NEW CHARACTERIZATIONS OF THE HELICOID IN A CYLINDER

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ABSTRACT. This paper characterizes a compact piece of the helicoid H_C in a solid cylinder $C \subset \mathbb{R}^3$ from the following two perspectives. First, under reasonable conditions, H_C has the smallest area among all immersed surfaces Σ with $\partial \Sigma \subset d_1 \cup d_2 \cup S$, where d_1 and d_2 are the diameters of the top and bottom disks of C and S is the side surface of C. Second, other than H_C , there exists no minimal surface whose boundary consists of d_1, d_2 , and a pair of rotationally symmetric curves γ_1, γ_2 on S along which it meets Sorthogonally. We draw the same conclusion when the boundary curves on S are a pair of helices of a certain height.

INTRODUCTION

The helicoid in \mathbb{R}^3 is a classic example of a minimal surface. It is a simply connected, complete, embedded, ruled minimal surface foliated by helices and having infinite total curvature. It is symmetric with respect to its central axis and any horizontal line it contains. Catalan [3] verified that, other than the plane, the helicoid is the unique ruled minimal surface. Collin and Krust [7] showed that a nonplanar minimal surface bounded by two lines, whose interior is a graph over a band, must be part of the helicoid. Colding and Minicozzi [6] proved that every embedded minimal disk in \mathbb{R}^3 is either a graph of a function or part of an appropriately scaled helicoid. Based on their work, Meeks and Rosenberg [15] showed that a complete, embedded, simply connected nonplanar minimal surface must be the helicoid. Recently, Choe and Hoppe [5] constructed a higher-dimensional helicoid, whose further generalization is demonstrated in [13].

Bernstein and Breiner [1] proved that part of the catenoid has the smallest area among the embedded minimal annuli in a slab spanning two parallel planes in \mathbb{R}^3 . Inspired by their work, Choe conjectured that part of the helicoid minimizes the area among the surfaces spanning two skew diameters of the top and bottom disks of a solid cylinder C, with other boundaries lying on the side of C. His question was natural, because the catenoid is locally isometric to the helicoid and, therefore, if a compact piece of the catenoid has an area-minimizing property in a certain setting, then a compact piece of the helicoid is highly likely to inherit a similar property correspondingly.

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Because of the existence of trivial counterexamples, however, we had to impose additional (yet reasonable) conditions to obtain meaningful results. We were able to prove two classical results on the least area property of a piece of helicoid and three theorems on the uniqueness of the helicoid in a certain setting. We should point out that our results concern some properties of a compact portion of the helicoid, whereas previous characterization results were mostly about the complete helicoid. Additionally, our proofs use classical methods, and are very geometrical.

The main results of this paper are twofold: concerning the least area property and the uniqueness of a piece of the helicoid centered in a cylinder in \mathbb{R}^3 . First, we investigate the conditions under which a piece of the helicoid H_C in a right circular compact cylinder C has the smallest area among the surfaces spanning two skew diameters d_1 and d_2 of the top and bottom disks and bounded by the curves on the side surface S of C. Second, we prove that H_C is the unique minimal surface spanning d_1 and d_2 and meeting S along the rotationally symmetric curves. Furthermore, we determine other conditions that guarantee the uniqueness of H_C .

Note that, in this paper, we consider every surface to be two-dimensional, orientable, and immersed in the three-dimensional Euclidean space \mathbb{R}^3 .

1. Helicoid as the least area surface



FIGURE 1. Candidates for the Area-minimizer

Let $C \subset \mathbb{R}^3$ be a right circular solid cylinder bounded by two disks D_1 and D_2 and a cylindrical surface S. Let H be the helicoid ruled by lines perpendicular to the axis of C. We define the compact helicoid as $H_C = H \cap C$, whose boundary consists of two diameters $d_1 \in D_1$ and $d_2 \in D_2$ and two symmetrical helices $h_1, h_2 \in S$. In fact, there are infinitely many such helicoids, but H_C is defined as the one whose boundary helices spiraling with an angle less than or equal to $\pi/2$. (See Figure 1(a)). Then, as a minimal surface, H_C is a strong candidate for the least area surface among the immersed surfaces passing through d_1 and d_2 who have their boundaries on S, if d_1 and d_2 are not parallel to each other. It is clear that the plane passing through d_1 and d_2 is the area-minimizer when d_1 and d_2 are parallel. Assume, therefore, that d_1 and d_2 are not parallel.

Depending on the height h and radius r of C, however, the surface composed of two half disks in D_1 and D_2 connected by a band of length $\epsilon > 0$ in S (Figure 1(b)) may have a smaller area than H_C if ϵ is sufficiently small. In fact, when h = r = 1 and the angle between d_1 and the vertical projection of d_2 onto D_2 is $\frac{\pi}{2}$, the area of the surface in Figure 1(b) is $\pi + \epsilon$, whereas the area of H_C is $\pi(\sqrt{2} + \log(1 + \sqrt{2})) > \pi + \epsilon$ for arbitrary small $\epsilon > 0$. Therefore, it is natural and meaningful to seek the conditions under which H_C has the smallest area. The next theorem provides a geometric condition on the candidate surfaces to rule out the trivial counterexample. By a disk-type surface we mean a surface topologically equivalent to a disk.

THEOREM 1.1. Let $\Sigma \subset C$ be an immersed disk-type surface with $\partial \Sigma \subset d_1 \cup d_2 \cup S$. Let $C_t = \{(x, y, z) \mid x^2 + y^2 \leq t^2\}$. If Σ contains the axis of C and Σ is transversal to ∂C_t for every $0 < t \leq r$, then $\mathcal{H}^2(\Sigma) \geq \mathcal{H}^2(\mathcal{H}_C)$. Equality holds if and only if Σ is congruent to \mathcal{H}_C .

To prove the theorem, we need the following lemma, which relates an integral of a function over a surface to the integral over the level sets.

LEMMA 1.2 (Co-area formula). If Σ is a Riemannian manifold and $g: \Sigma \to \mathbb{R}$ is a proper (i.e. $g^{-1}((-\infty, t])$ is compact for all $t \in \mathbb{R}$) Lipschitz function on Σ , then, for any locally integrable function f on Σ and $t \in \mathbb{R}$,

$$\int_{\{h \le t\}} f \left| \nabla_{\Sigma} g \right| dV = \int_{-\infty}^{t} \left(\int_{g=\tau} f \, dA_{\tau} \right) d\tau,$$

where ∇_{Σ} is the surface gradient on Σ .

PROOF. Take $N = \mathbb{R}$ in the co-area formula presented in [4].

Proof of Theorem 1.1. Without loss of generality, we may take $C = \{(x, y, z) \mid x^2 + y^2 \leq r^2, 0 \leq z \leq h\}$ and H_C is centered at z-axis. And let l be the axis of C. Then, for $d(x, y, z) = \sqrt{x^2 + y^2}$, the transversality condition implies that $|\nabla_{\Sigma} d| \neq 0$ on $\Sigma \setminus l$. Therefore, $f := \frac{1}{|\nabla_{\Sigma} d|}$ is locally integrable on $\Sigma \setminus l$. Let $\hat{\Sigma} = \Sigma \setminus C_{\epsilon}$ for an arbitrarily small $\epsilon > 0$. For $A_{\hat{\Sigma}}(t) = \mathcal{H}^2(\hat{\Sigma} \cap C_t)$ apply co-area formula to have

$$A_{\hat{\Sigma}}(t) = \int_{\hat{\Sigma}\cap C_t} dS = \int_{\epsilon}^t \int_{\hat{\Sigma}\cap\partial C_{\tau}} \frac{1}{|\nabla_{\hat{\Sigma}}d|} \, ds \, d\tau.$$
$$\therefore \frac{d}{dt} A_{\hat{\Sigma}}(t) = \int_{\hat{\Sigma}\cap\partial C_t} \frac{1}{|\nabla_{\hat{\Sigma}}d|} \, ds \ge \mathcal{H}^1(\hat{\Sigma}\cap\partial C_t),$$

since $|\nabla_{\hat{\Sigma}} d| \leq 1$.

$$\therefore \frac{d}{dt} A_{\hat{\Sigma}}(t) \ge \mathcal{H}^1(\hat{\Sigma} \cap \partial C_t).$$

Claim. If Σ contains the axis l of C, then $\mathcal{H}^1(\Sigma \cap \partial C_t) \geq \mathcal{H}^1(H_C \cap \partial C_t)$ for each $t \in (0, r]$.

To verify the Claim, let $P_{\tau} = \{(x, y, z) \mid z = \tau\} \{ 0 \leq \tau \leq h \}$ be the plane parallel to D_1 and D_2 . Given that Σ is of disk-type and contains the axis of C, $\Sigma \cap P_{\tau}$ should contain at least one curve joining two points on the circle $\{(x, y, z) \mid x^2 + y^2 = r^2, z = \tau\}$ and passing through the center. In other words, if we slice Σ by the planes parallel to D_i (i=1,2), then at each height at least one of the intersection curves should join two points on the boundary circle and pass through the center. Because this is true for any $\tau \in [0, h]$, if we slice C into the cylinders C_t with radius $t \in (0, r]$, then $\Sigma \cap \partial C_t$ must have curves $\gamma_{p,t}$ joining p_t^1 , p_t^2 and $\gamma_{q,t}$ joining q_t^1 , q_t^2 , where $d_i \cap C_t = \{p_t^i, q_t^i\}$, i = 1, 2. It is therefore clear that $\mathcal{H}^1(\Sigma \cap \partial C_t) \geq \mathcal{H}^1(\gamma_{p,t}) + \mathcal{H}^1(\gamma_{q,t}) \geq \mathcal{H}^1(h_{p,t}) + \mathcal{H}^1(h_{q,t})$ where $h_{p,t}$ and $h_{q,t}$ are helices joining p_t^1 , p_t^2 and q_t^1 , q_t^2 , respectively, because the helix gives the shortest path between two points on the cylindrical surface (other than the vertical line). This proves the claim.

Thus, we have

$$\frac{d}{dt}A_{\hat{\Sigma}}(t) \ge \mathcal{H}^1(\hat{H}_C \cap \partial C_t) = \int_{\hat{H}_C \cap \partial C_t} ds = \frac{d}{dt}A_{\hat{H}_C}(t).$$

where $\hat{H}_C = H_C \setminus C_{\epsilon}$.

Integrating both sides over $[\epsilon, r]$ yields $A_{\hat{\Sigma}}(r) - A_{\hat{\Sigma}}(\epsilon) \geq A_{\hat{H}_C}(r) - A_{\hat{H}_C}(\epsilon)$. Therefore, letting $\epsilon \to 0$ on both sides, we have $A_{\Sigma}(r) \geq A_{H_C}(r)$, in other words, $\mathcal{H}^2(\Sigma) \geq \mathcal{H}^2(H_C)$. Note that the equality holds if and only if $|\nabla_{\Sigma}d| = 1$ and $\mathcal{H}^1(\Sigma \cap \partial C_t) = \mathcal{H}^1(H_C \cap \partial C_t)$, which is only possible when $\Sigma \cap \partial C_t$ are helices, i.e. $\Sigma = H_C$.

Note that if our competitor surfaces are minimal, transversality condition can be dropped. However, the condition that Σ contains the central axis of C cannot be weakened. Without that condition, $\mathcal{H}^1(\Sigma \cap \partial C_t) \geq \mathcal{H}^1(H_C \cap \partial C_t)$ does not hold for each $t \in (0, r]$ in general; Figure 1(b) can be a counterexample. But if we have a certain condition on the boundary curves of Σ instead, we get the same area-minimization result:

THEOREM 1.3. Let $\Sigma \subset C$ be an immersed disk-type surface bounded by a Jordan curve $\Gamma = d_1 \cup d_2 \cup \gamma_1 \cup \gamma_2$ where γ_1 and γ_2 are rotationally symmetric C^2 curves on S. If the total curvature of γ_1 is less than π , then $\mathcal{H}^2(\Sigma) \geq \mathcal{H}^2(\mathcal{H}_C)$. Equality holds if and only if Σ is congruent to \mathcal{H}_C .

PROOF. Let Σ_0 be the Douglas-Radó solution for Γ . Then, $\int_{\Gamma} \kappa < 4\pi$ implies that Σ_0 is a unique disk-type minimal surface Σ_0 by the generalization of Nitsche's uniqueness theorem to curved polygons [20]. Moreover, Σ_0 is embedded by [8].

We claim that Σ_0 is rotationally symmetric *i.e.* $\rho_{\pi}(\Sigma_0) = \Sigma_0$, where $\rho(\cdot)$ is the rotation about the central axis l of C by π . To show this, suppose $\rho_{\pi}(\Sigma_0) \neq \Sigma_0$. Then, $\rho_{\pi}(\Sigma_0)$ and Σ_0 are two different minimal disks that share the same boundary because $\rho_{\pi}(\Gamma) = \Gamma$. This contradicts the fact that Γ bounds only one minimal disk. Therefore, $\rho_{\pi}(\Sigma_0) = \Sigma_0$. Moreover, the embeddedness and the rotational symmetry imply that Σ_0 should contain the axis of C.

We now show that Σ_0 is transversal to $\partial C_t = \{(x, y, z) \mid x^2 + y^2 \leq t^2\}$ for every $0 < t \leq r$. This transversality follows from the fact that a minimal surface cannot touch a cylinder from inside, as follows. Clearly Σ_0 is transversal to ∂C_t for sufficiently small t > 0. Also Σ_0 is transversal to ∂C_t for t sufficiently close to r because $\Sigma_0 \cap \partial C_r$ is a C^2 curve. Let Σ_0^1 be one of the two components of $\Sigma_0 \setminus l$. If $\Sigma_0^1 \cap \partial C_t$ is a connected curve on Σ_0^1 for every $0 < t \leq r$, then Σ_0^1 is transversal to ∂C_t for all $0 < t \leq r$ and $\Sigma_0^1 \cap \partial C_t$ is a naalytic curve for every 0 < t < r. This is because if Σ_0^1 is tangent to ∂C_t at a point $p \in \Sigma_0^1 \cap \partial C_t$ then in a neighborhood of p, $\Sigma_0^1 \cap \partial C_t$ is the union of at least two curves meeting at p, and then $\Sigma_0^1 \cap \partial C_{t\pm\delta}$ cannot be connected for small $\delta > 0$. Suppose $\Sigma_0^1 \cap \partial C_t$ is not connected for some 0 < t < r. Then there exists 0 < a < r such that

 $a = \inf\{b \mid \Sigma_0^1 \cap \partial C_t \text{ is connected for every } t \in (b, r)\}.$

Obviously $\Sigma_0^1 \cap \partial C_a$ is not connected. So it consists of at least two components A, B such that $E := \Sigma_0^1 \cap \{(x, y, z) \mid a^2 < x^2 + y^2 \leq r^2\}$ connects A and ∂C_r while B is disjoint from E. In fact, as t decreases from r to a, any point $p \in B$ must be a point of first touching between $\Sigma_0^1 \setminus \overline{E}$ and ∂C_t . Hence $\Sigma_0^1 \setminus \overline{E}$ is tangent to ∂C_a at p from inside, that is, $\Sigma_0^1 \setminus \overline{E} \subset C_a$. But this is not possible by the maximum principle. Therefore $\Sigma_0^1 \cap \partial C_t$ is a connected and analytic curve for every 0 < t < r and thus Σ_0 is transversal to ∂C_t for all $0 < t \leq r$. Therefore, by Theorem 1.1, we get $\mathcal{H}^2(\Sigma) \geq \mathcal{H}^2(\Sigma_0) \geq \mathcal{H}^2(\mathcal{H}_C)$, and it is clear that the equality holds if and only if $\Sigma = H_C$.

It is noteworthy that, according to Theorem 1.1, the area of H_C is smaller than the surface composed of a plane rectangle with a small piece of helicoid attached (Figure 2(a)), although most part of it is planar. Theorem 1.1 is also strong in the sense that it holds regardless of the height and radius of the given cylinder C.

Note that the topological condition on Σ being a disk-type surface is crucial. In fact, the surface with one handle depicted in Figure 2(b) may have a smaller area than a disk-type surface with the same boundary, even though it contains the central axis of C.

We close this section by mentioning the following two facts. First, in both theorems, the boundary condition that Σ contains d_1 and d_2 as its boundaries cannot be weakened. In fact, if we allow Σ to contain only one of the diameters, the surface consisting of a piece of a plane with small triangular regions on S attached on each side (see Figure 2(c)) has a smaller area than H_C .



FIGURE 2. (a) Plane attached to a small helicoidal region; (b) Genus-one surface containing the central axis; (c) Plane with two small triangular regions attached on each side

Second, Lawlor [12] showed a similar area-minimizing property for a compact portion of the half helicoid in a slightly different class of surfaces. Namely, he proved that half of the helicoid centered at the z-axis in \mathbb{R}^3 has the smallest area among all oriented surfaces that share the same boundary. From his result, we see that, by the reflection principle of minimal surfaces [11], a minimal surface containing the axis of a cylinder and bounded by two pair of helices on the side of cylinder has area greater than or equal to H_C . Theorem 1.1 and Theorem 1.3 not only coincide with this result, but generalize it into a broader class of surfaces: H_C is still the area-minimizer among the surfaces containing the central axis with their boundaries not necessarily helical.

2. Helicoid as the unique minimal surface

We now turn our attention to the uniqueness problem for minimal surfaces in C spanning $d_1 \cup d_2 \cup S$. In general, by Douglas' Existence Theorem (Douglas' solution) for the Plateau problem, every Jordan curve in $d_1 \cup d_2 \cup S$ bounds at least one minimal disk. Since there are infinitely many such curves lying on S, there exist infinitely many minimal surfaces bounded by $d_1 \cup d_2 \cup S$. But for partially free boundary solutions, meaning that d_1 and d_2 are the fixed boundaries and the other two boundary curves are free on S, H_C might be the unique suface under some conditions. It is clear that H_C satisfies the free boundary condition since it meets S orthogonally along the boundary helices. It is intringuing to investigate when H_C is a unique partially free boundary minimal surface.

In fact, with no further conditions, there is another minimal surface that meets S orthogonally; there should exist a (locally area-maximizing) minimal surface between H_C and the union U of the top and bottom half disks of C in the one-parameter family of minimal surfaces that are orthogonal to S. This is because H_C and U are the relative weak minima in the class of all neighboring surfaces with the same boundary. We will look

for some natural boundary conditions under which no partially free boundary solutions of minimal surfaces other than H_C exist.

Next, we turn out attention to the minimal surfaces in C spanning a pair of helices and will prove that H_C is unique if the angle of the projection of the boundary helices is bounded by the ratio of height and radius of C. Another uniqueness theorem follows with no assumption on height and radius of C. Note that these two results are not about partially free boundary problem. And, unlike in the previous section, the class of surfaces we consider here are minimal surfaces.

To make our discussion more accurate and general, we need the following definition concerning the curves on the side S of the cylinder C.

DEFINITION 2.1. Define the rotation angle $rot(\gamma)$ of a curve $\gamma \subset S$ connecting two points $p, q \in S$ as the oriented angle of the circular arc $proj(\gamma)$ joining proj(p) and proj(q), where $proj(\cdot)$ is the vertical projection of S onto ∂D_2 .



FIGURE 3. Rotation Angle

For example, if a curve $\gamma \subset S$ is part of a helix parameterized by $(r \cos \theta, r \sin \theta, b \theta)$ for some constants r and b and $0 \leq \theta \leq \theta_0$, then $rot(\gamma) = \theta_0$. In other words, the rotation angle of a helix measures how much the helix rotates as it sweeps from d_1 to d_2 .

We denote by H_C^{θ} the surface H_C with $rot(h_i) = \theta$ where $h_i \in H_C \cap S$ (i=1, 2) are the boundary helices. In fact, H_C in the previous section is actually H_C^{θ} with $0 < \theta \leq \frac{\pi}{2}$ or, equivalently, $-\frac{\pi}{2} \leq \theta < 0$. The notion of the rotation angle of the boundary curves was not needed in the previous section because the area of H_C is clearly smaller than that of H_C^{θ} with $\theta > \frac{\pi}{2}$ or $\theta < -\frac{\pi}{2}$, so that the area minimizer is H_C .

However, when we investigate the uniqueness of H_C , this notation makes it possible to show that not only H_C but the helicoids with boundary helices spiraling up more than $\frac{\pi}{2}$ have a uniqueness property; we shall see that H_C^{θ} is the only the minimal surface in Cwith the rotation angle of the boundary curves on S being θ under some conditions. Note

that thoughout this section we allow $\theta = 0$, so that d_1 and d_2 can be parallel and H_C^0 is a plane.

2.1. H_C^{θ} is the Unique Minimal Surface Having Symmetric Free Boundary. Hereafter, let $\Sigma \subset C$ be a minimal surface spanning d_1 , d_2 , and consider two C^2 curves γ_1 and γ_2 on S such that $\gamma_1 \cap \gamma_2 = \emptyset$ and $\partial(d_1 \cup d_2) = \partial(\gamma_1 \cup \gamma_2)$. With this setting, the question we originally posed was the following: if γ_1, γ_2 are symmetric with respect to the axis ℓ of C, is Σ also symmetric? This remains open, but the next theorem suggests not only an affirmative answer to this question, but leads to a stronger conclusion when Σ meets S at a right angle along γ_1 and γ_2 .

THEOREM 2.2. Let Σ be a minimal disk in C with $\partial \Sigma = d_1 \cup d_2 \cup \gamma_1 \cup \gamma_2$, where $\gamma_i \subset S$ are disjoint C^2 curves with $rot(\gamma_i) = \theta$ (i=1, 2) and Σ is orthogonal to S. If γ_1 and γ_2 are rotationally symmetric, then $\Sigma = H_C^{\theta}$.

The main idea of the proof is based on the maximum principle, which states that two different minimal surfaces cannot touch each other at an isolated interior point. If one minimal surface lies on one side of the other in the neighborhood of that point, the two surfaces must coincide. The same holds for a contact point on the boundary, provided the boundary curve is at least C^2 . More specifically, the following lemma explains how two different minimal surfaces behave in the neighborhood of their contact points.

LEMMA 2.3. Let M and N be two minimal surfaces in \mathbb{R}^3 that, at a common point p, have the same tangent plane P. Then, M and N either coincide, or the orthogonal projection of $M \cap N$ on P forms k curved rays emitting from p, making the same angles $2\pi/k$ for some even number $k \geq 4$.

Proof. See
$$[9]$$

Proof of Theorem 2.2. Suppose $\Sigma \neq H_C^{\theta}$. Consider $\mathcal{F} = \{\rho_{\alpha}(H_C^{\theta}) : \rho_{\alpha} \text{ to be a counter$ $clockwise rotation function about the axis <math>\ell$ of C by an angle $\alpha \in [0, \pi)\}$. Then, because \mathcal{F} foliates $C \sim \ell := C \setminus \ell$, \mathcal{F} induces a natural foliation on $\Sigma \sim \ell$ defined by $\mathcal{F} \mid_{\Sigma} = \{\rho_{\alpha}(H_C^{\theta}) \cap \Sigma : \alpha \in [0, \pi)\}$. In other words, Σ is filled with the disjoint curves (the leaves of the foliation) $\rho_{\alpha}(H_C^{\theta}) \cap \Sigma$. We will verify that these curves and Σ itself should satisfy the following four properties:

i) $\rho_0(H_C^{\theta}) \cap (int\Sigma)$ should contain a curve γ_0 connecting d_1 to d_2 . ii) $\alpha \in (0,\pi) \Rightarrow \rho_\alpha(H_C^{\theta}) \cap d_i = \{c_i\}, c_i$ is the center of d_i . (i = 1, 2)iii) $\alpha_1 \neq \alpha_2 \Rightarrow \rho_{\alpha_1}(H_C^{\theta}) \cap \Sigma$ and $\rho_{\alpha_2}(H_C^{\theta}) \cap \Sigma$ are disjoint except on ℓ . iv) $\exists \alpha_0 \in (0,\pi)$ such that

$$\begin{cases} 0 \leq \alpha \leq \alpha_0 \Rightarrow & \rho_{\alpha}(H_C^{\theta}) \cap \gamma_i \neq \emptyset \text{ and } \rho_{\alpha_0}(H_C^{\theta}) \text{ is tangent to } \Sigma \\ & \text{at some point } p_i \in \gamma_i. \\ \alpha > \alpha_0 & \Rightarrow & \rho_{\alpha}(H_C^{\theta}) \cap \gamma_i = \emptyset. \end{cases}$$

Property i) follows from the boundary conditions that $d_1 \cup d_2 \subset \Sigma \cap H_C^{\theta}$ and the γ_i are rotationally symmetric with each other. Properties ii) and iii) are straightforward because $\Sigma \sim \ell$ is foliated by $\{\rho_{\alpha}(H_C^{\theta}) \cap \Sigma : \alpha \in [0, \pi)\}$. To see iv), we proceed as follows: Let $\partial d_i = \{p^i, q^i\}$ (i = 1, 2). Develop S onto \mathbb{R}^2 after cutting it at the vertical plane passing through d_1 . Then, we have a rectangular region with points p^i and q^i on the top and bottom edges. Since $rot(\gamma_i) = rot(h_i) = \theta$, the developed curves of the γ_i and h_i have the same end points, connecting the top edge to the bottom. Note that the developed curves of h_i are actually line segments on the rectangular region. Hence, by the Mean Value Theorem, there exists at least one point, say p, on γ_1 and another point p' on γ_2 at which the tangent lines have the same slope as the h_i . Since p and p' are the common points of Σ and $\rho_{\alpha}(H_C^{\theta})$, and since Σ meets S orthogonally, p and p' become contact points of Σ and $\rho_{\alpha}(H_C^{\theta})$ at which they have the same tangent plane. Moreover, there will be no more intersection points on γ if we choose α_0 to be the maximum angle corresponding to the last contact point. This proves iv).

We will show that property iv) and the maximum principle for minimal surfaces imply the leaves $\rho_{\alpha}(H_C^{\theta}) \cap \Sigma$ of the foliation of Σ should behave in a special way, leading to a contradiction. Let us first look at the case of only one pair of boundary contact points $p_1 \in \gamma_1, p_2 \in \gamma_2$ of $\rho_{\alpha}(H_C^{\theta})$ and Σ at $\alpha = \alpha_0$. By ii) (that is, $\rho_{\alpha}(H_C^{\theta}) \cap d_i$ is a single point), there should be exactly one curve emitting from d_i for each i = 1, 2. Because of the axial symmetry of the $\gamma_i, \rho_{\alpha}(H_C^{\theta}) \cap \partial \Sigma$ are pairwise symmetric points on γ_1 and γ_2 with respect to the same α , converging to p_1 and p_2 as α goes to α_0 .

By the assumption that Σ is not congruent to H_C^{θ} and property iv), Lemma 2.3 implies that near p_1 , $\rho_{\alpha_0}(H_C^{\theta}) \cap \Sigma$ consists of k curves emitting from p_1 , making an angle of π/k $(k \geq 2)$. We assume, for now, that k = 2.

Note that the leaves $\rho_{\alpha}(H_C^{\theta}) \cap \Sigma$ should not be a closed curve in $\Sigma \sim \partial \Sigma$ for each α . Instead, every leaf should be connected to the boundary curves γ_i . This is because, if there is a region enclosed by the intersection curve, then those intersection curves should converge to an interior point as α increases. On that point, we can apply the maximum principle to get a contradiction.

It is now obvious that, under i), iii), and the previous observations, there exists $\alpha_C \in (\alpha_0, \pi)$ such that $\rho_{\alpha_C}(H_C^{\theta}) \cap int(\Sigma)$ is a single curve, say γ_C , along which $\rho_{\alpha_C}(H_C^{\theta})$ and Σ are tangent to each other. In other words, $\rho_{\alpha}(H_C^{\theta}) \cap int(\Sigma)$ converge to γ_C , which is one leaf of the foliation, such that $\rho_{\alpha_C}(H_C^{\theta})$ is located on one side of Σ along γ_C . However, this contradicts the boundary maximum principle. Therefore, we conclude that Σ must be equal to H_C^{θ} when k = 2.

If $k \geq 3$, we have the same conclusion, as the same argument can be applied to the component that contains the leaves $\rho_{\alpha}(H_C^{\theta}) \cap \Sigma$ of $\alpha > \alpha_0$. Specifically, near p_1 , Σ is divided by k + 1 regions such that, if one component is comprised of $\rho_{\alpha}(H_C^{\theta}) \cap \Sigma$ of $\alpha < \alpha_0$, then the adjacent component is of $\rho_{\alpha}(H_C^{\theta}) \cap \Sigma$ of $\alpha > \alpha_0$. The behavior of the

leaves of $\rho_{\alpha}(H_C^{\theta}) \cap \Sigma$ is the same as for k = 2, thus eliciting the same contradiction. In other words, we have $\Sigma = H_C^{\theta}$ when there are two boundary contact points of $\rho_{\alpha_0}(H_C^{\theta})$ and Σ (one at each γ_i).

In general cases, that is, when the number of the boundary contact points is greater than two, we get the same contradiction in the region containing p_i , the last boundary contact points of $\rho_{\alpha}(H_C^{\theta})$ and Σ .

Remark. From the proof, it is obvious that when d_1 and d_2 are parallel with $rot(\gamma_i) = 0$, Σ should be a piece of the plane under the same hypothesis. Also, note that the minimality of Σ is crucial in the proof: if Σ is not minimal, the intersection curves can be arbitrary, and one cannot assert the existence of the converging curve along which $\rho_{\alpha_C}(H_C^{\theta})$ is located on one side of Σ .

2.2. H_C^{θ} is the Unique Surface Spanning Helices. We now turn to the case in which $\partial \Sigma \cap S$ is a double helix and Σ does not necessarily meet S orthogonally. A double helix is a pair of helices that are symmetric to each other with respect to the axis of C. For simplicity, but without losing the generality, let us assume that the rotation angle θ is non-negative in this section. Then the next theorem asserts that such Σ should be equal to part of the helicoid if the height h is bigger than the product of the radius r and the rotation angle θ .

THEOREM 2.4. Suppose $\Sigma \subset C$ is a minimal surface spanning d_1, d_2 and a double helix h_1, h_2 with $rot(h_i) = \theta$. If $0 \leq \theta \leq h/r$, then $\Sigma = H_C^{\theta}$.

PROOF. Assume that $\Sigma \neq H_C^{\theta}$. Let $\Omega \in C$ be the surface obtained by the screw motion (along the z-axis) of a circular arc (in the xy-plane) with radius R > 0 and central angle $2\eta \in (0, \pi]$. Then, for $a := h/\theta$, Ω can be parametrized by

$$\Omega(\zeta,\phi) = (R\cos\eta\sin\zeta - R\cos(\pi/2 - \zeta + \eta - \phi), -R\cos\eta\cos\zeta + R\sin(\pi/2 - \zeta + \eta - \phi), a\zeta)$$
$$= (-R\sin(\zeta - \eta + \phi) + R\cos\eta\sin\zeta, R\cos(\zeta - \eta + \phi) - R\cos\eta\cos\zeta, a\zeta),$$

where $\sin \eta = \frac{r}{R}$. See Figure 4.

A computation yields

$$\begin{split} \Omega_{\zeta} &= (-R\cos(\zeta - \eta + \phi) + R\cos\eta\cos\zeta, -R\sin(\zeta - \eta + \phi) + R\cos\eta\sin\eta, a) \\ \Omega_{\phi} &= (-R\cos(\zeta - \eta + \phi), -R\sin(\zeta - \eta + \phi), 0) \\ \Omega_{\zeta} \times \Omega_{\phi} &= (aR\sin(\zeta - \eta + \phi), -aR\cos(\zeta - \eta + \phi), R^{2}\cos\eta\sin(\eta - \phi)) \\ \vec{N} &= \frac{(a\sin(\zeta - \eta + \phi), -a\cos(\zeta - \eta + \phi), R\cos\eta\sin(\eta - \phi))}{\sqrt{a^{2} + \rho^{2}\cos^{2}\eta\sin^{2}(\eta - \phi)}} \\ \Omega_{\zeta\zeta} &= (R\sin(\zeta - \eta + \phi) - R\cos\eta\sin\zeta, -R\cos(\zeta - \eta + \phi) + R\cos\eta\cos\zeta, 0) \\ \Omega_{\zeta\phi} &= (R\sin(\zeta - \eta + \phi), -R\cos(\zeta - \eta + \phi), 0) = \Omega_{\phi\phi}. \end{split}$$



FIGURE 4. Parameterization of Ω at height $a\zeta$

Therefore, the coefficients of the first and second fundamental forms of Ω are:

$$g_{11} = \langle \Omega_{\zeta}, \Omega_{\zeta} \rangle = R^{2} (1 + \cos^{2} \eta - 2 \cos \eta \cos(\eta - \phi)) + a^{2}$$

$$g_{12} = \langle \Omega_{\zeta}, \Omega_{\phi} \rangle = R^{2} - R^{2} \cos \eta \cos(\eta - \phi)$$

$$g_{22} = \langle \Omega_{\phi}, \Omega_{\phi} \rangle = R^{2}$$

$$b_{11} = \langle \Omega_{\zeta\zeta}, \vec{N} \rangle = \frac{aR(1 - \cos \eta \cos(\eta - \phi))}{\sqrt{a^{2} + R^{2} \cos^{2} \eta \sin^{2}(\eta - \phi)}}$$

$$b_{12} = \langle \Omega_{\zeta\phi}, \vec{N} \rangle = \frac{aR}{\sqrt{a^{2} + R^{2} \cos^{2} \eta \sin^{2}(\eta - \phi)}}$$

$$b_{22} = \langle \Omega_{\phi\phi}, \vec{N} \rangle = \frac{aR}{\sqrt{a^{2} + R^{2} \cos^{2} \eta \sin^{2}(\eta - \phi)}},$$

$$\therefore H(\Omega) = \frac{1}{2} \frac{g_{22}b_{11} - 2g_{12}b_{12} + g_{11}b_{22}}{g_{11}g_{22} - g_{12}^{2}}}$$

$$= \frac{a(a^{2} + R^{2} \cos^{2} \eta \sin^{2}(\eta - \phi))}{2R(a^{2} + R^{2} \cos^{2} \eta \sin^{2}(\eta - \phi))^{3/2}}$$

$$= \frac{a(a^{2} - r^{2} + R^{2}(1 - \cos \eta \cos(\eta - \phi)))}{2R(a^{2} + R^{2} \cos^{2} \eta \sin^{2}(\eta - \phi))^{3/2}}.$$

Substituting $\sin \eta = \frac{r}{R}$ induces the last equality.

Observe that the mean curvature $H(\Omega)$ of Ω is always greater than zero when $h \ge r\theta$, that is, $a = \frac{h}{\theta} \ge r$. Since the normal vector \vec{N}_{Ω} of Ω is always pointing toward Σ (see Figure 4), so is the mean curvature vector $\vec{H}(\Omega) = H(\Omega)\vec{N}_{\Omega}$ at any point of Ω . Therefore, if

there is a contact point of Ω and Σ inside C, we can apply the interior maximum principle there to get a contradiction. Fortunately, this is possible by choosing η appropriately and, if necessary, taking the surface Ω' obtained by reflecting Ω with respect to H_C^{θ} instead of Ω ; as η varies, Σ must have a contact point with either Ω or Ω' for some $\eta \in (0, \pi/2)$. This contradicts the maximum principle. In other words, $\Omega = H_C^{\theta}$ if $h \geq r\theta$.

One important remark is that our proof includes wider classes of surfaces. They do not have to be topological disks. Also, their boundary helices can have the total curvature bigger than 2π as long as $\theta \leq h/r$. In fact, if the total curvature of the boundary helices h_1, h_2 is less than 2π , the total curvature of $d_1 \cup d_2 \cup h_1 \cup h_2$ is less than 4π , which implies that the disk-type minimal surface bounded by $d_1 \cup d_2 \cup h_1 \cup h_2$ is unique by a generalization of Nitsche's uniqueness theorem [20] and therefore should be part of the helicoid.

If there is no restriction on the ratio of the height and radius of C, the method used in the proof of Theorem 2.4 can no longer be applied, because the mean curvature of Ω may be negative. However, we obtain the same uniqueness result with no conditions on the height and radius of C. Namely, if $0 \leq rot(h_i) \leq \pi$, Σ should be congruent to H_C^{θ} .

THEOREM 2.5. If $\Sigma \subset C$ is a disk-type minimal surface spanning d_1 , d_2 , h_1 , and h_2 with $0 \leq rot(h_i) \leq \pi$, then $\Sigma = H_C^{\theta}$ with $0 \leq \theta \leq \pi$.

PROOF. Let $rot(h_i) = \phi_0$. Without loss of generality, suppose $d_2(t) = (rt, 0, 0), d_1(t) = (rt \cos \phi_0, rt \sin \phi_0, b\phi_0), h_1(u) = (r \cos u, r \sin u, bu), and <math>h_2(u) = (-r \cos u, -r \sin u, bu)$ for $-1 \leq t \leq 1, 0 \leq u \leq \phi_0$ and for some positive constant b. Let π_{yz} be the projection map onto the yz-plane. Then, $\pi_{yz} \circ h_1(u) = (0, r \sin u, bu), \pi_{yz} \circ h_2(u) = (0, -r \sin u, bu),$ $\pi_{yz} \circ d_1(t) = (0, 0), \text{ and } \pi_{yz} \circ d_2(t) = (rt \sin \phi_0, b\phi_0).$ Hence, $rot(h_i) = \phi_0 \leq \pi$ implies that $\partial \Sigma = d_1 \cup d_2 \cup h_1 \cup h_2$ is a Jordan curve which has a monotonic orthogonal projection onto a convex plane Jordan curve. Then, by the Remark on p299 of [11] or by Theorem 2 of [14] (a generalized version of Rado's Theorem), $\partial \Sigma$ bounds a unique minimal disk. To be more specific, because Rado's Theorem remains true when vertical segments of a given Jordan curve are mapped onto single points of a plane convex curve by an orthogonal projection, we conclude that H_C^{θ} is the unique minimal disk spanning d_1, d_2, h_1 , and h_2 with $0 \leq rot(h_i) \leq \pi$.

Note that the method used in the proof of Theorem 2.5 is not valid when $rot(h_i) > \pi$, because the region bounded by the projection map is no longer convex. Theorem 2.5 is related to Theorem 2.4 in the sense that both provide an upper bound of θ to assert the uniqueness of H_C^{θ} . The differences are that Theorem 2.4 uses the ratio of h over r to bound θ with no topological restrictions, whereas Theorem 2.5 reaches the same conclusion regardless of h, r, or h/r with a topological restriction.



FIGURE 5. (a) The phase diagram of the helicoid of pitch 2π ; (b) The phase diagram of the helicoid as the pitch changes

2.3. **Remarks.** In [17](§111), Schwarz's result on the stable part of the helicoid is introduced. Namely, the area of the helicoid with pitch 2π : $\{(u \cos v, u \sin v, v) : -r \leq u \leq r, -\frac{\theta}{2} \leq v \leq \frac{\theta}{2}\}$ is a relatively weak minimum in the class of all neighboring surfaces with the same boundary if $\theta \leq \pi$ or $\theta > \pi$ and $r \leq r(\theta)$. The upper bound of $r(\theta)$ is given by $\bar{r} \approx 1.5088 \cdots$. Beware that the range of θ there is not exactly the same as ours.

Theorem 2.4 differs from this result in the sense that Schwarz's result was about the stability, whereas our focus is on uniqueness. To be more precise, a certain piece of the helicoid has, according to Schwarz's result, a smaller area than the *neighboring* surfaces that have helices as their boundary, whereas Theorem 2.4 guarantees that this piece of the helicoid is, in fact, unique among *all* minimal surfaces with the same boundary. In other words, Theorem 2.4 extends Schwarz's result by asserting that, when $r \leq 1$, the helicoid with pitch 2π is not only stable but also unique. See the shaded region in Figure 5(a). Although r is not covered up to $\bar{r} \approx 1.5088 \cdots$, our proof is more geometrical and avoids long calculations. Also, our result is independent of the pitch of th helicoid. We possibly reach \bar{r} if we can obtain the lower bound of the mean curvature $H(\Omega)$ in the proof of Theorem 2.4.

In [2], a similar result concerning the stable part of complete helicoid was studied in a slightly different setting: instead of fixing the pitch of the helicoid, the authors obtained the phase diagram of h/r with respect to θ for which the helicoid was stable, where hand r are the height and radius of the helicoid, respectively. Therefore, Theorem 2.4 is stronger than their result, because it asserts that part of the helicoid is not only stable,

but is actually unique when $h/r \ge \theta$ (see Figure 5(b)).

After completing the work reported in this paper, the author has learned that [19] contains a similar uniqueness result for the stable part of the helicoid. Namely, it was proved that the stable part of the helicoid is unique among the properly immersed minimal surfaces spanning a double helix, having bounded curvature and satisfying the following asymptotic condition

$$\lim_{(x,y,z) \to \infty} \frac{x^2 + y^2}{z^2} = 0.$$

The main differences are that again we do not fix the pitch of the boundary helices to 2π , nor do we require our surfaces to have bounded curvature or satisfy the asymptotic condition. Our method is also simpler in the sense that we use the surface Ω obtained by the screw motion of an arc in the plane as a barrier for applying the maximum principle, whereas they used the one-parameter family of associate minimal surfaces of the helicoid. Unlike [19], Σ shares the same boundary as Ω , and it is therefore geometrically clear that an interior touching point exists. In [19], it was necessary to prove the existence of the interior touching points, as the boundary of their barrier surfaces are not always on one side of C.

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