JOURNAL OF COMBINATORIAL THEORY, Series A 26, 210–214 (1979)

Note

New Bounds on the Number of Unit Spheres That Can Touch a Unit Sphere in n Dimensions

A. M. ODLYZKO AND N. J. A. SLOANE

Bell Laboratories, Murray Hill, New Jersey 07974

Communicated by the Managing Editors

Received November 3, 1978

New upper bounds are given for the maximum number, τ_n, of nonoverlapping unit spheres that can touch a unit sphere in n-dimensional Euclidean space, for $n \leqslant 24$. In particular it is shown that $\tau_8 = 240$ and $\tau_{24} = 196560$.

The problem of finding the maximum number, τ_3, of billiard balls that can touch another billiard ball has a long and fascinating history (see [2]); the answer is known to be 12. But up to now no corresponding numbers τ_n have been determined for higher dimensions.

We shall use the following theorem.

THEOREM. *Assume $n \geqslant 3$. If $f(t)$ is a real polynomial which satisfies*

(C1) *$f(t) \leqslant 0$ for $-1 \leqslant t \leqslant \frac{1}{2}$, and*

(C2) *the coefficients in the expansion of $f(t)$ in terms of Jacobi polynomials* [1, *Chap.* 22]

$$f(t) = \sum_{i=0}^{k} f_i P_i^{\alpha,\alpha}(t),$$

where $\alpha = (n-3)/2$, satisfy $f_0 > 0, f_1 \geqslant 0, \ldots, f_k \geqslant 0$, then τ_n is bounded by

$$\tau_n \leqslant \frac{f(1)}{f_0}.$$

This theorem may be found (implicitly or explicitly) in [3, 4, 6], but for completeness we sketch a simplified proof. A *spherical code C* is any finite

0097-3165/79/020210-05$02.00/0

subset of the unit sphere in n dimensions. For $-1 \leqslant t \leqslant 1$ let

$$A_t = \delta_t \cdot (1/|C|) \cdot (\text{number of ordered pairs } c, c' \in C \text{ such that } \langle c, c'\rangle = t),$$

where δ_t is a Dirac delta-function, $|C|$ is the cardinality of C, and $\langle\ ,\ \rangle$ is the usual inner product. Then $\int_{-1}^{1} A_t\,dt = |C|$. For all $k \geqslant 0$ we have

$$\int_{-1}^{1} A_t P_k^{\alpha,\alpha}(t)\,dt = \frac{1}{|C|} \sum_{c,c' \in C} P_k^{\alpha,\alpha}(\langle c, c'\rangle) \geqslant 0,$$

since the kernel $P_k^{\alpha,\alpha}(\langle x, y\rangle)$ is positive definite.

If there is an arrangement of τ unit spheres $S_1, \ldots, S_\tau$ touching another unit sphere S_0, the points of contact of S_0 with $S_1, \ldots, S_\tau$ form a spherical code C with $A_t = 0$ for $\frac{1}{2} < t < 1$. It follows that an upper bound to τ_n is given by the optimal solution to the following linear programming problem: choose the $A_t(-1 \leqslant t \leqslant \frac{1}{2})$ so as to maximize $\int_{-1}^{1/2} A_t\,dt$ subject to the constraints

$$A_t \geqslant 0 \qquad \text{for} \quad -1 \leqslant t \leqslant 1/2,$$

and

$$\int_{-1}^{1/2} A_t P_k^{\alpha,\alpha}(t)\,dt \geqslant -P_k^{\alpha,\alpha}(1), \qquad \text{for} \quad k = 0, 1, \ldots .$$

The theorem now follows by passing to the dual problem, and using the fact that any feasible solution to the dual problem is an upper bound to the optimal solution of the original problem.

For $n = 8$ we apply the theorem with

$$\begin{aligned} f(t) &= \frac{320}{3}(t+1)\left(t+\frac{1}{2}\right)^2 t^2 \left(t-\frac{1}{2}\right) \\ &= P_0 + \frac{16}{7}P_1 + \frac{200}{63}P_2 + \frac{832}{231}P_3 \\ &\quad + \frac{1216}{429}P_4 + \frac{5120}{3003}P_5 + \frac{2560}{4641}P_6, \end{aligned} \tag{1}$$

where P_i stands for $P_i^{2.5,2.5}(t)$, and obtain $\tau_8 \leqslant 240$. Similarly for $n = 24$ we take

$$f(t) = \frac{1490944}{15}(t+1)\left(t+\frac{1}{2}\right)^2\left(t+\frac{1}{4}\right)^2 t^2\left(t-\frac{1}{4}\right)^2\left(t-\frac{1}{2}\right)$$

$$= P_0 + \frac{48}{23}P_1 + \frac{1144}{425}P_2 + \frac{12992}{3825}P_3 + \frac{73888}{22185}P_4$$

$$+ \frac{2169856}{687735}P_5 + \frac{59062016}{25365285}P_6 + \frac{4472832}{2753575}P_7$$

$$+ \frac{23855104}{28956015}P_8 + \frac{7340032}{20376455}P_9 + \frac{7340032}{80848515}P_{10}, \quad (2)$$

where P_i stands for $P_i^{10.5,10.5}(t)$, and obtain $\tau_{24} \leqslant 196560$. Since each sphere in the E_8 lattice packing in 8 dimensions touches 240 others, and each sphere in the Leech lattice packing in 24 dimensions touches 196560 others [5], we have determined τ_8 and τ_{24}.

For other values of n below 24 we were unable to find such simple and effective polynomials. The best polynomial we have found for $n = 4$, for example, is $f(t) = P_0 + a_1P_1 + a_2P_2 + \cdots + a_9P_9$, where $a_1 = 2.412237$, $a_2 = 3.261973$, $a_3 = 3.217960$, $a_4 = 2.040011$, $a_5 = 0.853848$, $a_6 = a_7 = a_8 = 0$, $a_9 = 0.128520$ (shown to 6 decimal places, although we actually used 17 places), and P_i stands for $P_i^{0.5,0.5}(t)$. This implies $\tau_4 \leqslant 25.5585$. This polynomial was found by the following method. First replace (C1) by a finite set of inequalities at the points $t_j = -1 + 0.0015j$ $(0 \leqslant j \leqslant 1000)$. Second, choose a value of k, and use linear programming to find $f_1^*,..., f_k^*$ so as to minimize

$$\sum_{i=1}^{k} f_i^* P_i^{\alpha,\alpha}(1)$$

subject to the constraints

$$f_i^* \geqslant 0 \quad (1 \leqslant i \leqslant k), \qquad \sum_{i=1}^{k} f_i^* P_i^{\alpha,\alpha}(t_j) \leqslant -1 \quad (0 \leqslant j \leqslant 1000).$$

Let $f^*(t)$ denote the polynomial $1 + \sum_{i=1}^{k} f_i^* P_i^{\alpha,\alpha}(t)$. Of course this need not satisfy (C1) for *all* points t on the interval $[-1, \frac{1}{2}]$. Let ϵ be chosen to be greater than the maximum value of $f^*(t)$ on $[-1, \frac{1}{2}]$ (ϵ may be calculated by finding the zeros of the derivative of $f^*(t)$). Then $f(t) = f^*(t) - \epsilon$ satisfies (C1) and (C2), and so

$$\tau_n \leqslant \frac{f^*(1) - \epsilon}{1 - \epsilon}.$$

All the upper bounds shown in Table I, except for $n = 17$, were obtained in this way. The degree k was allowed to be as large as 30, but in all the cases considered the degree of the best polynomial (given in the third column of the table) did not exceed 14. For $n = 8$ and $n = 24$ the form of the polynomials obtained in this way led us to (1) and (2), but for the other values of n no such simple expression suggested itself.

For $n = 17$ we made use of the additional inequalities

$$\int_{-1}^{-3^{1/2}/2} A_t\, dt \leqslant 1 \qquad \text{and} \qquad \int_{-1}^{-(2/3)^{1/2}} A_t\, dt \leqslant 2$$

TABLE I

Range of Possible Values of τ_n, the Maximum Number of Unit Spheres That Can Touch a Unit Sphere in n Dimensions

n	τ_n	deg[a]
1	2	
2	6	
3	12	
4	24–25	9
5	40–46	10
6	72–82	10
7	126–140	10
8	240	6
9	306–380	11
10	500–595	11
11	582–915	11
12	840–1416	11
13	1130–2233	12
14	1582–3492	12
15	2564–5431	12
16	4320–8313	13
17	5346–12215	13
18	7398–17877	13
19	10668–25901	13
20	17400–37974	13
21	27720–56852	13
22	49896–86537	14
23	93150–128096	14
24	196560	10

[a] The degree of the polynomial used to obtain the upper bound.

to obtain $\tau_{17} \leqslant 12215$. Other inequalities of this type could probably be used to obtain further improvements of these results. Unfortunately for $n = 3$ our methods only give $\tau_3 \leqslant 13$.

These upper bounds are a considerable improvement over the old bounds [2, 5, 7]. For example, the bounds given in [5] (which are based on a still unproved conjecture of Coxeter [2]) are 26, 48, 85, 146, and 244 for $n = 4$, 5, 6, 7, and 8, respectively. The lower bounds in the table are taken from [5, 8, 9].

References

1. M. Abramowitz and I. A. Stegun, "Handbook of Mathematical Functions," National Bureau of Standards Applied Math. Series 55, U.S. Dept. Commerce, Washington, D.C., 1972.
2. H. S. M. Coxeter, An upper bound for the number of equal nonoverlapping spheres that can touch another of the same size, *in* "Proc. Symp. Pure Math.," Vol. VII, pp. 53–71, American Mathematical Society, Providence, R.I., 1963. Reprinted (with corrections) as Chapter 9, pp. 179–198 of "Twelve Geometric Essays," by H. S. M. Coxter, Southern Illinois University Press, Carbondale, Illinois, 1968.
3. P. Delsarte, J. M. Goethals, and J. J. Seidel, Spherical codes and designs, *Geometriae Dedicata* **6** (1977), 363–388.
4. G. A. Kabatiansky and V. I. Levenshtein, Bounds for packings on a sphere and in space, *Problemy Peredači Informacii* **14**, No. 1 (1978), 3–25; English translation: *Problems of Information Transmission* **14**, No. 1 (1978), 1–17.
5. J. Leech and N. J. A. Sloane, Sphere packings and error-correcting codes, *Canad. J. Math.* **23** (1971), 718–745.
6. S. P. Lloyd, Hamming association schemes and codes on spheres, preprint.
7. R. A. Rankin, The closest packing of spherical caps in *n* dimensions, *Proc. Glasgow Math. Assoc.* **2** (1955), 139–144.
8. N. J. A. Sloane, Binary codes, lattices, and sphere-packings, *in* "Combinatorial Surveys" (P. J. Cameron, Ed.), pp. 117–164, Academic Press, London/New York, 1977.
9. N. J. A. Sloane, Self-dual codes and lattices, *in* "Proc. Symp. Pure Math. on Relations Between Combinatorics and Other Parts of Mathematics," American Mathematical Society, Providence, R.I., Vol. XXXIV, pp. 273–308, 1979.

Printed in Belgium