New coherent states and modified heat equations

Antonio Pittelli^a and Lorenzo Sindoni^b

 ^a Università degli Studi di Trieste Via A. Valerio 2, 34127 Trieste, Italy
 ^b Max Planck Institute for Gravitational Physics, Albert Einstein Institute,
 Am Mühlenberg 1, 14476 Golm, Potsdam, Germany

Abstract

We clarify the relations between certain new coherent states for loop quantum gravity and the analytically continued heat kernel coherent states, highlighting the underlying general construction, the presence of a modified heat equation as well as the way in which the properties of the heat kernels are automatically inherited by these new states.

1 Introduction

In [1], extending previous work on semiclassical states in Loop Quantum Gravity (LQG) [2, 3, 4, 5, 6], a new family of coherent states (for the group SU(2)) has been introduced. Both choices present some advantages: for instance, $SL(2, \mathbb{C})$ heat kernels have got very simple convolution properties; on the other hand, flux coherent states provide correction-free expectation values for the flux operator \hat{E} . Therefore these states might provide an alternative, possibly more transparent, semiclassical treatment of the quantum operators associated to the classical phase space on which Loop Quantum Gravity [7] is based.

These new states are also naturally formulated in the flux representation of LQG, introduced in [8], where, using the concept of non-commutative Fourier transform [9, 10], a representation of the LQG building blocks based on functions over (several copies of) the $\mathfrak{su}(2)$ Lie algebra flux variables is developed.

The purpose of this paper is very specific: we want to reconsider the results of [1], with emphasis on the relationship with the coherent states they were originated from [2, 6] using the techniques of complexifiers and the analytic continuation of heat kernel states on SU(2). We will present the results in the group representation, so that the connection is even more clear. We will show, in particular that the states introduced in [1] are indeed heat kernels on their own right, associated to a different choice of heat equation, in particular a different Laplace operator. Not only this observation makes more transparent the results discussed in [1], but also shows how to construct further generalizations.

In this paper we will restrict the attention to the classical phase space of a single holonomy/flux pair, whether this is associated to a single link of a graph or to a more extended structure [11, 12]. The holonomy flux algebra is given by

$$\left[\hat{E}^{i},\hat{h}\right] = i\hbar(8\pi G\gamma)R^{i} \triangleright \hat{h}, \qquad \left[\hat{E}^{i},\hat{E}^{j}\right] = i\hbar(8\pi G\gamma)\epsilon_{k}^{ij}\hat{E}^{k}, \tag{1}$$

where $\hbar(8\pi G\gamma) = 8\pi l_p^2 \gamma$ has the dimension of a length squared¹ and \hat{R}^i is the i-th right invariant vector field on SU(2). For the present discussion we will not be concerned with dimensionful quantities, and hence we will use rescaled flux operators such that all the relevant variables are dimensionless.

¹We are using units in which c = 1.

Wavefunctions for kinematical states can be constructed by suitable choices of representations of several copies of this algebra (and by imposition of all the gauge invariances associated to gravity), thus leading to the concept of cylindrical functions [7] (see also [13] for a discussion of the feasibility of the construction within the flux representation). We will not embark in the construction of general states. Rather, we will focus on a single copy of this algebra of operators.

We will then construct wavefunctions as square integrable functions on one copy of the group, or, alternatively, on the Lie algebra $\mathfrak{su}(2)$. We will work in the group representation, with the following representation for the flux operators:

$$E^{j}(f(h)) = \lim_{\epsilon \to 0} \left[i \frac{d}{d\epsilon} f\left(e^{-i\sigma^{j}\epsilon} h \right) \right].$$
⁽²⁾

Heat kernels are the most natural generalization of the Gaussian functions to the case of more general (Riemannian) spaces. They are defined in terms of the heat equation

$$\partial_t \rho^t(x;y) = \Delta_x \rho^t(x;y); \qquad \rho^{t=0}(x;y) = (-g(x))^{-1/2} \delta(x-y),$$
(3)

where the Laplacian operator \triangle depends on the metric tensor, and t is an evolution parameter. Obviously, SU(2) can be seen as a Riemannian space (indeed, as S^3 with the canonical metric on it). Therefore, it is possible to naturally define a heat kernel on the group [14], as solutions of the heat equation

$$\partial_t K^t(gg_0^{-1}) = \triangle_{\mathrm{SU}(2)} K^t(gg_0^{-1}); \qquad K^{t=0}(gg_0^{-1}) = \delta(gg_0^{-1}), \tag{4}$$

where we are using the Laplace–Beltrami operator on SU(2),

$$\Delta_{\mathrm{SU}(2)} = -\delta_{ij} E^i E^j. \tag{5}$$

The Dirac delta on the group, providing the initial condition for the differential equation, completes the definition of these functions. The heat kernel K^t can be used to construct coherent states, with the appropriate identification of t with the spread of the wavefunction and of g_0 with the peak, its maximum. By construction,

$$\langle E^i \rangle = 0, \tag{6}$$

on these states, and therefore we need an additional step in order to define a state that is coherently peaking on a generic phase space point, labeled by $g_0 \in SU(2)$ and by $x_0 \in \mathfrak{su}(2)$.

In order to use them to construct semiclassical states peaking on a given point of the single link classical phase space, in [2, 3, 4, 5, 6] these functions (seen as functions of the peak element g_0) were analytically continued, turning the heat kernel (4) into a function on $SL(2, \mathbb{C})$,

$$K^{t}_{(g_0)}(h) = K^{t}(hg_0^{-1}), \qquad g_0 \in SU(2) \to K^{t}_{(G_0)}(h), \qquad G_0 \in SL(2, \mathbb{C}).$$
 (7)

We refer to the literature for a detailed discussion of the properties of the states defined in such a way. See also [15, 16, 17] for further discussions on related results.

The approach discussed so far gives particular relevance to the natural isomorphism between the phase space constructed on a single copy of SU(2), $T^*SU(2)$, and the analytic continuation of SU(2) itself, namely SL(2, \mathbb{C}). In [1] a different path has been followed, exploiting the point of view given by the definition of a Fourier transform on the group [9, 10] (see also [18] for a discussion of possible extensions to more general Lie groups). We will denote this operation as $\mathscr{F} : L^2(SU(2)) \to L^2(\mathfrak{su}(2), \star)$, mapping square integrable functions on the group onto square integrable functions on the Lie algebra $\mathfrak{su}(2)$, endowed with the ordinary Lebesgue measure but with a \star -product replacing the ordinary commutative product of functions.

As shown in [1], by modifying heat kernels with the multiplication by a plane wave,

$$\mathscr{K}^{t}_{(g_0,x_0)}(g) = e_g(x_0) \mathcal{K}^{t}_{(g_0)}(g), \tag{8}$$

one obtains states that have essentially the properties of coherent states, with respect to averages, peakedness properties, completeness and overlap. Here, plane waves are defined as complex valued functions over $SU(2) \times \mathfrak{su}(2)$:

$$(g,x) \in \mathrm{SU}(2) \times \mathfrak{su}(2) \mapsto e_g(x) \stackrel{\mathrm{def}}{=} \exp\left(-\frac{i}{2}\mathrm{Tr}\left(|g|x\right),\right), \qquad |g| = \mathrm{sign}(\chi(g))g,$$

$$(9)$$

where the group element g is assumed to be represented by its fundamental representation matrix acting on $\mathfrak{su}(2)$, in order for the expression above to make sense.

The results of [1] are based on the fact that a) the states originate from a heat kernel and b) that in terms of the Fourier transform, this state is just the ordinary heat kernel translated by x_0 in the Lie algebra representation, *i.e.*

$$\mathscr{F}\left(\mathscr{K}_{(g_0,x_0)}^t\right)(x) = \mathscr{F}\left(\mathbf{K}_{(g_0)}^t\right)(x-x_0).$$
(10)

The results were established by means of an explicit analysis of the various integrals which are involved in the calculations of the various expectation values and overlap functions. However, it was not clear whether it is possible to extend the results of the complexifier construction and the proof of the existence of an annihilation operator to this particular case, even though equation (10) suggests that this might be the case. In the next sections we will give a detailed derivation of these states $\mathscr{K}^t_{(g_0,x_0)}$ as heat kernels of suitably defined Laplace operators, thus making manifest the correspondence with the structure of the standard heat kernel coherent states, a cleaner and more automatic derivation of the results of [1] and a recipe for further generalizations.

2 Remarks on translations

As we have seen, one of the key ideas is to use the multiplication with a plane wave in order to achieve a translation in momentum space. Here we will give a more abstract description of this operation, which will give us the opportunity to better grasp the structure of the reasoning, of the results and to offer a path for possible future generalizations. Given $x_0 \in \mathfrak{su}(2)$, define the linear map

$$U_{x_0}: L^2(\mathrm{SU}(2)) \to L^2(\mathrm{SU}(2)); \qquad U_{x_0}(f)(h) := e_h(x_0)f(h).$$
 (11)

This map is obviously unitary. Indeed, it is invertible, with inverse $U_{x_0}^{-1} = U_{-x_0}$, since it preserves the canonical scalar product of $L^2(SU(2))$

$$\langle U_{x_0}f_1|U_{x_0}f_2\rangle = \langle f_1|f_2\rangle,\tag{12}$$

and its adjoint is the inverse:

$$U_{x_0}^{\dagger} = U_{-x_0} = U_{x_0}^{-1}.$$
(13)

Therefore, if we have an orthonormal basis for $L^2(SU(2))$, f_n with n in some index space, the functions

$$f_n^U = U_{x_0} f_n \tag{14}$$

will be an orthonormal basis for $L^2(SU(2))$ too². In particular, the Peter–Weyl theorem ensures us that the set of Wigner matrices is such a basis. Hence, also

$$\mathcal{D}_{ab}^{j}(h) = (U_{x_0} D_{ab}^{j})(h), \qquad j = 0, \frac{1}{2}, 1, \dots$$
 (15)

are an orthonormal set of functions. Indeed, while orthonormality is automatically guaranteed by unitarity, their role as basis functions is also clear from elementary considerations. For any $F \in \mathcal{L}^2(\mathrm{SU}(2))$, by construction, also $U_{x_0}^{-1}F$ belongs to this space. Therefore,

$$(U_{x_0}F)(g) = \sum_j M^j_{ab} D^j_{ba}(g) = \sum_j M^j_{ab} U^{-1}_{x_0} \mathcal{D}^j_{ba}(g) = U^{-1}_{x_0} \sum_j M^j_{ab} \mathcal{D}^j_{ba}(g),$$
(16)

or

$$F(g) = \sum_{j} M^{j}_{ab} \mathcal{D}^{j}_{ba}(g), \qquad (17)$$

for some matrices M_{ab}^{j} (of course, the sum might be, in fact, a series).

 $^{^{2}}$ The key of the argument is that, given the unitarity of U, Cauchy sequences are mapped onto Cauchy sequences, since the scalar product, and hence the norm and the distance derived from it are all conserved.

Of course, this result can be seen in a easier way when everything is written in terms of noncommutative functions on $\mathfrak{su}(2)$, the multiplication by the plane wave being just a translation.

It is clear that the construction of the unitary operator U can be generalized to the case in which the function that is used to define it is not necessarily a plane wave, but a sufficiently regular phase factor. All the results enumerated so far will carry over to the general case, with the exception of the picture in the algebra representation, where the translation will be replaced by a convolution. Define the Fourier transform of a general phase factor:

$$\int_{\mathrm{SU}(2)} dh \,\overline{e_h(x)} \exp(i\alpha(h)) = u_\alpha(x). \tag{18}$$

Then, the Fourier transform of the product of a phase factor with a function on SU(2) will be

$$\int_{\mathrm{SU}(2)} dh \overline{e_h(x)} \exp(i\alpha(h)) f(h) = \int_{\mathrm{SU}(2)} dh \int_{\mathfrak{su}(2)} d^3y \int_{\mathfrak{su}(2)} d^3z \overline{e_h(x)} u_\alpha(y) e_h(y) e_h(z) \mathscr{F}(f)(z), \tag{19}$$

whence we obtain the following convolution of Fourier transforms

$$\mathscr{F}(Uf)(x) = \int_{\mathfrak{su}(2)} d^3 y \, u_\alpha(y) \star_y \mathscr{F}(f)(x-y).$$
⁽²⁰⁾

We can easily conclude, then, that the flux coherent states introduced in [1] are just the image of the heat kernel coherent states³ under a family of these maps :

$$\mathscr{K}_{(g_0,x_0)}^t = U_{x_0} \mathbf{K}_{(g_0)}^t.$$
(21)

We will now use this to show what are the structures that are behind these new states, and to what extent we can generalize the construction of coherent states following this path.

3 The new coherent states as heat kernels

In the group representation, the flux operator acts as a derivative operator; especially, it obeys the standard Leibniz rule

$$E^{i}(f(h)g(h)) = (E^{i}f(h))g(h) + f(h)(E^{i}g(h)).$$
(22)

In particular, we can compute the action of the fluxes on the plane waves:

$$E^{j}(e_{g}(x_{0})) = \lim_{\epsilon \to 0} \left[i \frac{d}{d\epsilon} \exp\left(-\frac{i}{2} \operatorname{Tr}\left\{e^{-i\sigma^{j}\epsilon}|h|\sigma_{k}\right\} x_{0}^{k}\right) \right] = -\frac{i}{2} \operatorname{Tr}\left(\sigma^{j}|h|\sigma_{k}\right) x_{0}^{k}(e_{h}(x_{0})) = v_{x_{0}}^{j}(h)e_{h}(x_{0}), \quad (23)$$

where

$$v_{x_0}^j(h) = -\frac{\mathrm{i}}{2} \mathrm{Tr}\left(\sigma^j |h| \sigma_k\right) x_0^k.$$
(24)

Again, this is just a special case: the entire discussion can be generalized to phases which are not necessarily plane waves. In analogy with the case of gauge invariant (U(1)) field theories, we introduce a sort of gauge transformed version of the flux operator:

$$\mathcal{E}_{x_0}^i := E^i - v_{x_0}^i(h). \tag{25}$$

These operators are just giving the group representation of the operators corresponding to the translated fluxes in the algebra representation, with the residual dependence on the group element h being associated to the noncommutative nature of such a translation, once the structure of $\mathfrak{su}(2)$ is appropriately taken into account.

These new operators are obviously constructed in such a way that:

$$\mathcal{E}_{x_0}^i U_{x_0}(f) = U_{x_0}(E^i f), \qquad \mathcal{E}_{x_0}^i = U_{x_0} E^i U_{x_0}^{\dagger}.$$
(26)

³Here we are assuming that we are not analytically continuing them.

Using this definition, one immediately realizes that the algebra of the \mathcal{E}^i is actually the $\mathfrak{su}(2)$ one,

$$[\mathcal{E}^{i}, \mathcal{E}^{j}] = [U_{x_{0}}E^{i}U_{x_{0}}^{\dagger}, U_{x_{0}}E^{j}U_{x_{0}}^{\dagger}] = \mathrm{i}\epsilon_{ijk}U_{x_{0}}E^{k}U_{x_{0}}^{\dagger} = \mathrm{i}\epsilon_{ijk}\mathcal{E}^{k}, \qquad (27)$$

as expected from the fact that, in flux space, the \mathcal{E}^i are just translated fluxes, still belonging to $\mathfrak{su}(2)$. This definition allows us to understand better what these new coherent states are. Define the Casimir operator

$$\mathscr{D}_{x_0} = -\delta_{ij} \mathcal{E}^i_{x_0} \mathcal{E}^j_{x_0}.$$
(28)

Due to the properties elucidated in the previous section, this operator, a generalized Laplacian, is isospectral to the Laplace operator on SU(2), with the functions \mathcal{D}^l providing an orthonormal diagonalizing basis. Indeed:

$$\begin{aligned} \mathscr{D}_{x_{0}}\mathcal{D}_{ab}^{l} &= \delta_{ij}\mathcal{E}_{x_{0}}^{i}\mathcal{E}_{x_{0}}^{j}\mathcal{D}_{ab}^{l} \\ &= \delta_{ij}\mathcal{E}_{x_{0}}^{i}U_{x_{0}}(E^{j}D_{ab}^{l}) = U_{x_{0}}(\delta_{ij}E^{i}E^{j}D_{ab}^{l}) = U_{x_{0}}(\triangle_{\mathrm{SU}(2)}D_{ab}^{l}) = -l(l+1)U_{x_{0}}(D_{ab}^{l}) \\ &= -l(l+1)\mathcal{D}_{ab}^{l}. \end{aligned}$$

$$(29)$$

Of course this is an obvious result from the perspective of gauge field theories: we are using a Laplacian obtained by squaring a covariant derivative, in the case in which the connection one-form is a pure gauge. For this reason, we can immediately conclude that the flux coherent states are solutions of the heat equation

$$\partial_t \mathscr{K}^t_{(g_0, x_0)}(h) = \mathscr{D}_{x_0} \mathscr{K}^t_{(g_0, x_0)}(h), \tag{30}$$

with initial condition

$$\mathscr{K}_{(g_0,x_0)}^{t=0}(h) = \delta(hh_0^{-1})e_h(x_0).$$
(31)

This is the main result of the paper: the flux coherent states that were introduced in [1] are indeed heat kernel states, with the difference that now the Laplacian entering the heat equation is not the Laplace–Beltrami operator of SU(2), but a more general operator that includes the effect of the translation in the Lie algebra direction of the phase space. This is the group representation counterpart of the statement contained in (10). Notice that the operator \mathscr{D} depends on x_0 , and hence to obtain coherent states peaked on different Lie algebra elements we have to use heat kernel associated to different Laplacians. This is a key difference with respect to the analytically continued heat kernels, that are defined with respect to the same Laplace–Beltrami operator, whatever is the position of their peak.

4 Generalizations

The results discussed so far can be made even more clear with a more suggestive notation. Let us introduce ket vectors, labeled by SU(2) group elements,

$$|h_0\rangle: \qquad \langle h|h_0\rangle = \delta(hh_0^{-1}), \tag{32}$$

such that the heat kernel coherent states and the flux coherent states, denoted respectively as

$$|h_0\rangle_t = e^{t\Delta} |h_0\rangle, \qquad |h_0, x_0\rangle t = e^{t\mathscr{D}_{x_0}} U_{x_0} |h_0\rangle, \qquad (33)$$

have wavefunctions

$$\mathbf{K}_{(h_0)}^t(h) = \langle h | g_0 \rangle_t = \langle h | \exp(t\Delta) | h_0 \rangle, \qquad \mathscr{K}_{(g_0, x_0)}^t(h) = \langle h | h_0, x_0 \rangle t, \tag{34}$$

Using the relation with the modified/translated flux operator,

$$\mathscr{D}_{x_0}U_{x_0} = U_{x_0}\Delta,\tag{35}$$

we obtain immediately:

$$|h_0, x_0\rangle t = e^{t\mathscr{D}} U_{x_0} |h_0\rangle = U_{x_0} e^{t\Delta} |h_0\rangle = U_{x_0} |h_0\rangle_t,$$
(36)

which is exactly what we have already obtained, written in a different way. This rewriting allows us to show immediately how to generalize the results obtained previously to different operators U_{x_0} , possibly not plane waves, and to other groups. This rewriting also shows how the expectation values of operators can then be immediately inferred from the ones of the standard heat kernel, once we remember that $hU_{x_0} = U_{x_0}h$ and $\mathcal{E}_{x_0}^i = U_{x_0}E^iU_{x_0}^{\dagger}$:

$$\langle \varphi(h) \rangle = \langle h_0, x_0 | t\varphi(h) | h_0, x_0 \rangle t = {}_t (h_0 | \varphi(h) | h_0)_t;$$
(37)

$$\langle E^i \rangle = \langle h_0, x_0 | tE^i | h_0, x_0 \rangle t = {}_t (h_0 | U^{\dagger} E^i U | h_0)_t = {}_t (h_0 | \mathcal{E}^i_{-x_0} | h_0)_t = 0 + \langle v^i (x_0) \rangle_{h_0;t};$$
(38)

$$\langle E^{i}E^{j}\rangle = \langle h_{0}, x_{0} | tE^{i}E^{j} | h_{0}, x_{0}\rangle t = {}_{t}(h_{0} | U^{\dagger}E^{i}E^{j}U | h_{0})_{t} = {}_{t}(h_{0} | \mathcal{E}^{i}_{-x_{0}}\mathcal{E}^{j}_{-x_{0}} | h_{0})_{t}.$$
(39)

Hence, all the statistical properties, and in particular the behaviour with respect to the Heisenberg uncertainty relations, are consistently imported from the ones determined for the Hall states, as expected.

In [6] it was further considered the construction of the heat kernels in term of complexifiers and the associated creation/annihilation operators. In particular, it has been shown that it is natural to identify the annihilation operator with

$$A = e^{t\Delta} h e^{-t\Delta},\tag{40}$$

whose eigenvectors are indeed the heat kernels. In analogy with that case, we can construct similar annihilation operators for our case by replacing the Laplacian \triangle with the modified one \mathscr{D}_{x_0} . This is possible because the two operators possess the same spectrum; indeed, they are related by a unitary transformation. In a similar way, we can associate to the Laplacian (28) the annihilation operator

$$A(x_0) = e^{t\mathscr{D}} h e^{-t\mathscr{D}},\tag{41}$$

that by construction treats the flux coherent states as its eigenvector.

5 Concluding remarks

Let us briefly summarize the previous points. We have elucidated the structural similarities of the proposed flux coherent states with the complexifier coherent states, by their explicit representation in terms of group variables and operators acting on $L^2(SU(2))$.

We have explicitly shown the reason of their coherent behavior by tracing it back to their role as heat kernels for a modified Laplace operator, \mathscr{D} isospectral to $\triangle_{SU(2)}$. In particular, this might be used to reproduce the complexifier structure introduced for heat kernels. Furthermore, the mapping between these states and the heat kernels ensures us that the statistical properties of the latter will be imported automatically, allowing us to use them as coherent states.

At the same time, we have given a general recipe for the construction of coherent states based on a generalized notion of heat kernel on the group manifold, by highlighting the nature of the operation that is at the basis of the definition of the new states. Indeed, given any family of unitary maps U, if we define

$$\mathcal{E}^{i} = U E^{i} U^{\dagger}, \qquad \mathscr{D} = -\delta_{ij} \mathcal{E}^{i} \mathcal{E}^{j}, \tag{42}$$

we can construct arbitrary families of coherent states, provided that i) the map U can be parametrized by $\mathfrak{su}(2)$ elements, to which it will be associated the position of the peak of the state in the Lie algebra, ii) that the map between the coherent states and $T^*SU(2)$ defined by the expectation values of \hat{h} and \hat{E} is bijective, and iii) provided that the overcompleteness and overlap properties customarily valid fo coherent states are satisfied.

These indeed are the only conditions to be really checked explicitly as the conditions restricting the maps U to be of a specific form. For them one really needs to understand the dependence of the operators $\exp(t\mathscr{D})$ on the Lie algebra elements. While the proof of overcompleteness might be only a technical problem to find a measure with respect to which perform the integration in the Lie algebra elements (taken care of by the star product, in the specific case of the flux coherent states), the proof of overlap properties requires the understanding of integrals of the form

$${}_{t}\langle g_{0}, y_{0} | h_{0}, x_{0} \rangle t = {}_{t}(g_{0} | U_{y_{0}}^{\dagger} U_{x_{0}} | h_{0})_{t} = {}_{t}(g_{0} | U_{x_{0} - y_{0}} | h_{0})_{t},$$

$$(43)$$

for which of course the explicit form of the operators U is needed. A posteriori, in [1] these were the only two properties that needed to be verified, the others following in a straightforward manner from the properties of the heat kernel, already established in the literature.

Acknowledgments: The authors would like to thank S. Ansoldi for remarks on an earlier version of the draft and D. Oriti for fruitful discussions.

References

- [1] Daniele Oriti, Roberto Pereira, and Lorenzo Sindoni. Coherent states in quantum gravity: a construction based on the flux representation of LQG. J.Phys.A, A45:244004, 2012.
- [2] T. Thiemann. Gauge field theory coherent states (GCS): I. General properties. Classical and Quantum Gravity, 18:2025–2064, June 2001.
- [3] T. Thiemann and O. Winkler. Gauge field theory coherent states (GCS): II. Peakedness properties. Classical and Quantum Gravity, 18:2561–2636, July 2001.
- [4] T. Thiemann and O. Winkler. Gauge field theory coherent states (GCS): III. Ehrenfest theorems. Classical and Quantum Gravity, 18:4629–4681, November 2001.
- [5] T. Thiemann and O. Winkler. Gauge field theory coherent states (GCS): IV. Infinite tensor product and thermodynamical limit. *Classical and Quantum Gravity*, 18:4997–5053, December 2001.
- [6] T. Thiemann. Complexifier coherent states for quantum general relativity. Classical and Quantum Gravity, 23:2063–2117, March 2006.
- [7] C. Rovelli. Quantum Gravity. Cambridge University Press, November 2004.
- [8] A. Baratin, B. Dittrich, D. Oriti, and J. Tambornino. Non-commutative flux representation for loop quantum gravity. *Classical and Quantum Gravity*, 28(17):175011, September 2011.
- [9] L. Freidel and S. Majid. Noncommutative harmonic analysis, sampling theory and the Duflo map in 2+1 quantum gravity. *Classical and Quantum Gravity*, 25(4):045006, February 2008.
- [10] E. Joung, J. Mourad, and K. Noui. Three dimensional quantum geometry and deformed symmetry. Journal of Mathematical Physics, 50(5):052503, May 2009.
- [11] H. Sahlmann, T. Thiemann, and O. Winkler. Coherent states for canonical quantum general relativity and the infinite tensor product extension. *Nucl. Phys.*, B606:401–440, 2001.
- [12] Daniele Oriti, Roberto Pereira, and Lorenzo Sindoni. Coherent states for quantum gravity: towards collective variables. *Class. Quant. Grav.*, 29:135002, 2012.
- [13] Bianca Dittrich, Carlos Guedes, and Daniele Oriti. On the space of generalized fluxes for loop quantum gravity. 2012.
- [14] R. Camporesi. Harmonic analysis and propagators on homogeneous spaces. Phys. Rept., 196:1–134, 1990.
- [15] B. C. Hall and J. J. Mitchell. Coherent states on spheres. Journal of Mathematical Physics, 43:1211–1236, March 2002.
- [16] B. C. Hall and J. J. Mitchell. Erratum: Coherent states on spheres [J. Math. Phys. 43, 1211 (2002)]. Journal of Mathematical Physics, 46(5):059901, May 2005.
- [17] B. C. Hall and J. J. Mitchell. Coherent states for a 2-sphere with a magnetic field. Journal of Physics A Mathematical General, 45(24):244025, June 2012.
- [18] Matti Raasakka. Group Fourier transform and the phase space path integral for finite dimensional Lie groups. 2011.