separated from every connected component belonging to B_i by vertices in C_i. There are $w(A_i)d_i$ (resp. $w(B_i)d_i$) special boundary vertices in V_i, which belong to A_i (resp. B_i). It can be shown by an easy case analysis that the number of separating points $|C_i| \ge \min\{w(A_i), w(B_i)\}d_i$, and Claim 2 follows (see Fig. 2.).

In order to establish Theorem 6.2 (and hence Theorem 6), it remains to prove the following statement.

Claim 3. *The total number of edges between V_A and V_B satisfies*

$$|E(V_A, V_B)| \le 150(1 + g^{3/4}) \sqrt{\mathrm{cr}_g(G) + \sum_{i=1}^{n} d_i^2}.$$

To see this, denote by E_0 the set of all edges of H adjacent to at least one element of C_0. For any $1 \le i \le n$, define $E_i \subset E(H)$ as follows. If V_i is of type A (resp. type B),

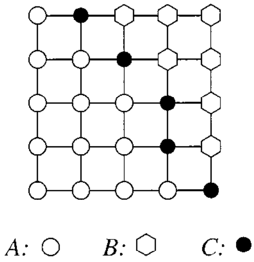

Fig. 2. The tripartition of V_i ($i \ge 1$).

let E_i consist of all edges leaving V_i and adjacent to a special boundary vertex belonging to B_i (resp. A_i). If V_i is of type C, let all edges leaving V_i belong to E_i.

For any $1 \le i \le n$, let E'_i denote the set of edges of G corresponding to the elements of E_i $(0 \le i \le n)$. Clearly, we have $|E'_i| \le |E_i|$, because distinct edges of G give rise to distinct edges of H. It is easy to see that every edge between V_A and V_B belongs to $\bigcup_{i=0}^{n} E'_i$.

Obviously, $|E'_0| \le |E_0| \le 4|C_0|$. By Claim 2, if V_i is of type A or of type B, then $|E'_i| \le |E_i| \le |C_i|$. If V_i is of type C, then $|E'_i| \le |E_i| = d_i \le 6|C_i|$. Therefore,

$$|E(V_A, V_B)| \le \left|\bigcup_{i=0}^{n} E'_i\right| \le \sum_{i=0}^{n} |E_i| \le 6|C| \le 150(1 + g^{3/4}) \sqrt{\mathrm{cr}_g(G) + \sum_{i=1}^{n} d_i^2}.$$

This concludes the proof of Claim 3 and hence Theorem 6.2 and Theorem 6. □

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