New Bounds on the Maximum Number of Edges in k-Quasi-Planar Graphs

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Abstract. A topological graph is *k-quasi-planar* if it does not contain k pairwise crossing edges. An old conjecture states that for every fixed k, the maximum number of edges in a k-quasi-planar graph on n vertices is $O(n)$. Fox and Pach showed that every k-quasi-planar graph with n vertices and no pair of edges intersecting in more than $O(1)$ points has at most $n(\log n)^{O(\log k)}$ edges. We improve this upper bound to $2^{\alpha(n)^c} n \log n$, where $\alpha(n)$ denotes the inverse Ackermann function, and c depends only on k. We also show that every k-quasi-planar graph with n vertices and every two edges have at most one point in common has at most $O(n \log n)$ edges. This improves the previously known upper bound of $2^{\alpha(n)^c} n \log n$ obtained by Fox, Pach, and Suk.

1 Introduction

A *topological graph* is a graph drawn in the plane so that its vertices are represented by points and its edges are represented by curves connecting the corresponding points. The curves are always *simple*, that is, they do not have self-intersections. The curves are allowed to intersect each other, but they cannot pass through vertices except for their endpoints. Furthermore, the edges are not allowed to have tangencies, that is, if two edges share an interior point, then they must properly cross at that point. We only consider graphs without parallel edges or loops. Two edges of a topological graph *cross* if their interiors share a point. A topological graph is *simple* if any two of its edges have at most one point in common, which can be either a common endpoint or a crossing.

It follows from Euler's polyhedral formula that every topological graph on n vertices and with no two crossing edges has at most $3n - 6$ edges. A graph

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is called *k-quasi-planar* if it can be drawn as a topological graph with no k pairwise crossing edges. Hence, a graph is 2-quasi-planar if and only if it is planar. According to an old conjecture (see Problem 1 in Section 9.6 of [4]), for any fixed $k \geq 2$ there exists a constant c_k such that every k-quasi-planar graph on n vertices has at most $c_k n$ edges. Agarwal, Aronov, Pach, Pollack, and Sharir [2] were the first to prove this conjecture for *simple* 3-quasi-planar graphs. Later, Pach, Radoičić, and Tóth [14] generalized the result to *all* 3-quasi-planar graphs. Ackerman [1] proved the conjecture for $k = 4$.

For larger values of k, several authors have proved upper bounds on the maximum number of edges in k-quasi-planar graphs under various conditions on how the edges are drawn. These include but are not limited to [5,7,8,15,19]. In 2008, Fox and Pach [7] showed that every k-quasi-planar graph with n vertices and no pair of edges intersecting in more than t points has at most $n(\log n)^{c \log k}$ edges, where c depends only on t. In this paper, we improve the exponent of the polylogarithmic factor from $O(\log k)$ to $1 + o(1)$ for fixed t.

Theorem 1. *Every k-quasi-planar graph with n vertices and no pair of edges intersecting in more than t points has at most $2^{\alpha(n)^c} n \log n$ edges, where $\alpha(n)$ denotes the inverse of the Ackermann function, and c depends only on k and t.*

Recall that the *Ackermann function $A(n)$* is defined as follows. Let $A_1(n) = 2n$, and $A_k(n) = A_{k-1}(A_k(n-1))$ for $k \geq 2$. In particular, we have $A_2(n) = 2^n$, and $A_3(n)$ is an exponential tower of n *two*'s. Now let $A(n) = A_n(n)$, and let $\alpha(n)$ be defined as $\alpha(n) = \min\{k \geq 1 : A(k) \geq n\}$. This function grows much slower than the inverse of any primitive recursive function.

For *simple* topological graphs, Fox, Pach, and Suk [8] showed that every k-quasi-planar simple topological graph on n vertices has at most $2^{\alpha(n)^c} n \log n$ edges, where c depends only on k. We establish the following improvement.

Theorem 2. *Every k-quasi-planar simple topological graph on n vertices has at most $c_k n \log n$ edges, where c_k depends only on k.*

We start the proofs of both theorems with a reduction to the case of topological graphs containing an edge that intersects every other edge. This reduction introduces the $O(\log n)$ factor for the bound on the number of edges. Then, the proof of Theorem 1 follows the approaches of Valtr [19] and Fox, Pach, and Suk [8], using a result on generalized Davenport-Schinzel sequences, which we recall in Section 3. Although the proofs in [19] and [8] heavily depend on the assumption that any two edges have at most one point in common, we are able to remove this condition by establishing some technical lemmas in Section 4. In Section 5, we finish the proof of Theorem 1. The proof of Theorem 2, which relies on a recent coloring result due to Lasoń, Micek, Pawlik, and Walczak [10], is given in Section 6.

2   Initial Reduction

We call a collection C of curves in the plane *decomposable* if there is a partition $C = C_1 \cup \cdots \cup C_w$ such that each C_i contains a curve intersecting all other curves

in C_i, and for $i \neq j$, no curve in C_i crosses nor shares an endpoint with a curve in C_j.

Lemma 1 (Fox, Pach, Suk [8]). *There is an absolute constant $c > 0$ such that every collection C of $m \geq 2$ curves such that any two of them intersect in at most t points has a decomposable subcollection of size at least $\frac{cm}{t \log m}$.*

In the proofs of both Theorem 1 and Theorem 2, we establish a (near) linear upper bound on the number of edges under the additional assumption that the graph has an edge intersecting every other edge. Once this is achieved, we use the following lemma to infer an upper bound for the general case.

Lemma 2 (implicit in [8]). *Let G be a topological graph on n vertices such that no two edges have more than t points in common. Suppose that for some constant β, every subgraph G' of G containing an edge that intersects every other edge of G' has at most $\beta |V(G')|$ edges. Then G has at most $c_t \beta n \log n$ edges, where c_t depends only on t.*

Proof. By Lemma 1, there is a decomposable subset $E' \subset E(G)$ such that $|E'| \geq c'_t |E(G)| / \log |E(G)|$, where c'_t depends only on t. Hence there is a partition $E' = E_1 \cup \cdots \cup E_w$, such that each E_i has an edge e_i that intersects every other edge in E_i, and for $i \neq j$, the edges in E_i are disjoint from the edges in E_j. Let V_i denote the set of vertices that are the endpoints of the edges in E_i, and let $n_i = |V_i|$. By the assumption, we have $|E_i| \leq \beta n_i$ for $1 \leq i \leq w$. Hence

$$\frac{c'_t |E(G)|}{\log |E(G)|} \leq |E'| \leq \sum_{i=1}^{w} \beta n_i \leq \beta n.$$

Since $|E(G)| \leq n^2$, we obtain $|E(G)| \leq 2(c'_t)^{-1} \beta n \log n$. □

3 Generalized Davenport-Schinzel Sequences

A sequence $S = (s_1, \ldots, s_m)$ is called *l-regular* if any l consecutive terms of S are pairwise different. For integers $l, m \geq 2$, the sequence $S = (s_1, \ldots, s_{lm})$ is said to be of *type* up(l, m) if the first l terms are pairwise different and $s_i = s_{i+l} = \cdots = s_{i+(m-1)l}$ for $1 \leq i \leq l$. In particular, every sequence of type up(l, m) is l-regular. For convenience, we will index the elements of an up(l, m) sequence as

$$S = (s_{1,1}, \ldots, s_{l,1}, \ s_{1,2}, \ldots, s_{l,2}, \ \ldots, \ s_{1,m}, \ldots, s_{l,m}),$$

where $s_{1,1}, \ldots, s_{l,1}$ are pairwise different and $s_{i,1} = \cdots = s_{i,m}$ for $1 \leq i \leq l$.

Theorem 3 (Klazar [9]). *For $l \geq 2$ and $m \geq 3$, every l-regular sequence over an n-element alphabet that does not contain a subsequence of type up(l, m) has length at most*

$$n \cdot l \cdot 2^{(lm-3)} \cdot (10l)^{10\alpha(n)^{lm}}.$$

For more results on generalized Davenport-Schinzel sequences, see [13,16,17].

4   Intersection Pattern of Curves

In this section, we will prove several technical lemmas on the intersection pattern of curves in the plane. We will always assume that no two curves are tangent, and that if two curves share an interior point, then they must properly cross at that point.

Lemma 3. *Let λ_1 and λ_2 be disjoint simple closed curves. Let C be a collection of m curves with one endpoint on λ_1, the other endpoint on λ_2, and no other common points with λ_1 or λ_2. If no k members of C pairwise cross, then C contains $\lceil m/(k-1)^2 \rceil$ pairwise disjoint members.*

Proof. Let G be the intersection graph of C. Since G does not contain a clique of size k, by Turán's theorem, $|E(G)| \leq (1-1/(k-1))m^2/2$. Hence there is a curve $a \in C$ and a subset $S \subset C$, such that $|S| \geq m/(k-1) - 1$ and a is disjoint from every curve in S. We order the elements in $S \cup \{a\}$ as $a_0, a_1, \ldots, a_{|S|}$ in clockwise order as their endpoints appear on λ_1, starting with $a_0 = a$. Now we define the partial order $\prec$ on the pairs in S so that $a_i \prec a_j$ if $i < j$ and a_i is disjoint from a_j. A simple geometric observation shows that $\prec$ is indeed a partial order. Since S does not contain k pairwise crossing members, by Dilworth's theorem [6], $S \cup \{a\}$ contains $\lceil m/(k-1)^2 \rceil$ pairwise disjoint members. □

A collection of curves with a common endpoint v is called a *fan* with *apex* v. Let $C = \{a_1, \ldots, a_m\}$ be a fan with apex v, and $\gamma = \gamma_1 \cup \cdots \cup \gamma_m$ be a curve with endpoints p and q partitioned into m subcurves $\gamma_1, \ldots, \gamma_m$ that appear in order along γ from p to q. We say that C is *grounded* by $\gamma_1 \cup \cdots \cup \gamma_m$, if

  (i) γ does not contain v,
  (ii) each a_i has its other endpoint on γ_i.

We say that C is *well-grounded* by $\gamma_1 \cup \cdots \cup \gamma_m$ if C is grounded by $\gamma_1 \cup \cdots \cup \gamma_m$ and each a_i intersects γ only within γ_i. Note that both notions depend on a particular partition $\gamma = \gamma_1 \cup \cdots \cup \gamma_m$.

Lemma 4. *Let $C = \{a_1, \ldots, a_m\}$ be a fan grounded by a curve $\gamma = \gamma_1 \cup \cdots \cup \gamma_m$. If each a_i intersects γ in at most t points, then there is a subfan $C' = \{a_{i_1}, \ldots, a_{i_r}\} \subset C$ with $i_1 < \cdots < i_r$ and $r = \lfloor \log_{t+1} m \rfloor$ that is grounded by a subcurve $\gamma' = \gamma'_1 \cup \cdots \cup \gamma'_r \subset \gamma$. Moreover,*

  (i) $\gamma'_j \supset \gamma_{i_j}$ *for $1 \leq j \leq r$,*
  (ii) a_{i_j} *intersects γ' only within $\gamma'_1 \cup \cdots \cup \gamma'_j$ for $1 \leq j \leq r$.*

Proof. We proceed by induction on m. The base case $m \leq t$ is trivial. Now assume that $m \geq t+1$ and the statement holds up to $m-1$. Since a_1 intersects γ in at most t points, there exists an integer j such that a_1 is disjoint from $\gamma_j \cup \gamma_{j+1} \cup \cdots \cup \gamma_{j+\lfloor m/(t+1) \rfloor - 1}$. By the induction hypothesis applied to $\{a_j, a_{j+1}, \ldots, a_{j+\lfloor m/(t+1) \rfloor - 1}\}$ and the curve $\gamma_j \cup \gamma_{j+1} \cup \cdots \cup \gamma_{j+\lfloor m/(t+1) \rfloor - 1}$, we obtain a subfan $C^* = \{a_{i_2}, \ldots, a_{i_r}\}$ of $r-1 = \lfloor \log_{t+1} \lfloor m/(t+1) \rfloor \rfloor = \lfloor \log_{t+1} m \rfloor - 1$ curves, and a subcurve $\gamma^* = \gamma'_2 \cup \cdots \cup \gamma'_r \subset \gamma_j \cup \gamma_{j+1} \cup \cdots \cup \gamma_{j+\lfloor m/(t+1) \rfloor - 1}$ with

the desired properties. Let γ_1' be the subcurve of γ obtained by extending the endpoint of γ_1 to the endpoint of γ^* along γ so that $\gamma_1' \supset \gamma_1$. Set $\gamma' = \gamma_1' \cup \gamma^*$. Hence the collection of curves $C' = \{a_1\} \cup C^*$ and γ' have the desired properties. $\qquad\square$

Lemma 5. *Let $C = \{a_1, \ldots, a_m\}$ be a fan grounded by a curve $\gamma = \gamma_1 \cup \cdots \cup \gamma_m$. If each a_i intersects γ in at most t points, then there is a subfan $C' = \{a_{i_1}, \ldots, a_{i_r}\} \subset C$ with $i_1 < \cdots < i_r$ and $r = \lfloor \log_{t+1} \log_{t+1} m \rfloor$ that is well-grounded by a subcurve $\gamma' = \gamma_1' \cup \cdots \cup \gamma_r' \subset \gamma$. Moreover, $\gamma_j' \supset \gamma_{i_j}$ for $1 \le j \le r$.*

Proof. We apply Lemma 4 to C and $\gamma = \gamma_1 \cup \cdots \cup \gamma_m$ to obtain a subcollection $C^* = \{a_{j_1}, a_{j_2}, \ldots, a_{j_{m^*}}\}$ of $m^* = \lfloor \log_{t+1} m \rfloor$ curves, and a subcurve $\gamma^* = \gamma_1^* \cup \cdots \cup \gamma_{m^*}^* \subset \gamma$ with the properties listed in Lemma 4. Then we apply Lemma 4 again to C^* and γ^* with the elements in C^* in reverse order. By the second property of Lemma 4, the resulting subcollection $C' = \{a_{i_1}, \ldots, a_{i_r}\}$ of $r = \lfloor \log_{t+1} \log_{t+1} m \rfloor$ curves is well-grounded by a subcurve $\gamma' = \gamma_1' \cup \cdots \cup \gamma_r' \subset \gamma$, and by the first property we have $\gamma_j' \supset \gamma_{i_j}$ for $1 \le j \le r$. $\qquad\square$

We say that fans $C_1, \ldots, C_l$ are *simultaneously grounded* (*simultaneously well-grounded*) by a curve $\gamma = \gamma_1 \cup \cdots \cup \gamma_m$ to emphasize that they are grounded (well-grounded) by γ with *the same partition* $\gamma = \gamma_1 \cup \cdots \cup \gamma_m$.

Lemma 6. *Let $C_1, \ldots, C_l$ be l fans with $C_i = \{a_{i,1}, \ldots, a_{i,m}\}$, simultaneously grounded by a curve $\gamma = \gamma_1 \cup \cdots \cup \gamma_m$. If each $a_{i,j}$ intersects γ in at most t points, then there are indices $j_1 < \cdots < j_r$ with $r = \lfloor \log_{t+1}^{(2l)} m \rfloor$ ($2l$-times iterated logarithm of m) and a subcurve $\gamma' = \gamma_1' \cup \cdots \cup \gamma_r' \subset \gamma$ such that*

(i) *the subfans $C_i' = \{a_{i,j_1}, \ldots, a_{i,j_r}\} \subset C_i$ for $1 \le i \le l$ are simultaneously well-grounded by $\gamma_1' \cup \cdots \cup \gamma_r'$,*

(ii) *$\gamma_s' \supset \gamma_{j_s}$ for $1 \le s \le r$.*

Proof. We proceed by induction on l. The base case $l = 1$ follows from Lemma 5. Now assume the statement holds up to $l - 1$. We apply Lemma 5 to the fan $C_1 = \{a_{1,1}, \ldots, a_{1,m}\}$ and the curve $\gamma = \gamma_1 \cup \cdots \cup \gamma_m$, to obtain a subfan $C_1^* = \{a_{1,w_1}, \ldots, a_{1,w_s}\} \subset C_1$ with $w_1 < \cdots < w_s$ and $s = \lfloor \log_{t+1} \log_{t+1} m \rfloor$ that is well-grounded by a subcurve $\gamma^* = \gamma_1^* \cup \cdots \cup \gamma_s^* \subset \gamma$ and satisfies $\gamma_i^* \supset \gamma_{w_i}$ for $1 \le i \le s$. For $2 \le i \le l$, let $C_i^* = \{a_{i,w_1}, \ldots, a_{i,w_s}\} \subset C_i$. Now we apply the induction hypothesis on the collection of $l - 1$ fans $C_2^*, \ldots, C_l^*$ that are simultaneously grounded by the curve $\gamma^* = \gamma_1^* \cup \cdots \cup \gamma_s^*$. Hence we obtain indices $j_1 < \cdots < j_r$ with $r = \lfloor \log_{t+1}^{(2l-2)} s \rfloor = \lfloor \log_{t+1}^{(2l)} m \rfloor$ and a subcurve $\gamma' = \gamma_1' \cup \cdots \cup \gamma_r' \subset \gamma^*$ such that each subfan $C_i' = \{a_{i,j_1}, \ldots, a_{i,j_r}\} \subset C_i$ with $2 \le i \le l$ is well-grounded by $\gamma_1' \cup \cdots \cup \gamma_r'$, and moreover $\gamma_z' \supset \gamma_z^* \supset \gamma_z$ for $1 \le z \le r$. By setting $C_1' = \{a_{1,j_1}, \ldots, a_{1,j_r}\} \subset C_1^*$, the collection $C_1', \ldots, C_l'$ is simultaneously well-grounded by the subcurve $\gamma' = \gamma_1' \cup \cdots \cup \gamma_r' \subset \gamma$. $\qquad\square$

Let $C = \{a_1, \ldots, a_m\}$ be a fan with apex v grounded by a curve $\gamma = \gamma_1 \cup \cdots \cup \gamma_m$ with endpoints p and q. We say that a_i is *left-sided* (*right-sided*) if moving along a_i from v until we reach γ for the first time, and then turning left (right)

onto the curve γ, we reach the endpoint q (p). We say that C_i is *one-sided*, if the curves in C_i are either all left-sided or all right-sided.

Lemma 7. *Let $C_1, \ldots, C_l$ be l fans with $C_i = \{a_{i,1}, \ldots, a_{i,m}\}$, simultaneously grounded by a curve γ. Then there are indices $j_1 < \cdots < j_r$ with $r = \lceil m/2^l \rceil$ such that the subfans $C_i' = \{a_{i,j_1}, \ldots, a_{i,j_r}\} \subset C_i$ for $1 \leq i \leq l$ are one-sided.*

Proof. We proceed by induction on l. The base case $l = 1$ is trivial since at least half of the curves in $C_1 = \{a_{1,1}, \ldots, a_{1,m}\}$ form a one-sided subset. For the inductive step, assume that the statement holds up to $l - 1$. Let $C_1^* = \{a_{1,w_1}, \ldots, a_{1,w_{\lceil m/2 \rceil}}\}$ with $w_1 < \cdots < w_{\lceil m/2 \rceil}$ be a subset of $\lceil m/2 \rceil$ curves that is one-sided. For $i \geq 2$, set $C_i^* = \{a_{i,w_1}, \ldots, a_{i,w_{\lceil m/2 \rceil}}\}$. Then apply the induction hypothesis on the $l - 1$ fans $C_2^*, \ldots, C_l^*$, to obtain indices $j_1 < \cdots < j_r$ with $r = \lceil \lceil m/2 \rceil / 2^{l-1} \rceil = \lceil m/2^l \rceil$ such that the subfans $C_i' = \{a_{i,j_1}, \ldots, a_{i,j_r}\} \subset C_i^*$ for $2 \leq i \leq l$ are one-sided. By setting $C_1' = \{a_{1,j_1}, \ldots, a_{1,j_r}\} \subset C_1^*$, the collection $C_1', \ldots, C_l'$ have the desired properties. $\qquad\square$

Since at least half of the fans obtained from Lemma 7 are either left-sided or right-sided, we have the following corollary.

Corollary 1. *Let $C_1, \ldots, C_{2l}$ be $2l$ fans with $C_i = \{a_{i,1}, \ldots, a_{i,m}\}$, simultaneously grounded by a curve γ. Then there are indices $i_1 < \cdots < i_l$ and $j_1 < \cdots < j_r$ with $r = \lceil m/2^{2l} \rceil$ such that the subfans $C_{i_w}' = \{a_{i_w,j_1}, \ldots, a_{i_w,j_r}\} \subset C_{i_w}$ for $1 \leq w \leq l$ are all left-sided or all right-sided.*

By combining Lemma 6 and Corollary 1, we easily obtain the following lemma which will be used in Section 5.

Lemma 8. *Let $C_1, \ldots, C_{2l}$ be $2l$ fans with $C_i = \{a_{i,1}, \ldots, a_{i,m}\}$, simultaneously grounded by a curve $\gamma = \gamma_1 \cup \cdots \cup \gamma_m$. Then there are indices $i_1 < \cdots < i_l$ and $j_1 < \cdots < j_r$ with $r = \lceil \lfloor \log_{t+1}^{(4l)} m \rfloor / 2^{2l} \rceil$ and a subcurve $\gamma^* = \gamma_1^* \cup \cdots \cup \gamma_r^* \subset \gamma$ such that*

(i) *the subfans $C_{i_w}' = \{a_{i_w,j_1}, \ldots, a_{i_w,j_r}\} \subset C_{i_w}$ for $1 \leq w \leq l$ are simultaneously well-grounded by $\gamma_1^* \cup \cdots \cup \gamma_r^*$,*

(ii) *$\gamma_s^* \supset \gamma_{j_s}$ for $1 \leq s \leq r$,*

(iii) *the subfans $C_{i_1}', \ldots, C_{i_l}'$ are all left-sided or all right-sided.*

5   Proof of Theorem 1

By Lemma 2 and the fact that the function $\alpha(n)$ is non-decreasing, it is enough to prove that every k-quasi-planar topological graph on n vertices such that no two edges have more than t points in common and there is an edge that intersects every other edge has at most $2^{\alpha(n)^c} n$ edges, where c depends only on k and t.

Let G be a k-quasi-planar graph on n vertices with no two edges intersecting in more than t points. Let $e_0 = pq$ be an edge that intersects every other edge of G. Let $V_0 = V(G) \setminus \{p, q\}$ and E_0 be the set of edges with both endpoints in

V_0. Hence we have $|E_0| > |E(G)| - 2n$. Assume without loss of generality that no two elements of E_0 cross e_0 at the same point.

By a well-known fact (see e.g. Theorem 2.2.1 in [3]), there is a bipartition $V_0 = V_1 \cup V_2$ such that at least half of the edges in E_0 connect a vertex in V_1 to a vertex in V_2. Let E_1 be the set of these edges. For each vertex $v_i \in V_1$, consider the graph G_i whose each vertex corresponds to the subcurve γ of an edge $e \in E_1$ such that

(i) e is incident to v_i,
(ii) the endpoints of $\gamma \subset e$ are v_i and the first intersection point in $e \cap e_0$ as moving from v_i along e.

Two vertices are adjacent in G_i if the corresponding subcurves cross. Each graph G_i is isomorphic to the intersection graph of a collection of curves with one endpoint on a simple closed curve λ_1 and the other endpoint on a simple closed curve λ_2 and with no other points in common with λ_1 or λ_2. To see this, enlarge the point v_i and the curve e_0 a little, making them simple closed curves λ_1 and λ_2, and shorten the curves γ appropriately, so as to preserve all crossings between them. Since no k of these curves pairwise intersect, by Lemma 3, G_i contains an independent set of size $\lceil |V(G_i)|/(k-1)^2 \rceil$. We keep all edges corresponding to the elements of this independent set, and discard all other edges incident to v_i. After repeating this process for all vertices in V_1, we are left with at least $\lceil |E_1|/(k-1)^2 \rceil$ edges, forming a set E_2. We continue this process on the vertices in V_2 and the edges in E_2. After repeating this process for all vertices in V_2, we are left with at least $\lceil |E_2|/(k-1)^2 \rceil$ edges, forming a set E'. Thus $|E(G)| < 2(k-1)^4|E'| + 2n$. Now, for any two edges $e_1, e_2 \in E'$ that share an endpoint, the subcurves $\gamma_1 \subset e_1$ and $\gamma_2 \subset e_2$ described above must be disjoint.

For each edge $e \in E'$, fix an arbitrary intersection point $s \in e \cap e_0$ to be the *main intersection point* of e and e_0. Let $e_1, \ldots, e_{|E'|}$ denote the edges in E' listed in the order their main intersection points appear on e_0 from p to q, and let $s_1, \ldots, s_{|E'|}$ denote these points respectively. We label the endpoints of each e_i as p_i and q_i, as follows. As we move along e_0 from p to q until we arrive at s_i, then we turn left and move along e_i, we finally reach p_i, while as we turn right at s_i and move along e_i, we finally reach q_i. We define sequences $S_1 = (p_1, \ldots, p_{|E'|})$ and $S_2 = (q_1, \ldots, q_{|E'|})$. They are sequences of length $|E'|$ over the $(n-2)$-element alphabet V_0.

Lemma 9 (Valtr [19]). *For $2l \geq 1$, at least one of the sequences S_1, S_2 defined above contains a $2l$-regular subsequence of length at least $\lceil |E'|/(8l) \rceil$.*

The proof of Lemma 9 can be copied almost verbatim from the proof of Lemma 5 in [19]. Indeed, the only fact about the sequences S_1 and S_2 it uses is that the edges $e_{j_1}, e_{j_1+1}, \ldots, e_{j_2}$ are spanned by the vertices $p_{j_1}, \ldots, p_{j_2}$ and $q_{j_1}, \ldots, q_{j_2}$, for any $j_1 < j_2$.

For the rest of this section, we set $l = 2^{k^2+2k}$ and m to be such that $(\log_{l+1}^{(4l)} m)/2^{2l} = 3 \cdot 2^k - 4$.

Lemma 10. *Neither of the sequences S_1 and S_2 has a subsequence of type* up$(2l, m)$.

Proof. By symmetry, it suffices to show that S_1 does not contain a subsequence of type up$(2l, m)$. We will prove that the existence of such a subsequence would imply that G has k pairwise crossing edges. Let

$$S = (s_{1,1}, \ldots, s_{2l,1},\ s_{1,2}, \ldots, s_{2l,2},\ \ldots,\ s_{1,m}, \ldots, s_{2l,m})$$

be a subsequence of S_1 of type up$(2l, m)$ such that the first $2l$ terms are pairwise distinct and $s_{i,1} = \cdots = s_{i,m} = v_i$ for $1 \leq i \leq 2l$. For $1 \leq j \leq m$, let $a_{i,j}$ be the subcurve of the edge corresponding to the entry $s_{i,j}$ in S_1 between the vertex v_i and the main intersection point with e_0. Let $C_i = \{a_{i,1}, \ldots, a_{i,m}\}$ for $1 \leq i \leq 2l$. Hence $C_1, \ldots, C_{2l}$ are $2l$ fans with apices $v_1, \ldots, v_{2l}$ respectively. Clearly, there is a partition $e_0 = \gamma_1 \cup \cdots \cup \gamma_m$ such that $C_1, \ldots, C_{2l}$ are simultaneously grounded by $\gamma_1 \cup \cdots \cup \gamma_m$.

We apply Lemma 8 to the fans $C_1, \ldots, C_{2l}$ that are simultaneously grounded by $\gamma_1 \cup \cdots \cup \gamma_m$ to obtain indices $i_1 < \cdots < i_l$ and $j_1 < \cdots < j_r$ with $r = (\log_{t+1}^{(4l)} m)/2^{2l} = 3 \cdot 2^k - 4$ and a subcurve $\gamma^* = \gamma_1^* \cup \cdots \cup \gamma_r^* \subset e_0$ such that

(i) the subfans $C'_{i_w} = \{a_{i_w,j_1}, \ldots, a_{i_w,j_r}\} \subset C_{i_w}$ for $1 \leq w \leq l$ are simultaneously well-grounded by $\gamma_1^* \cup \cdots \cup \gamma_r^*$,

(ii) $\gamma_z^* \supset \gamma_{j_z}$ for $1 \leq z \leq r$,

(iii) the subfans $C'_{i_1}, \ldots, C'_{i_l}$ are all left-sided or all right-sided.

We will only consider the case that $C'_{i_1}, \ldots, C'_{i_l}$ are left-sided, the other case being symmetric.

Now for $1 \leq w \leq l$ and $1 \leq z \leq r$, we define the subcurve $a_{w,z}^* \subset a_{i_w,j_z}$ whose endpoints are v_{i_w} and the first point from $a_{i_w,j_z} \cap \gamma^*$ as moving from v_{i_w} along a_{i_w,j_z}. Hence the interior of $a_{w,z}^*$ is disjoint from γ^*. Let $A_w^* = \{a_{w,1}^*, \ldots, a_{w,r}^*\}$ for $1 \leq w \leq l$. Note that any two curves in A_w^* do not cross by construction, and all curves in A_w^* enter γ^* from the same side. For simplicity, we will call this the *left side* of γ^* and we will relabel the apices of the fans $A_1^*, \ldots, A_l^*$ from $v_{i_1}, \ldots, v_{i_l}$ to $v_1, \ldots, v_l$. To finally reach a contradiction, we prove the following.

Claim 1. *For $l = 2^{k^2+2k}$ and $r = 3 \cdot 2^k - 4$, among the l fans $A_1^*, \ldots, A_l^*$ with the properties above, there are k pairwise crossing curves.*

The proof follows the argument of Lemma 4.3 in [8]. We proceed by induction on k. The base case $k = 1$ is trivial. For the inductive step, assume the statement holds up to $k - 1$. For simplicity, we let $a_{i,j}^* = a_{i,j'}^*$ for all $j \in \mathbb{Z}$, where $j' \in \{1, \ldots, r\}$ is such that $j \equiv j' \pmod{r}$. Consider the fan A_1^*, which is of size r. By construction of A_1^*, the arrangement $A_1^* \cup \{\gamma^*\}$ partitions the plane into r regions. By the pigeonhole principle, there is a subset $V' \subset \{v_1, \ldots, v_l\}$ of size

$$|V'| = \frac{l-1}{r} = \frac{2^{k^2+2k} - 1}{3 \cdot 2^k - 4},$$

such that all the vertices in V' lie in the same region. Let $j_0 \in \{1, \ldots, r\}$ be an integer such that V' lies in the region bounded by a_{1,j_0}^*, a_{1,j_0+1}^*, and γ^*.

Let $v_i \in V'$ and $1 < j_1 < r$, and consider the curve $a_{i,j_0+j_1}^*$. Recall that $a_{i,j_0+j_1}^*$ is disjoint from $\gamma_{j_0}^* \cup \gamma_{j_0+1}^*$ and thus intersects $a_{1,j_0}^* \cup a_{1,j_0+1}^*$. Let $a \subset a_{i,j_0+j_1}^*$

be the maximal subcurve with an endpoint on γ^* whose interior is disjoint from $a_{1,j_0}^* \cup a_{1,j_0+1}^*$. If a intersects a_{1,j_0+1}^* (i.e. the second endpoint of a lies on a_{1,j_0+1}^*), then v_i and the left side of $\gamma_{j_0+2}^* \cup \cdots \cup \gamma_{j_0+j_1-1}^*$ lie in different connected components of $\mathbb{R}^2 \setminus (a_{1,j_0+1}^* \cup \gamma^* \cup a)$. Likewise, if a intersects a_{1,j_0}^*, then v_i and the left-side of $\gamma_{j_0+j_1+1}^* \cup \cdots \cup \gamma_{j_0+r-1}^*$ lie in different connected components of $\mathbb{R}^2 \setminus (a_{1,j_0}^* \cup \gamma^* \cup a)$.

If a intersects a_{1,j_0+1}^*, then all curves $a_{i,j_0+2}^*, \ldots, a_{i,j_0+j_1-1}^*$ must also cross a_{1,j_0+1}^*. Indeed, they connect v_i with the left-side of $\gamma_{j_0+2}^* \cup \cdots \cup \gamma_{j_0+j_1-1}^*$, but their interiors are disjoint from γ^* and $a_{i,j_0+j_1}^*$. Likewise, if a intersects a_{i,j_0}^*, then all curves $a_{i,j_0+j_1+1}^*, \ldots, a_{i,j_0+r-1}^*$ must also cross a_{1,j_0}^*. Therefore, we have the following.

Claim 2. *For half of the vertices $v_i \in V'$, the curves emanating from v_i satisfy one of the following:*

(i) $a_{i,j_0+2}^*, a_{i,j_0+3}^*, \ldots, a_{i,j_0+r/2}^*$ *all cross* a_{1,j_0+1}^*,

(ii) $a_{i,j_0+r/2+1}^*, a_{i,j_0+r/2+2}^*, \ldots, a_{i,j_0+r-1}^*$ *all cross* a_{1,j_0}^*.

We keep all curves satisfying Claim 2, and discard all other curves. Since $r/2-2 = 3 \cdot 2^{k-1} - 4$ and

$$\frac{|V'|}{2} \geq \frac{l-1}{2r} = \frac{2^{k^2+2k}-1}{6 \cdot 2^k - 8} \geq 2^{(k-1)^2+2(k-1)},$$

by Claim 2, we can apply the induction hypothesis on these remaining curves which all cross a_{1,j_0+1}^* or a_{1,j_0}^*. Hence we have found k pairwise crossing edges, and this completes the proof of Claim 1 and thus Lemma 10.   □

We are now ready to prove Theorem 1.

Proof (Theorem 1). By Lemma 9 we know that, say, S_1 contains a $2l$-regular subsequence of length $\lceil |E'|/(8l) \rceil$. By Theorem 3 and Lemma 10, this subsequence has length at most

$$n \cdot 2l \cdot 2^{(2lm-3)} \cdot (20l)^{10\alpha(n)^{2lm}}.$$

Therefore, we have

$$\left\lceil \frac{|E'|}{8l} \right\rceil \leq n \cdot 2l \cdot 2^{(2lm-3)} \cdot (20l)^{10\alpha(n)^{2lm}},$$

which implies

$$|E'| \leq 8n \cdot 2l^2 \cdot 2^{(2lm-3)} \cdot (20l)^{10\alpha(n)^{2lm}}.$$

Since $l = 2^{k^2+2k}$ and m depends only on k and t, for sufficiently large c (depending only on k and t) and $\alpha(n) \geq 2$, we have

$$|E(G)| < 2(k-1)^4 |E'| + 2n \leq 2^{\alpha(n)^c} n,$$

which completes the proof of Theorem 1.   □

6 Proof of Theorem 2

A family of curves in the plane is *simple* if any two of them share at most one point. A family C of curves is K_k*-free* if the intersection graph of C is K_k-free, that is, no k curves in C pairwise intersect. By $\chi(C)$ we denote the chromatic number of the intersection graph of C, that is, the minimum number of colors that suffice to color the curves in C so that no two intersecting curves receive the same color.

Let ℓ be a horizontal line in the plane. Our proof of Theorem 2 is based on the following result, proved in [10] in a more general setting, for simple K_k-free families of compact arc-connected sets in the plane whose intersections with a line ℓ are non-empty segments.

Theorem 4 (Lasoń, Micek, Pawlik, Walczak [10]). *Every simple K_k-free family of curves C all intersecting ℓ at exactly one point satisfies $\chi(C) \le a_k$, where a_k depends only on k.*

Special cases of Theorem 4 have been proved by McGuinness [11] for $k = 3$ and by Suk [18] for y-monotone curves and any k. We will also use the following graph-theoretical result.

Lemma 11 (McGuinness [12]). *Let G be a graph, $\prec$ be a total ordering of $V(G)$, and $a, b \ge 0$. For $u, v \in V(G)$, let $G(u, v)$ denote the subgraph of G induced by the vertices strictly between u and v in $\prec$. If $\chi(G) > 2^{a+b+1}$, then there is an induced subgraph H of G such that $\chi(H) > 2^a$ and $\chi(G(u, v)) \ge 2^b$ for any $uv \in E(H)$.*

Let β be a segment in ℓ. We will consider curves crossing β at exactly one point, always assuming that this intersection point is distinct from the endpoints of β. Any such curve γ is partitioned by β into two subcurves: γ^+ that enters β from above and γ^- that enters β from below, both including the intersection point of β and γ.

Lemma 12. *Let C be a simple K_k-free family of curves all crossing β at exactly one point. If $\gamma_1^+ \cap \gamma_2^+ = \emptyset$ and $\gamma_1^- \cap \gamma_2^- = \emptyset$ for any $\gamma_1, \gamma_2 \in C$, then $\chi(C) \le 2^{3k-6}$.*

Proof. We proceed by induction on k. The base case $k = 2$ is trivial. For the induction step, assume $k \ge 3$ and the statement holds up to $k-1$. Assume for the sake of contradiction that $\chi(C) > 2^{3k-6}$. Let $\prec$ be the ordering of C according to the left-to-right order of the intersection points with β. Apply Lemma 11 with $a = 0$ and $b = 3k - 7$. It follows that there are two intersecting curves $\delta_1, \delta_2 \in C$ such that $\chi(C(\delta_1, \delta_2)) \ge 2^{3k-7}$, where $C(\delta_1, \delta_2) = \{\gamma \in C : \delta_1 \prec \gamma \prec \delta_2\}$. The curves β, δ_1 and δ_2 together partition the plane into two regions R^+ and R^- so that for $\gamma \in C(\delta_1, \delta_2)$, γ^+ enters β from the side of R^+, while γ^- enters β from the side of R^-. Take any $\gamma_1, \gamma_2 \in C(\delta_1, \delta_2)$ that intersect at a point p. It follows from the assumptions of the lemma that $p \in \gamma_1^+ \cap \gamma_2^-$ or $p \in \gamma_1^- \cap \gamma_2^+$. If $p \in R^+$, then one of γ_1^-, γ_2^- (whichever contains p) must intersect δ_1 or δ_2. Similarly,

if $p \in R^-$, then one of γ_1^+, γ_2^+ must intersect δ_1 or δ_2. In both cases, one of γ_1, γ_2 intersects δ_1 or δ_2. Let C_1 and C_2 consist of those members of $C(\delta_1, \delta_2)$ that intersect δ_1 and δ_2, respectively. Clearly, both C_1 and C_2 are K_{k-1}-free, and thus the induction hypothesis yields $\chi(C_1) \leq 2^{3k-9}$ and $\chi(C_2) \leq 2^{3k-9}$. Moreover, $\chi(C(\delta_1, \delta_2) \setminus (C_1 \cup C_2)) \leq 1$ as $C(\delta_1, \delta_2) \setminus (C_1 \cup C_2)$ is independent by the assumption $\gamma_1^+ \cap \gamma_2^+ = \emptyset$ and $\gamma_1^- \cap \gamma_2^- = \emptyset$ for any $\gamma_1, \gamma_2 \in C$. To conclude, $\chi(C(\delta_1, \delta_2)) \leq 2 \cdot 2^{3k-9} + 1 < 2^{3k-7}$, which is a contradiction. $\qquad\square$

Now we prove the following theorem, which can also be generalized to simple K_k-free families of compact arc-connected sets in the plane whose intersections with a segment β are non-empty subsegments.

Theorem 5. *Every simple K_k-free family of curves C all crossing β at exactly one point satisfies $\chi(C) \leq b_k$, where b_k depends only on k.*

Proof. Assume without loss of generality that no curve in C passes through the endpoints of β. One can transform the family $C^+ = \{\gamma^+ : \gamma \in C\}$ into a family $\tilde{C}^+ = \{\tilde{\gamma}^+ : \gamma \in C\}$ so that

- $\tilde{C}^+$ is simple,
- each $\tilde{\gamma}^+$ is entirely contained in the upper half-plane delimited by ℓ,
- $\tilde{\gamma}_1^+$ and $\tilde{\gamma}_2^+$ intersect if and only if γ_1^+ and γ_2^+ intersect.

Similarly, one can transform the family $C^- = \{\gamma^- : \gamma \in C\}$ into a family $\tilde{C}^- = \{\tilde{\gamma}^- : \gamma \in C\}$ so that

- $\tilde{C}^-$ is simple,
- each $\tilde{\gamma}^-$ is entirely contained in the lower half-plane delimited by ℓ,
- $\tilde{\gamma}_1^-$ and $\tilde{\gamma}_2^-$ intersect if and only if γ_1^- and γ_2^- intersect.

The curves $\tilde{\gamma}^+$ and $\tilde{\gamma}^-$ are respectively the upper and lower parts of the curve $\tilde{\gamma} = \tilde{\gamma}^+ \cup \tilde{\gamma}^-$ intersecting ℓ at exactly one point. The family $\tilde{C} = \{\tilde{\gamma} : \gamma \in C\}$ is clearly simple and K_k-free. Therefore, by Theorem 4, $\chi(\tilde{C}) \leq a_k$. Fix a proper a_k-coloring ϕ of $\tilde{C}$ and consider the set C_i consisting of those $\gamma \in C$ for which $\phi(\tilde{\gamma}) = i$. It follows that $\gamma_1^+ \cap \gamma_2^+ = \emptyset$ and $\gamma_1^- \cap \gamma_2^- = \emptyset$ for any $\gamma_1, \gamma_2 \in C_i$. Therefore, by Lemma 12, $\chi(C_i) \leq 2^{3k-6}$. Summing up over all colors used by ϕ we obtain $\chi(C) \leq 2^{3k-6} a_k$. $\qquad\square$

Proof (Theorem 2). By Lemma 2, it is enough to prove that every k-quasi-planar simple topological graph on n vertices that contains an edge intersecting every other edge has at most $c_k n$ edges, where c_k depends only on k.

Let G be a k-quasi-planar simple topological graph on n vertices, and let pq be an edge that intersects every other edge. Remove all edges with an endpoint at p or q except the edge pq. Shorten each curve representing a remaining edge by a tiny bit at both endpoints, so that curves sharing an endpoint become disjoint, while all crossings are preserved. The resulting set of curves C is simple and K_k-free and contains a curve γ crossing every other curve in C. Therefore, $C \setminus \{\gamma\}$ is K_{k-1}-free and $|C \setminus \{\gamma\}| > |E(G)| - 2n$. Since C can be transformed into an equivalent set of curves so that γ becomes the horizontal segment β, Theorem 5

yields $\chi(C \setminus \{\gamma\}) \leq b_{k-1}$. Consequently, $C \setminus \{\gamma\}$ contains an independent set S of size

$$|S| \geq \frac{|C \setminus \{\gamma\}|}{b_{k-1}} > \frac{|E(G)| - 2n}{b_{k-1}}.$$

The edges of G corresponding to the curves in S form a planar subgraph of G, which implies $|S| < 3n$. The two inequalities give $|E(G)| < (3b_{k-1} + 2)n$.   □

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