New classical properties of quantum coherent states

Gérard G. Emch[a)]
Institut für Theoretische Physik, Universität Göttingen, Göttingen, West Germany
Gerhard C. Hegerfeldt[b)]
A. P. Sloan Laboratory of Mathematics and Physics, California Institute of Technology, Pasadena, California 91125

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A noncommutative version of the Cramer theorem is used to show that if two quantum systems are prepared independently, and if their center of mass is found to be in a coherent state, then each of the component systems is also in a coherent state, centered around the position in phase space predicted by the classical theory. Thermal coherent states are also shown to possess properties similar to classical ones.

I. INTRODUCTION

The coherent states ϕ to be studied in this paper have expectation values of the form

$$\langle \phi; e^{-i(uP+vQ)}\rangle = \exp\{-\Theta(\lambda u^2 + \lambda^{-1}v^2)/4\}e^{-i(u\langle P\rangle + v\langle Q\rangle)}, \quad (1.1)$$

where P and Q are the momentum and position operator for a quantum particle in one dimension; the generalization to $\mathbb{R}^n$ is straightforward. The physical interpretation of the parameters $\langle P\rangle$, $\langle Q\rangle$, Θ, and λ, characterizing the state ϕ, is obtained from (1.1) by differentiation; namely,

$$\begin{aligned}\langle P\rangle &= \langle \phi; P\rangle,\\ \langle Q\rangle &= \langle \phi; Q\rangle,\\ \Theta\lambda/2 &= \langle \phi; (P-\langle P\rangle)^2\rangle,\\ \Theta\lambda^{-1}/2 &= \langle \phi; (Q-\langle Q\rangle)^2\rangle,\end{aligned} \quad (1.2)$$

so that

$$\langle (P-\langle P\rangle)^2\rangle\langle (Q-\langle Q\rangle)^2\rangle = \Theta^2/4. \quad (1.3)$$

We must therefore have

$$\lambda > 0 \quad \text{and} \quad \Theta \geqslant \hbar. \quad (1.4)$$

The case $\Theta = \hbar$ corresponds to the class of coherent states introduced by Schrödinger[1]: they have minimal dispersion, compatible with the Heisenberg uncertainty relation, around the point $(\langle P\rangle, \langle Q\rangle)$ of the classical phase space $T^*\mathbb{R} \simeq \mathbb{R}^2$. They are pure states and are characterized by the existence of a vector $\widehat{\Phi} \in \mathscr{H} \equiv \mathscr{L}^2(\mathbb{R}, dx)$ such that,

$$\langle \phi; e^{-i(uP+vQ)}\rangle = (\widehat{\Phi}, e^{-i(uP+vQ)}\widehat{\Phi}) \quad (1.5)$$

and

$$\hat{a}\widehat{\Phi} = 0, \quad (1.6)$$

where

$$\hat{a} = \widehat{Q} + i\lambda^{-1}\widehat{P} \quad (1.7)$$

with

$$\widehat{P} = P - \langle P\rangle, \quad \widehat{Q} = Q - \langle Q\rangle. \quad (1.8)$$

Note that (1.6)–(1.8) is equivalent to saying that $\widehat{\Phi}$ is the wave function for the ground state of the harmonic oscillator with Hamiltonian

$$\widehat{H} = (1/2m)\widehat{P}^2 + (1/2)k\widehat{Q}^2, \quad (1.9)$$

where m and k satisfy the relations

$$\lambda = m\omega \quad \text{with} \quad \omega^2 = k/m. \quad (1.10)$$

When we further have

$$\langle P\rangle = 0 = \langle Q\rangle, \quad (1.11)$$

let us denote by Φ_0 the vector $\widehat{\Phi}$ characterized by (1.5)–(1.8). Since the Schrödinger representation of the canonical commutation relations is irreducible, every vector $\Phi \in \mathscr{H}$ is cyclic. In particular we thus have that the (algebraic) vector space

$$\text{Span}\{e^{-i(uP+vQ)}\Phi_0 | u,v \in \mathbb{R}\} \quad (1.12)$$

is dense in $\mathscr{H}$, and for general values of $\langle P\rangle$ and $\langle Q\rangle$ the corresponding vector $\widehat{\Phi}$ is linked to Φ_0 by

$$\widehat{\Phi} = e^{-i(\langle Q\rangle P - \langle P\rangle Q)/\hbar}\Phi_0. \quad (1.13)$$

In this sense, the vectors $\widehat{\Phi}$, obtained by letting $(\langle P\rangle, \langle Q\rangle)$ run over the classical phase space $T^*\mathbb{R} \simeq \mathbb{R}^2$, form an overcomplete basis in $\mathscr{H}$, a mathematical property that has been given much attention[2] in connection with the theory of reproducing kernel Hilbert spaces.

When $\Theta > \hbar$, the change of variables

$$\Theta \equiv \hbar\coth(\beta\hbar\omega/2), \quad \lambda \equiv m\omega, \quad (1.14)$$

allows one, as explained in Sec. III, to interpret the corresponding coherent state as the canonical equilibrium state, at inverse temperature β, for a quantum harmonic oscillator (1.9) with frequency defined as in (1.10). These states are therefore not pure.

All coherent states ($\Theta \geqslant \hbar$) have in common the property that they allow one,[3] upon controlling the limit $\hbar \to 0$, to derive from Mackey's formulation of quantum mechanics the formalism of classical mechanics, complete with its Jordan and Lie products, i.e., with the algebraic structures corresponding to (a) the pointwise multiplication of functions on the classical phase space T^*M, and (b) the Poisson bracket associated with the canonical symplectic form on T^*M.

In this paper we focus our attention on other classical

[a)] Permanent address: Department of Mathematics, The University of Florida, Gainesville, Florida 32601.

[b)] Permanent address: Institut für Theoretische Physik, Universität Göttingen, Göttingen, West Germany.

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properties of quantum coherent states, exploring what can be said about the individual states of two quantum systems when these are prepared independently and when their center of mass is found to be in a coherent state. In Sec. II we limit our attention to the usual case of pure coherent states ($\Theta = \hbar$); the general case ($\Theta \geqslant \hbar$) is presented in Sec. II.

The mathematical motivation for this paper is a quantum version of the classical Cramer theorem.[5] The latter asserts that *if* the sum of two independent random variables is normally distributed, *then* each of the two random variables entering in this sum must also be normally distributed. Lemma 4.1 allows a simple derivation of a quantum version of this theorem adapted to the case $\Theta = \hbar$; in Sec. III, however, we need the general quantum version established by one of us in Refs. 6 and 7. The mathematical proofs, pertinent to the results stated in Secs. II and III are collected in Sec. IV.

II. PURE COHERENT STATES

If two *classical* particles, Σ_1 and Σ_2 say, are prepared independently and if their center of mass is found to be at the point $\{p_{\rm CM}, q_{\rm CM}\}$ of phase space one concludes immediately that the state of each of the component systems is described by a point $\{p_\kappa, q_\kappa\}$ ($\kappa = 1, 2$) and that

$$p_1 + p_2 = p_{\rm CM}, \quad \mu_1 q_1 + \mu_2 q_2 = q_{\rm CM}, \tag{2.1}$$

with

$$\mu_\kappa = m_\kappa / m_{\rm CM} \quad \text{and} \quad m_{\rm CM} = m_1 + m_2, \tag{2.2}$$

where m_κ is the mass of the κth particle.

If, however, the two particles are *quantum* systems, and one knows the wave function $\Psi_{\rm CM}$ describing the state of their center of mass, one cannot in general conclude anything about the shape of the wave function Ψ_κ ($\kappa = 1, 2$) of the two component systems, beyond consistency relations between expectation values, e.g.,

$$\langle P_1 \rangle + \langle P_2 \rangle = \langle P_{\rm CM} \rangle, \quad \mu_1 \langle Q_1 \rangle + \mu_2 \langle Q_2 \rangle = \langle Q_{\rm CM} \rangle, \tag{2.3}$$

$$\begin{aligned} &\langle (P_1 - \langle P_1 \rangle)^2 \rangle + \langle (P_2 - \langle P_2 \rangle)^2 \rangle \\ &\quad = \langle (P_{\rm CM} - \langle P_{\rm CM} \rangle)^2 \rangle, \\ &\mu_1^2 \langle (Q_1 - \langle Q_1 \rangle)^2 \rangle + \mu_2^2 \langle (Q_2 - \langle Q_2 \rangle)^2 \rangle \\ &\quad = \langle (Q_{\rm CM} - \langle Q_{\rm CM} \rangle)^2 \rangle, \end{aligned} \tag{2.4}$$

where μ_κ ($\kappa = 1,2$) are as in (2.2). To establish (2.3) one uses the linearity of the state, while to establish (2.4) one also uses the fact that when the two systems Σ_1 and Σ_2 are prepared independently, there are (by definition) no correlations between the observables A_1 relative to Σ_1 and the observable A_2 relative to Σ_2.

The purpose of this section is to show that if in addition $\Psi_{\rm CM}$ describes a coherent state, centered around the point $\{\langle P_{\rm CM} \rangle, \langle Q_{\rm CM} \rangle\}$ in the classical center-of-mass phase space, then each of the component system must be in a pure coherent state, centered precisely around the points $\{\langle P_\kappa \rangle, \langle Q_\kappa \rangle\}$ ($\kappa = 1,2$) satisfying (2.3), and with dispersion parameter λ_κ given by the now unique solution of (2.4), namely,

$$\lambda_\kappa = \mu_\kappa \lambda \quad (\text{with } \kappa = 1, 2), \tag{2.5}$$

where λ is the dispersion parameter of $\Psi_{\rm CM}$, determined uniquely from

$$\begin{aligned} \langle (P_{\rm CM} - \langle P_{\rm CM} \rangle)^2 \rangle &= \lambda \hbar / 2, \\ \langle (Q_{\rm CM} - \langle Q_{\rm CM} \rangle)^2 \rangle &= \lambda^{-1} \hbar / 2. \end{aligned} \tag{2.6}$$

As a consequence, the wave functions Ψ_1 and Ψ_2 will inherit both the Gaussian character of $\Psi_{\rm CM}$ *and* minimal dispersion, i.e., equality sign in the Heisenberg uncertainty relation,

$$\langle (P_\kappa - \langle P_\kappa \rangle)^2 \rangle \langle (Q_\kappa - \langle Q_\kappa \rangle)^2 \rangle = \hbar^2 / 4, \tag{2.7}$$

for $\kappa = 1,2$. We have, in fact,

$$\begin{aligned} \langle (P_\kappa - \langle P_\kappa \rangle)^2 \rangle &= \lambda_\kappa (\hbar / 2), \\ \langle (Q_\kappa - \langle Q_\kappa \rangle)^2 \rangle &= \lambda_\kappa^{-1} (\hbar / 2), \end{aligned} \tag{2.8}$$

with λ_κ as in (2.5).

As discussed in Sec. I, this is the closest one can possibly come to the classical result: when the scale of the phenomena one observes is such that $\hbar$ can be neglected, our quantum states are well approximated by the corresponding, dispersion-free classical states.

We now turn to the mathematical formulation of these results. For the general mathematical concepts underlying the following brief presentation, see, e.g., §8.3 and §9.1 in Ref. 4. In order to streamline our nomenclature, we systematically use the following abbreviations. By an "algebra" $\mathscr{A}$ we mean a W^*-algebra, with unit denoted by I, i.e., a C^*-algebra (with unit) that is the dual of a Banach space $\mathscr{A}_*$; by a "state" ϕ on $\mathscr{A}$, we mean a completely additive state, i.e., a positive linear functional

$$\phi\colon A \in \mathscr{A} \mapsto \langle \phi; A \rangle \in \mathbb{C} \tag{2.9}$$

that is normalized to 1 and belongs to $\mathscr{A}_*$ (these states are called "normal" in the literature on von Neumann algebras; we will, however, avoid this adjective here, as it may create confusion with the concept of "normal" distribution, familiar in the literature on classical statistics to which we also refer). The following particular case will be of central interest in the sequel: if $\mathscr{A}$ is (isomorphic, as a W^*-algebra, to) the algebra $\mathscr{B}(\mathscr{H})$ of all bounded linear operators on a separable Hilbert space $\mathscr{H}$, then completely additive states ϕ on $\mathscr{A}$ characterized by the fact that they are of the following form, familiar to physicists:

$$\phi\colon A \in \mathscr{A} \mapsto \operatorname{tr}(\rho A) \in \mathbb{C}, \tag{2.10}$$

where ρ is a density matrix, i.e., ρ is a positive trace-class operator on $\mathscr{H}$, of trace 1, uniquely determined by ϕ. In the present section, we are primarily concerned with (completely additive!) states ϕ on $\mathscr{A} \simeq \mathscr{B}(\mathscr{H})$ which are *pure*, i.e., states for which ρ is a one-dimensional projector, and we denote by Φ any unit vector in the range of ρ. For this section and the next, it is nevertheless useful to recall that for every state (whether pure or not) ϕ on $\mathscr{A}$, there exists a representation, unique up to unitary equivalence,

$$\pi_\phi\colon A \in \mathscr{A} \mapsto \pi_\phi(A) \in \mathscr{B}(\mathscr{H}_\phi) \tag{2.11}$$

called the GNS representation canonically associated to ϕ and characterized by the existence of a vector $\Phi \in \mathscr{H}_\phi$, such that

$$(\Phi, \pi_\phi(A)\Phi) = \langle \phi; A \rangle, \quad \forall A \in \mathscr{A}, \tag{2.12}$$

$$\overline{\mathrm{Span}}\,\{\pi_\phi(A)\Phi|A\in\mathscr{A}\}=\mathscr{H}_\phi. \tag{2.13}$$

While the existence of the GNS representation does not require that ϕ be completely additive, the latter property ensures that π_ϕ is ultraweakly continuous so that $\pi_\phi(\mathscr{A})$ is a W^*-algebra. If in addition the state ϕ on $\mathscr{A}\simeq\mathscr{B}(\mathscr{H})$ is faithful, i.e., if

$$\langle\phi;A^*A\rangle=0 \quad \text{implies} \quad A=0, \tag{2.14}$$

then $\pi_\phi(\mathscr{A})$ is a factor, i.e.,

$$\pi_\phi(\mathscr{A})\cap\pi_\phi(\mathscr{A})'=\mathbb{C}I, \tag{2.15}$$

isomorphic to its commutant

$$\pi_\phi(\mathscr{A})'\equiv\{B\in\mathscr{B}(\mathscr{H}_\phi)\,|\,[B,\pi_\phi(A)]=0,\ \forall A\in\mathscr{A}\}. \tag{2.16}$$

Moreover, every (completely additive!) state ψ on $\mathscr{A}\simeq\mathscr{B}(\mathscr{H})$ is then a vector state for this representation, i.e., there exists a vector $\Psi\in\mathscr{H}_\phi$ such that

$$(\Psi,\pi_\phi(A)\Psi)=\langle\psi;A\rangle, \quad \forall A\in\mathscr{A}. \tag{2.17}$$

Finally, by the Weyl CCR algebra for a particle with one degree of freedom, we mean the abstract W^*-algebra, defined by its realization on $L^2(\mathbb{R},dx)$, namely,

$$\mathscr{A}=\{e^{i(uP+vQ)}|u,v\in\mathbb{R}\}'', \tag{2.18}$$

where P and Q are the self-adjoint operators defined by their restriction to the Schwartz space $\mathscr{S}(\mathbb{R})$, i.e.,

$$(P\Psi)(x)=-i\hbar(\partial_x\Psi)(x),$$
$$(Q\Psi)(x)=x\Psi(x).$$

For two particles of mass m_1 and m_2 with Weyl CCR algebras $\mathscr{A}_1$ and $\mathscr{A}_2$, the Weyl CCR algebra $\mathscr{A}_{\mathrm{CM}}$ for the center-of-mass motion is the subalgebra of $\mathscr{A}_1\otimes\mathscr{A}_2$ generated, in the L^2-realization, by

$$\{e^{i(uP_{\mathrm{CM}}+vQ_{\mathrm{CM}})}|u,v\in\mathbb{R}\}, \tag{2.19}$$

where, in analogy to (2.1) and (2.2),

$$\begin{aligned}P_{\mathrm{CM}}&\equiv P_1\otimes I+I\otimes P_2,\\ Q_{\mathrm{CM}}&\equiv\mu_1Q_1\otimes I+\mu_2I\otimes Q_2.\end{aligned} \tag{2.20}$$

The results of this section can now be expressed mathematically as follows.

Theorem 2.1: Let $\mathscr{A}_\kappa$ $(\kappa=1,2)$ be the Weyl CCR algebras for two particles with one degree of freedom; let $\mathscr{A}_{\mathrm{CM}}$ be the Weyl CCR algebra for the center-of-mass motion; let ϕ_κ be a state on $\mathscr{A}_\kappa$ $(\kappa=1,2)$; let

$$\phi_0\equiv\phi_1\otimes\phi_2 \quad \text{on} \quad \mathscr{A}_0\equiv\mathscr{A}_1\otimes\mathscr{A}_2; \tag{2.21}$$

and let ϕ_{CM} be the restriction of ϕ_0 to $\mathscr{A}_{\mathrm{CM}}\subset\mathscr{A}_0$, i.e.,

$$\phi_{\mathrm{CM}}\equiv\phi_0\restriction\mathscr{A}_{\mathrm{CM}}\equiv\phi_1*\phi_2. \tag{2.22}$$

If ϕ_{CM} is a pure coherent state, then ϕ_1 and ϕ_2 are also pure coherent states.

The relations between the characteristic parameters of ϕ_1 and ϕ_2 and those of ϕ_{CM} are now specified.

Corollary 2.2 With the notation of the theorem, the pure coherent state ϕ_{CM} is completely described by

$$\langle\phi_{\mathrm{CM}};e^{-i(uP_{\mathrm{CM}}+vQ_{\mathrm{CM}})}\rangle =\exp\{-\hbar(\lambda_{\mathrm{CM}}u^2+\lambda_{\mathrm{CM}}^{-1}v^2)/4\}e^{-i(u\langle P_{\mathrm{CM}}\rangle+v\langle Q_{\mathrm{CM}}\rangle)} \tag{2.23}$$

valid for all u, $v\in\mathbb{R}$; the characteristic parameters $\langle P_{\mathrm{CM}}\rangle$, $\langle Q_{\mathrm{CM}}\rangle$, and λ_{CM} of ϕ_{CM} are determined by the relations

$$\langle P_{\mathrm{CM}}\rangle=\langle\phi_{\mathrm{CM}};P_{\mathrm{CM}}\rangle,\quad \langle Q_{\mathrm{CM}}\rangle=\langle\phi_{\mathrm{CM}};Q_{\mathrm{CM}}\rangle, \tag{2.24}$$

$$\begin{aligned}\langle\phi_{\mathrm{CM}};(P_{\mathrm{CM}})-\langle P_{\mathrm{CM}}\rangle)^2\rangle&=\lambda_{\mathrm{CM}}\hbar/2,\\ \langle\phi_{\mathrm{CM}};(Q_{\mathrm{CM}}-\langle Q_{\mathrm{CM}}\rangle)^2\rangle&=\lambda_{\mathrm{CM}}^{-1}\hbar/2.\end{aligned} \tag{2.25}$$

The ϕ_κ $(\kappa=1,2)$ are then of the form

$$\langle\phi_\kappa;e^{-i(uP_{\mathrm{CM}}+vQ_{\mathrm{CM}})}\rangle =\exp\{-\hbar(\lambda_\kappa u^2+\lambda_\kappa^{-1}v^2)/4\}e^{-i(u\langle P_\kappa\rangle+v\langle Q_\kappa\rangle)}, \tag{2.26}$$

where

$$\lambda_\kappa=\mu_\kappa\lambda_{\mathrm{CM}},\quad \mu_\kappa=m_\kappa/m_{\mathrm{CM}},\quad m_{\mathrm{CM}}=m_1+m_2, \tag{2.27}$$

$$\langle P_\kappa\rangle=\langle\phi_\kappa;P_\kappa\rangle,\quad \langle Q_\kappa\rangle=\langle\phi_\kappa;Q_\kappa\rangle, \tag{2.28}$$

and

$$\begin{aligned}\langle P_1\rangle+\langle P_2\rangle&=\langle P_{\mathrm{CM}}\rangle,\\ \mu_1\langle Q_1\rangle+\mu_2\langle Q_2\rangle&=\langle Q_{\mathrm{CM}}\rangle.\end{aligned} \tag{2.29}$$

The following information on the state of relative motion is also available.

Corollary 2.3: With the notation and assumptions of the theorem and with ϕ_{rel} the restriction of ϕ_0 to $\mathscr{A}_{\mathrm{rel}}\subset\mathscr{A}_0$, we have that ϕ_0 is also a pure coherent state, and

$$\phi_0=\phi_{\mathrm{CM}}\otimes\phi_{\mathrm{rel}}. \tag{2.30}$$

Note that (2.30) means physically that there are no correlations between the observables for the center of mass and those for the relative motion.

The above three results follow directly from the noncommutative extension of the classical Cramer theorem obtained in Ref. 6, the essence of which, for the case of interest here, is captured in Lemma 4.1 below.

III. THERMAL COHERENT STATES

Let Σ be a *classical* ideal gas in canonical equilbrium at inverse temperature β; its partition function Z and density function f are thus, by definition

$$Z=\int\cdots\int dp_1\cdots dp_N\,dq_1\cdots dq_N \times\exp[-\beta H(p_1,\ldots,p_N,q_1,\ldots,q_N)], \tag{3.1}$$

$$f(p_1,\ldots,p_N,q_1,\ldots,q_N) =Z^{-1}\exp[-\beta H(p_1,\ldots,p_N,q_1,\ldots,q_N)], \tag{3.2}$$

with

$$H(p_1,\ldots,p_N,q_1,\ldots,q_N)=\sum_{\kappa=1}^N H_\kappa(p_\kappa,q_\kappa) \tag{3.3}$$

and, for $\kappa=1,2,\ldots,N$,

$$H_\kappa(p_\kappa,q_\kappa)=(1/2m_\kappa)p_\kappa^2+V_\kappa(x_\kappa). \tag{3.4}$$

Suppose now that the center of mass of this ideal gas is observed to be distributed according to the canonical equilibrium density of a harmonic oscillator, i.e,

$$f_{\mathrm{CM}}(p_{\mathrm{CM}},q_{\mathrm{CM}})=Z_{\mathrm{CM}}^{-1}e^{-\beta H_{\mathrm{CM}}(p_{\mathrm{CM}},q_{\mathrm{CM}})}, \tag{3.5}$$

with

$$H_{CM}(p_{CM},q_{CM}) = (1/2m_{CM})p_{CM}^2 + \tfrac{1}{2}k_{CM}q_{CM}^2, \tag{3.6}$$

$$Z_{CM} = \int\int dp_{CM}\, dq_{CM} \exp\{-\beta H_{CM}(p_{CM},q_{CM})\}. \tag{3.7}$$

It then follows, by repeated application of the classical Cramer theorem, that the situation described by (3.5)–(3.7) occurs if and only if the individual particles of the ideal gas are displaced harmonic oscillators, in equilibrium at the inverse temperature β. Specifically, one finds, for $\kappa = 1,2,...,N$,

$$H_\kappa = (1/2m_\kappa)(p_\kappa - \langle p_\kappa\rangle)^2 + \tfrac{1}{2}k_\kappa(q_\kappa - \langle q_\kappa\rangle)^2, \tag{3.8}$$

with

$$\sum_{\kappa=1}^{N} m_\kappa = m_{CM}, \tag{3.9}$$

$$\sum_{\kappa=1}^{N} \mu_\kappa \omega_\kappa^{-2} = \omega_{CM}^{-2}, \tag{3.10}$$

$$\sum_{\kappa=1}^{N} \mu_\kappa \langle q_\kappa\rangle = 0 \quad\text{and}\quad \sum_{\kappa=1}^{N} \langle p_\kappa\rangle = 0, \tag{3.11}$$

where

$$\mu_\kappa = m_\kappa/m_{CM} \tag{3.12}$$

$$\omega_\kappa^2 = k_\kappa/m_\kappa \quad\text{and}\quad \omega_{CM}^2 = k_{CM}/m_{CM}. \tag{3.13}$$

Note that

$$\omega = \omega_{CM}, \quad \forall \kappa = 1,2,...,N, \tag{3.14}$$

is always a solution of (3.10) with μ_κ defined by (3.12) and (3.9). Mathematically, this particular solution is characterized by the condition that the independent, $\mathbb{R}^2$-valued random variables

$$(\tilde{p}_\kappa,\tilde{q}_\kappa), \quad \kappa = 1,2,...,N, \tag{3.15}$$

defined by

$$\tilde{p}_\kappa = \mu_\kappa^{-1/2} p_\kappa, \tag{3.16}$$

$$\tilde{q}_\kappa = \mu_\kappa^{1/2}\hat{q}_\kappa \quad\text{with}\quad \hat{q}_\kappa = q_\kappa - \langle q_\kappa\rangle \tag{3.17}$$

be identically distributed, with density

$$f(p,q) = (2\pi)^{-1}\beta\omega_{CM}^{1/2} \times \exp[-\beta((1/2m_{CM})p^2 + \tfrac{1}{2}k_{CM}q^2)]. \tag{3.18}$$

Alternatively, this condition can be expressed by saying that for any pair $\kappa_1 \neq \kappa_2$ of indices 1, 2,...,N, the two $\mathbb{R}^2$-valued random variables

$$(\bar{p}_{CM},\bar{q}_{CM}) \quad\text{and}\quad (\bar{p}_{rel},\bar{q}_{rel}) \tag{3.19}$$

are statistically independent, where

$$\bar{p}_{CM} = p_{\kappa_1} + p_{\kappa_2}, \tag{3.20}$$

$$\bar{q}_{CM} = \bar{\mu}_1\hat{q}_{\kappa_1} + \bar{\mu}_2\hat{q}_{\kappa_2}, \tag{3.21}$$

$$\bar{p}_{rel} = \bar{m}_{rel}\left(\frac{1}{m_{\kappa_1}}p_{\kappa_1} - \frac{1}{m_{\kappa_2}}p_{\kappa_2}\right), \tag{3.22}$$

$$\bar{q}_{rel} = \hat{q}_{\kappa_1} - \hat{q}_{\kappa_2} \tag{3.23}$$

with

$$\bar{\mu}_\kappa = m_\kappa/\bar{m}_{CM} \quad (\kappa = \kappa_1,\kappa_2), \tag{3.24}$$

$$\bar{m}_{CM} = m_{\kappa_1} + m_{\kappa_2}, \tag{3.25}$$

$$m_{rel} = m_{\kappa_1}m_{\kappa_2}/\bar{m}_{CM}. \tag{3.26}$$

Physically, the condition (3.14) means thus that for every pair $\Sigma_{\kappa_1} \neq \Sigma_{\kappa_2}$ of oscillators in the gas one has

$$H_{\kappa_1}(p_{\kappa_1},q_{\kappa_1}) + H_{\kappa_2}(p_{\kappa_2},q_{\kappa_2}) = \bar{H}_{CM}(\bar{p}_{CM},\bar{q}_{CM}) + \bar{H}_{rel}(\bar{p}_{rel},\bar{q}_{rel}), \tag{3.27}$$

where $\bar{H}_{CM}$ and $\bar{H}_{rel}$ are harmonic oscillator Hamiltonians. Specifically

$$\bar{H}_{CM} = (1/2\bar{m}_{CM})\bar{p}_{CM}^{\,2} + \tfrac{1}{2}\bar{k}_{CM}\bar{q}_{CM}^{\,2}, \tag{3.28}$$

$$\bar{H}_{rel} = (1/2\bar{m}_{rel})\bar{p}_{rel}^{\,2} + \tfrac{1}{2}\bar{k}_{rel}\bar{q}_{rel}^{\,2}, \tag{3.29}$$

where the masses $\bar{m}_{CM}$ and $\bar{m}_{rel}$ are defined in (3.25) and (3.26) and the oscillator strengths $\bar{k}_{CM}$ and $\bar{k}_{rel}$ are given by

$$\bar{k}_{CM}/\bar{m}_{CM} = \omega_{CM}^2 = \omega_{rel}^2 = k_{rel}/m_{rel}. \tag{3.30}$$

The purpose of this section is to analyze the corresponding *quantum* situation. Let

$$V\colon x\in\mathbb{R} \mapsto V(x)\in\mathbb{R} \tag{3.31}$$

be such that

$$H = -\hbar^2\frac{1}{2m}\frac{d^2}{dx^2} + V(x) \tag{3.32}$$

defines a self-adjoint operator in $\mathscr{H} = \mathscr{L}^2(\mathbb{R},dx)$ with $\exp(-\beta H)$ of trace class for all $\beta > 0$.

The density matrix

$$\rho = Z^{-1}\exp(-\beta H) \tag{3.33}$$

with

$$Z = \mathrm{Tr}\exp(-\beta H) \tag{3.34}$$

is then interpreted as the canonical equilibrium state, at inverse temperature β, of a quantum particle in the potential V. In particular, for a harmonic oscillator

$$H = -\hbar^2\frac{1}{2m}\frac{d^2}{dx^2} + \frac{1}{2}kx^2, \tag{3.35}$$

the state

$$\phi\colon B\in\mathscr{B}(\mathscr{H}) \mapsto \mathrm{Tr}\,\rho B\in\mathbb{C} \tag{3.36}$$

is faithful and is uniquely determined by its restriction on the Weyl algebra; specifically, with P and Q defined as in (2.19 and 2.20) one has[4]

$$\langle\phi;\exp[-i(uP + vQ)]\rangle = \exp\{-\Theta(\lambda u^2 + \lambda^{-1}v^2)/4\}, \tag{3.37}$$

where

$$\Theta = \hbar\coth(\beta\hbar\omega/2) \tag{3.38}$$

$$\lambda = m\omega \quad\text{with}\quad \omega^2 = k/m. \tag{3.39}$$

It is worthwhile for the sequel to note that (i) ϕ is Gaussian; (ii) one recovers the classical result

$$\lim_{\hbar\to 0}\langle P^2\rangle = \lim_{\hbar\to 0}\Theta\lambda/2 = m/\beta, \qquad \lim_{\hbar\to 0}\langle Q^2\rangle = \lim_{\hbar\to 0}\Theta\lambda^{-1}/2 = 1/k\beta; \tag{3.40}$$

and (iii) one recovers the low-temperature limit of Sec. II, namely,

$$\lim_{\beta\to\infty} \langle P^2\rangle\langle Q^2\rangle = \hbar^2/4. \tag{3.41}$$

We therefore extend the definition of coherent states on the Weyl CCR algebra for one degree of freedom to include states that satisfy

$$\langle\phi;\exp[-i(uP+vQ)]\rangle = \exp\{-\Theta(\lambda u^2+\lambda^{-1}v^2)/4\}e^{-i(u\langle P\rangle + v\langle Q\rangle)} \tag{3.42}$$

with $\Theta/\hbar \geqslant 1$; such states are pure coherent states (in the sense of Sec. II) if and only if $\Theta/\hbar = 1$, i.e., $\beta = \infty$ in Eq. (3.38). The following result is stated for two-particle systems although it extends trivially, as does its classical counterpart, to an n-particle system.

Theorem 3.1: Let $\mathscr{A}_\kappa$ (with $\kappa = 1,2,\mathrm{CM}$) be as in Theorem 2.1. For $\kappa = 1,2$ let ϕ_κ be a state on $\mathscr{A}_\kappa$, and let

$$\phi_0 \equiv \phi_1 \otimes \phi_2 \quad \text{on} \quad \mathscr{A}_0 \equiv \mathscr{A}_1 \otimes \mathscr{A}_2. \tag{3.43}$$

Then the restriction ϕ_{CM} of ϕ_0 to $\mathscr{A}_{\mathrm{CM}}$ is a coherent state of the form

$$\begin{aligned}\langle\phi_{\mathrm{CM}};\exp[-i(uP_{\mathrm{CM}}+vQ_{\mathrm{CM}})]\rangle &= \exp\{-\Theta_{\mathrm{CM}}(\lambda_{\mathrm{CM}}u^2 \\ &\quad + \lambda_{\mathrm{CM}}^{-1}v^2)/4\}e^{-i(u\langle P_{\mathrm{CM}}\rangle + v\langle Q_{\mathrm{CM}}\rangle)}, \\ &\quad \text{with} \quad \Theta_{\mathrm{CM}}/\hbar \geqslant 1 \quad \text{and} \quad \lambda_{\mathrm{CM}} > 0,\end{aligned} \tag{3.44}$$

if and only if ϕ_κ $(\kappa = 1,2)$ are coherent states of the form

$$\begin{aligned}\langle\phi_\kappa;\exp[-i(uP_\kappa+vQ_\kappa)]\rangle &= \exp\{-\Theta_\kappa(\lambda_\kappa u^2+\lambda_\kappa^{-1}v^2)/4\} \\ &\quad \times\exp[-i(u\langle P_\kappa\rangle + v\langle Q_\kappa\rangle)], \\ &\quad \text{with} \quad \Theta_\kappa/\hbar \geqslant 1 \quad \text{and} \quad \lambda_\kappa > 0\end{aligned} \tag{3.45}$$

with the compatibility relations

$$\begin{aligned}&\langle P_1\rangle + \langle P_2\rangle = \langle P_{\mathrm{CM}}\rangle, \\ &\mu_1\langle Q_1\rangle + \mu_2\langle Q_2\rangle = \langle Q_{\mathrm{CM}}\rangle,\end{aligned} \tag{3.46}$$

$$\begin{aligned}&\Theta_1\lambda_1 + \Theta_2\lambda_2 = \Theta_{\mathrm{CM}}\lambda_{\mathrm{CM}}, \\ &\mu_1^2\Theta_1\lambda_1^{-1} + \mu_2^2\Theta_2\lambda_2^{-1} = \Theta_{\mathrm{CM}}\lambda_{\mathrm{CM}}^{-1},\end{aligned} \tag{3.47}$$

where

$$\mu_\kappa = m_\kappa/m_{\mathrm{CM}} \quad \text{and} \quad m_{\mathrm{CM}} = m_1 + m_2. \tag{3.48}$$

The physical meaning of the compatibility condition (3.47) is given by the following result.

Scholium 3.2: With (3.43)–(3.46) taken into account, (3.47) is equivalent to

$$\begin{aligned}&\langle(P_1-\langle P_1\rangle)^2\rangle + \langle(P_2-\langle P_2\rangle)^2\rangle \\ &\quad = \langle(P_{\mathrm{CM}}-\langle P_{\mathrm{CM}}\rangle)^2\rangle, \\ &\mu_1^2\langle(Q_1-\langle Q_1\rangle)^2 + \mu_2^2\langle(Q_2-\langle Q_2\rangle)^2\rangle \\ &\quad = \langle(Q_{\mathrm{CM}}-\langle Q_{\mathrm{CM}}\rangle)^2\rangle,\end{aligned} \tag{3.49}$$

where μ_1 and μ_2 are given by (3.48)

Note that these results are in conformity with the classical results; see in particle (3.9) and (3.10).

The results of Sec. II ("low-temperature limit") are recovered from (3.45) and (3.46) and the following consequence of (3.47).

Scholium 3.3: With the notation of Theorem 3.1, the following two conditions are equivalent:

$$\Theta_{\mathrm{CM}} = \hbar, \tag{3.50}$$

$$\text{for } \kappa = 1,2, \quad \Theta_\kappa = \hbar \quad \text{and} \quad \lambda_\kappa = \mu_\kappa\lambda. \tag{3.51}$$

The following change of variables allows us to interpret our results in terms of canonical equilibrium states of harmonic oscillators, in particular in the nontrivial Corollary 3.5.

Scholium 3.4: With the notation of Theorem 3.1, there exist (for $\kappa = 1,2,\mathrm{CM}$) $\beta_\kappa \in (0,\infty]$ and $\omega_\kappa \in (0,\infty)$ such that

$$\Theta_\kappa = \hbar\coth(\beta_\kappa\hbar\omega_\kappa/2), \quad \lambda_\kappa = m_\kappa\omega_\kappa. \tag{3.52}$$

Corollary 3.5: With the notations of Theorem 3.1 and Scholium 3.4 assume that

$$\beta_1 = \beta_2 \equiv \beta \in (0,\infty), \tag{3.53}$$

and

$$\text{either} \quad \beta_{\mathrm{CM}} = \beta \quad \text{or} \quad \omega_1 = \omega_2 \equiv \omega. \tag{3.54}$$

Then

$$\omega_1 = \omega_2 = \omega_{\mathrm{CM}} \quad \text{and} \quad \beta_1 = \beta_2 = \beta_{\mathrm{CM}}, \tag{3.55}$$

and

$$\phi_0 = \phi_{\mathrm{CM}} \otimes \phi_{\mathrm{rel}}, \tag{3.56}$$

where ϕ_{rel} is the coherent state

$$\begin{aligned}\langle\phi_{\mathrm{rel}};\exp[-i(uP_{\mathrm{rel}}+vQ_{\mathrm{rel}})]\rangle &= \exp\{-\Theta_{\mathrm{rel}}(\lambda_{\mathrm{rel}}u^2+\lambda_{\mathrm{rel}}^{-1}v^2)/4\} \\ &\quad \times\exp[-i(u\langle P_{\mathrm{rel}}\rangle + v\langle Q_{\mathrm{rel}}\rangle)],\end{aligned} \tag{3.57}$$

with

$$\begin{aligned}&\Theta_{\mathrm{rel}} = \hbar\coth(\beta_{\mathrm{rel}}\hbar\omega_{\mathrm{rel}}/2), \\ &\omega_{\mathrm{rel}} = \omega_{\mathrm{CM}}, \quad \beta_{\mathrm{rel}} = \beta_{\mathrm{CM}}, \\ &\lambda_{\mathrm{rel}} = [m_1m_2/(m_1+m_2)]\lambda_{\mathrm{CM}}.\end{aligned} \tag{3.58}$$

IV. PROOFS

The proofs of Theorem 2.1 and of its Corollaries 2.2 and 2.3 follow directly from the introductory remarks presented in Sec. I–see in particular (1.5) and (1.6) and (1.9) and (1.10)–and the next simple lemma, an analog of Lemma 2.2 in Ref. 6. The reader interested in domain questions may consult Lemma 2.1 in Ref. 6.

Lemma 4.1: With the notation and assumptions of Theorem 2.1, let (for $\kappa = 1, 2$) π_κ be the GNS representation of $\mathscr{A}_\kappa$ associated to ϕ_κ, let Φ_κ be the corresponding cyclic vector, and let

$$\begin{aligned}&\widehat{P}_\kappa \equiv \pi_\kappa(P_\kappa) - \langle P_\kappa\rangle, \\ &\widehat{Q}_\kappa \equiv \pi_\kappa(Q_\kappa) - \langle Q_\kappa\rangle, \\ &\hat{a}_\kappa \equiv \widehat{Q}_\kappa + i\lambda_\kappa^{-1}\widehat{P}_\kappa.\end{aligned}$$

Then, for λ_κ as in (2.27), one has

$$a_\kappa\Phi_\kappa = 0 \quad (\kappa = 1,2).$$

Proof: Let π_κ $(\kappa = 1, 2, 0)$ be the GNS representation of $\mathscr{A}_\kappa$ associated with ϕ_κ, and let Φ_κ be the corresponding cyclic vector. Note that

$$\Phi_0 = \Phi_1 \otimes \Phi_2. \tag{4.1}$$

Define

$$\hat{P}_{\rm CM} \equiv \pi_0(P_{\rm CM}) - \langle P_{\rm CM} \rangle,$$
$$\hat{Q}_{\rm CM} \equiv \pi_0(Q_{\rm CM}) - \langle Q_{\rm CM} \rangle, \quad (4.2)$$
$$\hat{a}_{\rm CM} \equiv \hat{Q}_{\rm CM} + i\lambda_{\rm CM}^{-1}\hat{P}_{\rm CM},$$

and, for $\kappa = 1,2$,

$$\hat{P}_\kappa \equiv \pi_\kappa(P_\kappa) - \langle P_\kappa \rangle,$$
$$\hat{Q}_\kappa \equiv \pi_\kappa(Q_\kappa) - \langle Q_\kappa \rangle, \quad (4.3)$$
$$\hat{a}_\kappa \equiv \hat{Q}_\kappa + i\lambda_\kappa^{-1}P_\kappa,$$

where

$$\langle P_\kappa \rangle \equiv \langle \phi_\kappa; P_\kappa \rangle, \quad \langle Q_\kappa \rangle \equiv \langle \phi_\kappa; Q_\kappa \rangle,$$
$$\lambda_\kappa \equiv \mu_\kappa \lambda_{\rm CM}, \quad \mu_\kappa \equiv m_\kappa/(m_1 + m_2). \quad (4.4)$$

We then have

$$\hat{a}_{\rm CM} = \mu_1 \hat{a}_1 \otimes I + \mu_2 I \otimes \hat{a}_2, \quad (4.5)$$

$$(\Phi_\kappa, \hat{a}_\kappa \Phi_\kappa) = 0 \quad (\kappa = 1,2) \quad (4.6)$$

and, from the fact that $\phi_{\rm CM}$ is a pure coherent state,

$$\hat{a}_{\rm CM}\Phi_0 = 0. \quad (4.7)$$

Upon inserting (4.5) and (4.7), taking the norm of the resulting expression, and taking (4.6) into account, we obtain

$$\mu_1^2 ||\hat{a}_1\Phi_1||^2 + \mu_2^2 ||\hat{a}_2\Phi_2||^2 = 0 \quad (4.8)$$

and thus, since $\mu_\kappa > 0$,

$$\hat{a}_\kappa \Phi_\kappa = 0 \quad (\kappa = 1,2). \quad (4.9)$$

This proves Lemma 4.1.

The proof of (3.45) in Theorem (3.1) is a straightforward application of the general quantum version of Cramer's Theorem established by one of us.[6,7] The consistency relations (3.46)–(3.48) follow then by inspection, we replace $P_{\rm CM}$ and $Q_{\rm CM}$ in (3.44) by their definition (2.20) and match then (3.44) and (3.45), taking into account (3.43).

Scholium 3.2 follows immediately from (3.44), (3.45), and (1.2).

Proof of Scholium 3.3: We multiply the two equations in (3.47) by one another to obtain

$$(\mu_1\Theta_1 + \mu_2\Theta_2)^2 + \mu_1\mu_2\Theta_1\Theta_2(\lambda_1\lambda_2)^{-1}(\mu_2\lambda_1 - \mu_1\lambda_2)^2 = \Theta_{\rm CM}^2. \quad (4.10)$$

From the facts that $\Theta_{\rm CM} = \hbar$, $\mu_1 + \mu_2 = 1$, and $\Theta_\kappa \geqslant \hbar$ ($\kappa = 1, 2$), we conclude from (4.10) that

$$\Theta_\kappa = \hbar \quad (\kappa = 1,2) \quad (4.11)$$

and

$$\mu_1^{-1}\lambda_1 = \mu_2^{-1}\lambda_2. \quad (4.12)$$

Upon inserting (4.12) in the first (or the second) of the consistency relations (3.47), upon taking into account that $\mu_1 + \mu_2 = 1$ and that $\Theta_1 = \Theta_2 = \Theta_{\rm CM}$, we obtain

$$\lambda_\kappa = \mu_\kappa \lambda \quad (\kappa = 1,2). \quad (4.13)$$

This completes the proof.

Scholium 3.4 is only an adaptation of the change of variables (1.14) to the situation now under consideration. Note that $\beta_\kappa = \infty$ corresponds to the pure case $\Theta_\kappa = 1$.

Proof of Corollary 3.5: With the change of variables (3.52) the consistency relation (3.47) reads

$$\mu_1\Theta_1\omega_1 + \mu_2\theta_2\omega_2 = \Theta_{\rm CM}\omega_{\rm CM},$$
$$\mu_1\Theta_1\omega_1^{-1} + \mu_2\Theta_2\omega_2^{-1} = \Theta_{\rm CM}\omega_{\rm CM}^{-1}, \quad (4.14)$$

with

$$\Theta_\kappa = \hbar \coth(\beta_\kappa \hbar\omega_\kappa/2) \quad (\kappa = 1,2,{\rm CM}). \quad (4.15)$$

In case we assume

$$\beta_1 = \beta_2 = \beta_{\rm CM} \equiv \beta \in (0,\infty), \quad (4.16)$$

it is useful to write the consistency equations (4.14) in the vector form

$$\mu_1 X(\omega_1) + \mu_2 X(\omega_2) = X(\omega_{\rm CM}), \quad (4.17)$$

with

$$X(\omega) \equiv \begin{pmatrix} \Xi(\omega) \\ \xi(\omega) \end{pmatrix} \quad (4.18)$$

and

$$\Xi\colon \omega \in (0,\infty) \mapsto \omega \coth(\beta\hbar\omega/2) \in (0,\infty),$$
$$\xi\colon \omega \in (0,\infty) \mapsto \omega^{-1} \coth(\beta\hbar\omega/2) \in (0,\infty). \quad (4.19)$$

Upon noticing that ξ is bijective, we can use ξ as a variable, and define

$$\hat{\Xi}(\xi) \equiv \Xi \circ \omega(\xi) \quad (4.20)$$

and

$$\hat{X}(\xi) \equiv \begin{pmatrix} \hat{\Xi}(\xi) \\ \xi \end{pmatrix}. \quad (4.21)$$

We then verify that

$$\frac{d^2}{d\xi^2}\hat{\Xi}(\xi) > 0, \quad (4.22)$$

i.e., that $\hat{\Xi}$ is strictly convex. As a consequence, the equation

$$\mu_1 \hat{X}(\xi_1) + \mu_2 \hat{X}(\xi_2) = \hat{X}(\xi_{\rm CM}), \quad (4.23)$$

where

$$\mu_\kappa > 0 \quad \text{and} \quad \mu_1 + \mu_2 = 1, \quad (4.24)$$

admits a unique solution, namely,

$$\xi_1 = \xi_2 = \xi_{\rm CM}, \quad (4.25)$$

i.e.,

$$\omega_1 = \omega_2 = \omega_{\rm CM}. \quad (4.26)$$

We have thus proven the first part of (3.54) and (3.55).

If we now assume

$$\beta_1 = \beta_2 \equiv \beta \in (0,\infty] \quad \text{and} \quad \omega_1 = \omega_2 \equiv \omega \in (0,\infty), \quad (4.27)$$

we have

$$\Theta_1 = \Theta_2 \equiv \Theta \equiv \hbar \coth(\beta\hbar\omega/2), \quad (4.28)$$

so that (4.26) reduces to

$$\Theta\omega = \Theta_{\rm CM}\omega_{\rm CM}, \quad \Theta\omega^{-1} = \Theta_{\rm CM}\omega_{\rm CM}^{-1}, \quad (4.29)$$

from which we obtain, upon using (4.15) and (4.28),

$$\Theta_{\rm CM} = \Theta, \quad \omega_{\rm CM} = \omega, \quad \beta_{\rm CM} = \beta. \quad (4.30)$$

We have thus proved (3.55). The remainder of the corollary follows then by straightforward inspection. Q.E.D.

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