NEW CLASS OF WAVELETS FOR SIGNAL APPROXIMATION

Jan E. Odegard *C. Sidney Burrus*

Department of Electrical and Computer Engineering
Rice University, Houston, TX 77251-1892, USA
odegard@rice.edu *csb@rice.edu*

ABSTRACT

This paper develops a new class of wavelets for which the classical Daubechies zero moment property has been relaxed. The advantages of relaxing higher order wavelet moment constraints is that within the framework of compact support and perfect reconstruction (orthogonal and biorthogonal) one can obtain wavelet basis with new and interesting approximation properties. This paper investigates a new class of wavelets that is obtained by setting a few lower order moments to zero and using the remaining degrees of freedom to minimize a larger number of higher order moments. The resulting wavelets are shown to be robust for representing a large classes of inputs. Robustness is achieved at the cost of exact representation of low order polynomials but with the advantage that higher order polynomials can be represented with less error compared to the maximally regular solution of the same support.

1. INTRODUCTION

A fundamental property of the classical compactly supported Daubechies [2] wavelets is that a maximal number of wavelet moments is set to zero (hence then name *maximally regular wavelets*). In addition to the fact that setting lower order moments to zero guarantees a certain degree of smoothness of the wavelet basis, Daubechies was also able to derive a closed form expression for designing compactly supported (maximally regular) wavelets. Furthermore, it is also easy to show that K-regular wavelets can be used to represent polynomials up to order $K-1$ exactly. However, since most signals are not polynomials (albeit locally they can be modeled by a sufficiently hight order polynomial c.f., Weierstrass Approximation theorem of 1885) it is not clear how significant it is to have exact zero moments for signal processing applications. Furthermore, fast algorithms for computing using wavelets require that the wavelet filters be implemented efficiently typically involving coefficient quantization and approximation. As a result of the fast but approximate implementation exactly zero moments are not preserved [4] with the exception for the zeroth order moment (which must be treated carefully to preserve the wavelet existence condition).

Although regularity and smoothness of the wavelet basis are related, using moments to achieve smoothness is not very efficient. Several recent publications [5, 6, 10, 9] have looked at the design and analysis of maximally smooth wavelet basis. However, most of this work has been on optimization of the Hölder or Sobolev regularity by systematically reducing the number of zero moments. To date, the authors are, however, not aware of any results successfully answering the following important question: *Is smoothness of the higher order derivatives of the wavelet basis important – or is it sufficient that the functions "appear" to be smooth?*

This work was supported in part by AFORS grant no. F49620–1–0006 funded by ARPA and in part by the Texas Advanced Technology Program, grant no. TX–ATP 003604–002.

Appears in *Proc. ISCAS-96*, Atlanta, GA.

It seems somewhat contradictory to pose this question since clearly if $\psi^{(m)}(x) \in C$, where $\psi^{(m)}(x)$ denotes the mth derivative of $\psi(x)$, then $\psi(x) \in C^m$ almost everywhere. However, keeping in mind that most signal processing applications deal with discrete signals one might wonder whether the behavior of the basis function between the sample points truly matters [8, 4]. Based on this discussion several questions arise: What "kind" of smoothness is required for a given application? Is it sufficient that the functions themselves be smooth up to a fixed scale (e.g., not require that higher order derivatives be smooth)? Do typical DSP applications require that moments be exactly zero?

To come to grasp with several of the above questions it is important to fully understand the possible importance of exact or approximate zero moments. To address this we investigate and design a new class of wavelets which do not have $\frac{N}{2}$ exact zero moments, instead, the free parameters are used to minimize a larger number of (weighted) higher order moments. Clearly there are applications for which a large number of exactly vanishing moments are important [1] but for which smoothness might not be. Similarly, there are problems (probably most practical signal processing problems) for which exact zero moments are not important but for which some form of smoothness of the wavelet and scaling function is important [11]. Notice that while requiring that the wavelet basis be Hölder smooth of order m then necessarily the function will have $m+1$ zero moments. However, it is not clear how this relates to discretely sampled functions for which derivatives do not exist [8].

2. WAVELET REVIEW

2-band compactly supported orthonormal wavelet bases are uniquely characterized by a length $N = 2K$ scaling filter, $h_0(n)$ satisfying the linear and quadratic condition

$$\sum_{n=0}^{N-1} h_0(n) = \sqrt{2} \quad (1)$$

$$\sum_{n=0}^{N-1} h_0(n) h_0(n-2l) = \delta(l). \quad (2)$$

Given h_0 the associated orthogonal wavelet filter h_1 is given by (moduli shifts by multiples of 2)

$$h_1(n) = (-1)^n h_0(N-1-n). \quad (3)$$

Given $h_0(n)$ and $h_1(n)$ the scaling function, $\psi_0(t)$, and the wavelet, $\psi_1(t)$, exist and are defined by the dyadic difference equation

$$\psi_i(t) = \sqrt{2} \sum_{n=0}^{N-1} h_i(n) \psi_0(2t-n). \quad (4)$$

Furthermore, let the continuous and discrete scaling and wavelet moments be defined by

$$m(i,k) = \int x^k \psi_i(x)dx \tag{5}$$

and

$$\mu(i,k) = \sum_{n=0}^{N-1} n^k h_i(n) \tag{6}$$

respectively. Then by substitution of (4) in to (5) it is easy to show that $m(1,k) = 0 \iff \mu(1,k) = 0$ for $k = 0, \ldots, K-1$ [3].

Several methods for finding a set of filter coefficients, $h_0(n)$ satisfying (1) and (2) have been proposed. However, none of the existing methods have investigated the problem of setting a few low order wavelet moments to zero and using the remaining free parameters to minimize several higher order wavelet moments. That is, for $L > \frac{N}{2}$ and $K < \frac{N}{2}$ minimize some norm on $\mu(1,k)$ for $k = 0, \ldots, L-1$ subject to $\mu(1,k) = 0$ for $k = 0, \ldots, K-1$. A more general problem is obtained by introducing a weighting function $w(i)$ on the moments, hence if

$$\boldsymbol{\mu} = [\mu(1,K), \ldots, \mu(1,L-1)]^T$$

and

$$\mathbf{W} = \mathrm{diag}\left(\mathrm{w(K)}, \ldots, \mathrm{w(1, L-1)}\right)$$

then the desired cost function is given by

$$\|\mathbf{W}\boldsymbol{\mu}\|_p \tag{7}$$

Note that maximally regular wavelets are obtained by letting the weights be infinite for $i = 0, 1, \ldots, \frac{N}{2} - 1$ and zero for $i \geq \frac{N}{2}$.

3. DESIGN ALGORITHM

Given (7) an optimal wavelet solution can be obtained by solving the following nonlinear constrained optimization problem

$$\min_{h_0} \|\mathbf{W}\boldsymbol{\mu}\|_p \tag{8}$$

subject to

a) $\mu(1,i) = 0 \quad \text{for } i = 0, \ldots, K-1$

b) $\displaystyle\sum_{n=0}^{N-1} h_0(n)h_0(n-2l) = \delta(l)$

with $1 \leq K \leq \frac{N}{2} \leq L$. The constrained optimization problem posed by (8) can be solved using a generic constrained optimization package such as `constr` from the software package MATLAB. For shorthand notation we will refer to a particular solution to the above problem by $\Omega_{k,l}$ where k indicates the number of wavelet moments explicitly set to zero and l denotes the number of minimized wavelet moments. With this notation the maximally regular Daubechies solution is denoted by $\Omega_{\frac{N}{2},0}$. Also notice that $\Omega_{\frac{N}{2}-i,i}$ is equivalent to $\Omega_{\frac{N}{2},0}$ for all $i = 0, \ldots, \frac{N}{2}$.

4. EXAMPLES

This section gives an example of the kinds of wavelet solutions that can be obtained by solving the above nonlinear constrained optimization problem for $N = 6$. Results indicate that by choosing an appropriate set of weights the corresponding optimal wavelet basis are "robust." By robust we mean that the polynomial approximation error for a range of polynomial orders is small, on the average, compared to the approximation error obtained by using the maximally regular solution.

Specifically, we will consider the design of a compactly supported wavelet basis (of length $N = 6$) with $K = 2$ zero wavelet moments (e.g., $\mu(1,k) = 0$ for $k = 0, 1$). Requiring that 2 moments be exactly zero leaves one free parameter to be used in the optimization of higher order moments. For the purpose of illustration we choose to use the free parameter to minimize (jointly) $\mu(1,2)$ and $\mu(1,3)$. Hence, $\boldsymbol{\mu} = [\mu(1,2)\ \mu(1,3)]$ and by choosing $\mathbf{W} = \mathrm{diag}\,(1, \alpha)$, $0 \leq \alpha < \infty$ all possible weights can be realized with one single parameter. The corresponding set of optimal solutions (one for each α) is denoted by $\Omega_{2,2}$. Notice that choosing $\alpha = 0$ yields the maximally regular solution. Experimental results indicate that the prefered norm to use in the optimization is the ℓ^∞ norm and hence the desired cost function is given by

$$\max_{k \in [K, L-1]} |w(k)\mu(1,k)| = \|\mathbf{W}\boldsymbol{\mu}\|_\infty .$$

Although beyond the scope of this paper [7], the ℓ^∞ norm can also be obtained by showing that the L^2 wavelet (polynomial) approximation error is bounded above by an expression involving the ℓ^∞ norm of the higher order moments.

Using a Taylor series type argument it is also reasonable to expect that the most interesting set of weights is obtained by restricting α to be in the range $[0, 1]$. Furthermore, let $p_{L-1}(x) = \sum_{k=0}^{L-1} x^k$ be the desired test polynomial. In Fig. 1 we plot the wavelet approximation error as a function of α. Notice from Fig. 1a that for $L = 3$ (e.g., a second order polynomial) the optimal solution is indeed the maximally regular solution due to Daubechies (e.g., the minimum of the bottom curve occurs at $\alpha = 0$). However, keeping N and K fixed, the optimal wavelet system for approximating the given polynomial for $L > 3$ is not the maximally regular wavelet basis. In fact, by carefully investigating Fig. 1a and b we notice that choosing $\alpha > 0$ a non-maximally regular solution is obtained that gives rise to better performance and is hence more robust. This can be more easily observed from Fig. 2 where the polynomial approximation error is plotted versus the order of the input polynomial (still keeping N and K fixed). Each line in Fig. 2 corresponds to the wavelet basis obtained by choosing the optimal α from Fig. 1 (e.g., chose the α that minimize the error for each L in Fig. 1). Observe from Fig. 2b that for $0 \leq L < 5$ the maximally regular solution (the dashed line) is near optimal. Furthermore, for $L > 9$ the maximally regular solution gives rise to the worst approximation error. Hence, clearly choosing $\alpha \neq 0$ will give rise to a more robust solution. Finally, in Fig. 3a the corresponding scaling functions are plotted. It is interesting to note that the differences between the various functions are rather insignificant and can only be observed by zooming in on points of large variation as seen from Fig. 3b and c.

In Table 1 the corresponding wavelet moments are tabulated and finally, in Table 2 the Hölder exponent of each solution tabulated in Table 1 is given. It is interesting to note that for $\alpha = 0.06$ and $\alpha = 0.068$ the estimated Hölder exponents are larger than the corresponding Hölder exponent for the maximally regular solution (e.g., the functions are smoother in the Hölder sense, albeit marginally).

5. SUMMARY

In this paper a new family of wavelet approximants of compact supported are introduced. The new family is obtained by relaxing the stringent requirement that a maximal number of moments be exactly zero (e.g., maximally regular Daubechies wavelets) in favor of obtaining a larger number of small, but non-zero, moments. A design algorithm is posed using the scaling filter coefficients as the parameter space, giving rise to a nonlinear constrained optimization problem. A solution of the desired optimization problem was obtained using readily available optimization routines (e.g., MATLAB's `constr` algorithm). The wavelet basis obtained by solving the given optimization problem are shown to be robust in the sense that they can be used to approximate both low order polynomials as well as high order polynomials with less error, on the average, compared to maximally regular wavelets of the same support.

REFERENCES

[1] G. Beylkin, R. Coifman, and V. Rokhlin. Fast wavelet transforms and numerical algorithms. *Comm. Pure Applied Math.*, 44:141–183, 1991.

[2] I. Daubechies. Orthonormal bases of compactly supported wavelets. *Comm. Pure Applied Math.*, XLI(41):909–996, November 1988.

[3] R. A. Gopinath and C. S. Burrus. On the moments of the scaling function ψ_0. In *Proc. of the IEEE ISCAS*, volume 2, pages 963–966, San Diego, CA, May 1992. Also Tech. Report CML TR91-05.

[4] J. Götze, J. E. Odegard, P. Rieder, and C. S. Burrus. Approximate moments and regularity of efficiently implemented orthonormal wavelet basis. In *IEEE Proc. Int. Symp. Circuits and Systems*, Atlanta, GA, May 1996. IEEE. Also Rice University Tech. Report CML TR95-17.

[5] P. Heller and R. O. Wells, Jr. Spectral theory of multiresolution operators and applications. In *Proceedings of the International Conference on Wavelets*, Taormina, Italy, 1993. Also Rice University Tech. Report CML TR93-09.

[6] M. Lang and P. N. Heller. The design of maximally smooth wavelets. In *IEEE Proc. Int. Conf. Acoust., Speech, Signal Processing*, volume 3, pages 1463–1466, Atlanta, GA, May 1996.

[7] J. E. Odegard. *Moments, smoothness and optimization of wavelet systems*. PhD thesis, Rice University, Houston, TX 77251, USA, May 1996.

[8] J. E. Odegard and C. S. Burrus. Discrete finite variation: A new measure of smoothness for the design of wavelet basis. In *IEEE Proc. Int. Conf. Acoust., Speech, Signal Processing*, volume 3, pages 1467–1470, Atlanta, GA, May 1996.

[9] O. Rioul. Simple regularity criteria for subdivision schemes. *SIAM J. Math Anal.*, 23(6):1544–1576, November 1992.

[10] O. Rioul. Regular wavelets: A discrete-time approach. *IEEE Trans. SP*, 41(12):3572–3579, December 1993.

[11] M. J. Smith and T. P. Barnwell III. Exact reconstruction techniques for tree-structured subband coders. *IEEE Trans. ASSP*, 34:434–441, 1986.

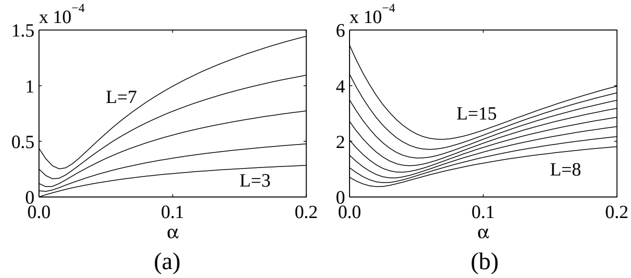

Figure 1: Wavelet polynomial approximation error as a function of α for $N = 6$ and $K = 2$. The vertical axis represents the approximation error normalized by the energy of the polynomial

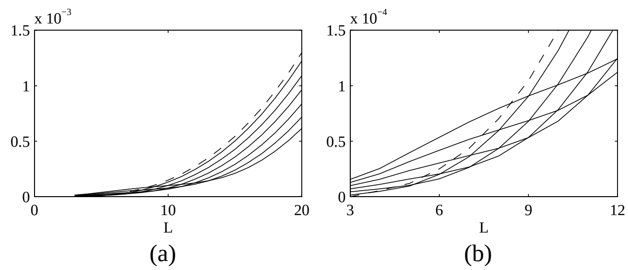

Figure 2: Wavelet polynomial approximation error as a function of L for $N = 6$ and $K = 2$. The vertical axis represents the approximation error normalized by the energy of the polynomial. Dashed line corresponds to the maximally regular solution.

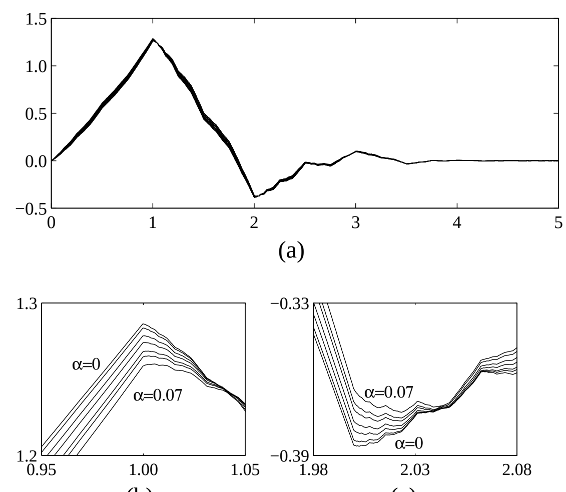

Figure 3: Comparison of scaling functions, $\psi_0(x)$ for $\alpha \in \{0, 0.007, 0.016, 0.026, 0.038, 0.052, 0.068\}$. (b,c) The max/min value of $\psi_0(x)$ decrease/increase as α increase.

Table 1: Discrete wavelet moments.

α	$\mu(1,k)$					
	0	1	2	3	4	5
0	0	0	0	3.354	40.680	329.324
0.004	0	0	-0.013	3.236	39.904	324.807
0.007	0	0	-0.022	3.153	39.355	321.610
0.012	0	0	-0.036	3.023	38.499	316.624
0.016	0	0	-0.047	2.927	37.862	312.910
0.021	0	0	-0.060	2.815	37.119	308.578
0.026	0	0	-0.071	2.711	36.430	304.555
0.032	0	0	-0.083	2.595	35.666	300.093
0.038	0	0	-0.095	2.489	34.963	295.985
0.045	0	0	-0.107	2.376	34.211	291.586
0.052	0	0	-0.118	2.272	33.523	287.560
0.060	0	0	-0.130	2.164	32.805	283.359
0.068	0	0	-0.141	2.066	32.152	279.531

Table 2: Numerical estimates of Holder exponents.

α	Hölder
0	1.0816
0.004	1.0277
0.007	1.0323
0.012	1.0394
0.016	1.0447
0.021	1.0509
0.026	1.0566
0.032	1.0629
0.038	1.0687
0.045	1.0749
0.052	1.0805
0.060	1.0863
0.068	1.0915