Combinatorial Formula for Modified Hall-Littlewood Polynomials

Anatol N. Kirillov

CRM-2538
April 1998

CRM, Université de Montréal C.P. 6128, Succursale Centre-ville, Montréal (Quebec) H3C 3J7, Canada and Steklov Mathematical Institute, Fontanka 27, St.Petersburg, 191011, Russia

Abstract

We obtain new combinatorial formulae for modified Hall–Littlewood polynomials, for matrix elements of the transition matrix between the elementary symmetric polynomials and Hall-Littlewood's ones, and for the number of rational points over the finite field of unipotent partial flag variety. The definitions and examples of generalized mahonian statistic on the set of transport matrices and dual mahonian statistic on the set of transport (0,1)–matrices are given. We also review known q–analogues of Littlewood–Richardson numbers and consider their possible generalizations. Some conjectures about multinomial fermionic formulae for homogeneous unrestricted one dimensional sums and generalized Kostka–Foulkes polynomials are formulated. Finally we suggest two parameter deformations of polynomials $\mathcal{P}_{\lambda\mu}(t)$ and one dimensional sums.

Dedicated to Richard Askey on the occasion of his 65th birthday

Résumé

Nous obtenons des nouvelles formules combinatoires concernant les polynômes de Hall-Littlewood modifiés, la matrice de transition entre les functions symétriques élémentaires et celles de Hall-Littlewood, et le nombre de points rationnels sur la sous-variété de points fixes d'un élément unipotent d'une variété de drapeau sur un corps fini. On définie la notion de statistique mahonienne généralisee sur l'ensemble des matrices de transport, ainsi que son duale, et on en fournit des exemples. Ces definitions généralisent naturellement la définition de statistique mahonienne introduite par D. Foata.

Contents

§0. Introduction.

In this paper certain combinatorial and algebraic applications and generalizations of fermionic formulae for unrestricted one dimensional sums, obtained in [HKKOTY], are studied. The main applications of the fermionic formulae for unrestricted one dimensional sums considered in [HKKOTY] are related to the fermionic formulae for the branching functions and characters of some integrable representations of the affine Lie algebra $\widehat{sl}(n)$.

Among applications considered in the present paper are the following ones:

• New combinatorial formula for modified Hall–Littlewood polynomials $Q'_{\lambda}(X_n;t)$ (Theorem 3.1). Let λ be a partition, $l(\lambda)\le n$, then

$$Q'_{\lambda}(X_n;t)=\sum_{\mu}\mathcal{P}_{\lambda\mu}(t)m_{\mu}(X_n), \tag{0.1}$$

where $m_{\mu}(X_n)$ denotes the monomial symmetric function corresponding to partition μ, $X_n=(x_1,\ldots,x_n)$, and

$$\mathcal{P}_{\lambda\mu}(t)=\sum_{\{\nu\}}t^{c(\nu)}\prod_{k=1}^{n-1}\prod_{i\ge 1}\begin{bmatrix}\nu_i^{(k+1)}-\nu_{i+1}^{(k)}\\ \nu_i^{(k)}-\nu_{i+1}^{(k)}\end{bmatrix}_t, \tag{0.2}$$

summed over all flags of partitions $\nu=\{0=\nu^{(0)}\subset\nu^{(1)}\subset\cdots\subset\nu^{(n)}=\lambda'\}$, such that $|\nu^{(k)}|=\mu_1+\cdots+\mu_k$, $1\le k\le n$, and $c(\nu)=\displaystyle\sum_{k=0}^{n-1}\sum_{i\ge 1}\binom{\nu_i^{(k+1)}-\nu_i^{(k)}}{2}$.

• New combinatorial formula for the transition matrix $M(e,P)$ (Theorem 3.4). Let λ,μ be partitions, then

$$M(e,P)_{\lambda\mu}=\sum_{\eta}K_{\eta\lambda}K_{\eta'\mu}(t):=\mathcal{R}_{\mu\lambda}(t),\quad\text{where}$$
$$\mathcal{R}_{\lambda\mu}(t)=\sum_{\{\nu\}}\prod_{k=1}^{r-1}\prod_{i\ge 1}\begin{bmatrix}\nu_i^{(k+1)}-\nu_{i+1}^{(k+1)}\\ \nu_i^{(k)}-\nu_{i+1}^{(k+1)}\end{bmatrix}_t, \tag{0.3}$$

summed over all flags of partitions $\{\nu\}=\{0=\nu^{(0)}\subset\nu^{(1)}\subset\cdots\subset\nu^{(r)}=\lambda'\}$ such that $\nu^{(k)}/\nu^{(k-1)}$ is a horizontal μ_k–strip, $1\le k\le r$, $r=l(\mu)$.

• New combinatorial formula for the number of rational points $\mathcal{F}_{\mu}^{\lambda}(\mathbf{F}_q)$ over the finite field $\mathbf{F}_q$ of the unipotent partial flag variety $\mathcal{F}_{\mu}^{\lambda}$ (Section 1.4):

$$\mathcal{F}_{\mu}^{\lambda}(\mathbf{F}_q)=q^{n(\lambda)}\mathcal{P}_{\lambda\mu}(q^{-1}), \tag{0.4}$$

where polynomial $\mathcal{P}_{\lambda\mu}(t)$ is given by (0.2).

• New interpretation of the number $\alpha_\lambda(S;p)$ of chains of subgroups

$$\{e\} \subseteq H^{(1)} \subseteq \cdots \subseteq H^{(m)} \subseteq G$$

in a finite abelian p–group of type λ such that each subgroup $H^{(i)}$ has order p^{a_i}, $1 \le i \le m$ (Subsection 1.6):

$$p^{n(\lambda)}\alpha_\lambda(S;p^{-1}) = \mathcal{P}_{\lambda\mu}(p). \tag{0.5}$$

Here $S = \{a_1 < a_2 < \cdots < a_m\}$ is a subset of the set $[1, |\lambda| - 1]$, and $\mu := \mu(S) = (a_1, a_2 - a_1, \ldots, a_m - a_{m-1}, |\lambda| - a_m)$.

• New interpretation of the Schilling–Warnaar t–supernomial coefficients $\begin{bmatrix} \mathbf{L} \\ a \end{bmatrix}_t$ and $T(L,a)$, [ScW], and Example 1, Subsection 3.1. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition, then

$$\begin{bmatrix} \mathbf{L} \\ a \end{bmatrix}_t = \sum_\eta K_{\eta\mu}\widetilde{K}_{\eta\lambda}(t) = t^{n(\lambda)}\mathcal{P}_{\lambda\mu}(t^{-1}), \tag{0.6}$$

where $L_i = \lambda'_i - \lambda'_{i+1}$, $1 \le i \le k$, $\lambda_{k+1} = 0$, $\mu = \left(\frac{1}{2}|\lambda| - a,\ \frac{1}{2}|\lambda| + a\right)$, and

$$\begin{bmatrix} \mathbf{L} \\ a \end{bmatrix}_q = \sum_{j_1+\cdots+j_k = a+\frac{|\lambda|}{2}} t^{\sum_{l=2}^{k} j_{l-1}(L_l + \cdots + L_k - j_l)} \begin{bmatrix} L_k \\ j_k \end{bmatrix} \begin{bmatrix} L_{k-1} + j_k \\ j_{k-1} \end{bmatrix} \cdots \begin{bmatrix} L_1 + j_2 \\ j_1 \end{bmatrix}.$$

$$T(\mathbf{L}, a) := t^{\frac{1}{4}L^tC_k^{-1}L - \frac{a^2}{k}} \begin{bmatrix} \mathbf{L} \\ a \end{bmatrix}_{1/t} = t^{-B}\mathcal{P}_{\lambda\mu}(t), \tag{0.7}$$

where $B = \frac{1}{2}n(\lambda) + \frac{1}{k}n(\mu') - \frac{1}{4k}\left(|\lambda|^2 - (k+2)|\lambda|\right) = \frac{1}{4}L^tC_k^{-1}L + \frac{1}{k}\left(n(\mu') + \frac{|\mu|}{2}\right)$; $\left(C_k^{-1}\right)_{ij} := \min(i,j) - \frac{ij}{k}$, $1 \le i,j \le k-1$, stands for the inverse of the Cartan matrix C_k of the Lie algebra of type A_{k-1}.

We introduce also the $SU(n)$–analogue of t–multinomial coefficients (0.6) and (0.7) (Definition 3.2).

• Definition and examples of the generalized mahonian statistics on the set of transport matrices $\mathcal{P}_{\lambda\mu}$ (Section 2). This is a natural generalization of notion of *mahonian statistic* on the set of *words* introduced and studied by D. Foata in particular case $\lambda = (1^n)$, [F], see also [Ma], [An], [ZB], [FZ], [GaW].

• Connection between the rigged configurations polynomials $RC_{\lambda R}(t)$ for a sequence of rectangular partitions $R = (R_1, \ldots, R_p)$, cf. [Ki1], and the classically restricted one dimensional sums $f_R^{\mathrm{cl}}(b_{T_{\max}}; \mu)$ corresponding to the tensor product of "rectangular" crystals $B_{R_1} \otimes \cdots \otimes B_{R_p}$ (Section 7).

• Definition, examples and properties of the two parameter deformation $B_{\lambda\mu}(q,t)$ of the unrestricted one dimensional sum $\mathcal{P}_{\lambda\mu}(t)$ (Section 8).

The paper is organized as follows:

In Section 1 we recall the definition of modified Hall–Littlewood polynomials, and explain a connection between the character of level 1 basic representation of the affine Lie algebra $\widehat{sl}(n)$, and the limit $N \to \infty$ of the modified Hall–Littlewood function corresponding to partition $\mu = (1^N)$, see [Ki2]. This result was extended to more general cases in [Ki2], [NY] and [HKKOTY]. In Subsections 1.4 and 1.5 we explain a connection between the modified Hall–Littlewood polynomials and the unipotent partial flag varieties [HS], [LLT], [Sh], the Demazure characters [Ka2], [HKMOTY1,2], and the number of chains of subgroups in a finite abelian p–group, [Bu1,2,3], [F], [St].

In Section 2 we introduce the generalized mahonian and dual mahonian statistics on the set of transport matrices and on the set of (0,1)–transport matrices, respectively, and give few examples of such statistics.

In Section 3 we state the fermionic formulae for polynomials $\mathcal{P}_{\lambda\mu}(t) = \sum_{\eta} K_{\eta\mu}K_{\eta\lambda}(t)$ (Theorem 3.1) and $\mathcal{R}_{\lambda\mu}(t) = \sum_{\eta} K_{\eta\mu}K_{\eta'\lambda}(t)$ (Theorem 3.4), and study their special cases. In particular, we show that polynomials $\mathcal{P}_{\lambda\mu}(t)$ and $t^{n(\lambda)}\mathcal{P}_{\lambda\mu}(t^{-1})$ give a natural generalization of generalized p–binomial coefficients introduced and studied by F. Regonati [R], L. Butler [Bu1], S. Fishel [F], ..., and supernomial and multinomial coefficients introduced by A. Schilling and S.O. Warnaar, [Sc], [ScW], [W], and A.N. Kirillov [Ki2].

In Section 4 we give algebraic proofs of main results, formulated in Section 3, namely proofs of Theorems 3.1 and 3.4. A combinatorial proof of these theorems will appear elsewhere.

The main purpose of Section 5 is to show frequent apparitions of the one dimensional sums related to the tensor product of crystals $B_{R_1} \otimes \cdots \otimes B_{R_p}$ in different branches of Mathematics such as: representation theory, combinatorics, algebraic geometry, theory of finite abelian groups, and integrable systems. In our opinion, the fundamental role played by one dimensional sums in Mathematics and Mathematical Physics may be explained by the fact that one dimensional sums can be considered as a natural q–analog of the tensor product multiplicities. In the literature there exist at least 4 or 5 ways to define a q–analog of the Littlewood–Richardson numbers, see, e.g., [GoW], [BKMW], [CL], [LLT], [KLLT], [KS], ...

In Section 6 we overview several known ways to define the q–analogues of the tensor product multiplicities, and formulate conjectures (Conjectures 6.4, 6.5, 6.8 and 6.9) which relate the classically restricted one dimensional sums $f_R^{\mathrm{cl}}(b_{T_{\min}};\mu) := CK_{\mu R}(t)$, the ribbon Kostka polynomials $K_{\lambda\mu}^{(p)}(t)$, introduced by A. Lascoux, B. Leclerc and J.-Y. Thibon, [LLT], and the generalized Kostka polynomials $K_{\lambda R}(t)$, introduced by M. Shimozono and J. Weyman. We expect that only in the case of dominant sequence of rectangular partitions R

the crystal $CK_{\mu R}(t)$, the ribbon $K^{(p)}_{(R)\mu}(t)$ and the generalized Kostka polynomials $K_{\mu R}(t)$ give the equivalent q–analogues of the tensor product multiplicities. We also formulate some unsolved problems.

In Section 7 we formulate few conjectures about multinomial fermionic formulae for homogeneous unrestricted one dimensional sums, and generalized Kostka–Foulkes polynomials corresponding to a sequence of rectangles.

In Section 8 we suggest two parameter deformations of polynomials $\mathcal{P}_{\lambda\mu}(t)$ and one dimensional sums.

Acknowledgments. This paper presents an extended version of my talk "Fermionic formulae for the branching functions of the affine Lie algebra $\widehat{sl}(n)$" delivered at PhD Centennial Conference, Department of Mathematics, University of Wisconsin–Madison, May 22–24, 1997. I would like to thank Richard Askey and Georgia Benkart for invitation to this conference. I am thankful to Goro Hatayama, Atsuo Kuniba, Franklin M. Maley, Masato Okado, Mark Shimozono, Taichiro Takagi, Jean–Yves Thibon and Yasuhiko Yamada for very fruitful discussions and suggestions. I wish to thank L. Vinet for hospitality at the CRM, University of Montreal, where this work was completed.

§1. Modified Hall-Littlewood polynomials.

1.1. Definition.

Let λ be a partition, $l(\lambda) \le n$, $Q_\lambda(X_n;q)$ and $P_\lambda(X_n;q)$ be the Hall–Littlewood polynomials corresponding to λ, see e.g. [M], Chapter III.

Definition 1.1. *A modified Hall–Littlewood polynomial $Q'_\lambda(X_n;q)$ is defined to be*

$$Q'_\lambda(X_n;q) = Q_\lambda(X_n/(1-q);q) := Q_\lambda(X'_n;q), \tag{1.1}$$

where the variables X'_n are the products xq^{j-1}, $j \ge 1$, $x \in X_n := (x_1,\ldots,x_n)$.

The $Q'_\lambda(X;q)$ serve to interpolate between the Schur functions s_λ and the complete homogeneous symmetric functions h_λ, because

$$Q'_\lambda(X;0) = s_\lambda(X)$$

as it is clear from (1.1), and

$$Q'_\lambda(X;1) = h_\lambda(X)$$

as it is clear from Cauchy's identity (1.3) below.

Proposition 1.2.

$$Q'_\lambda(X;q) = \sum_\mu \left(\sum_\eta K_{\eta\mu} K_{\eta\lambda}(q) \right) m_\mu(X) = \sum_\eta s_\eta(X) K_{\eta\lambda}(q). \tag{1.2}$$

Proof. Let us remind that the Hall-Littlewood polynomials Q_λ and P_λ satisfy the following orthogonality condition (see, e.g. [M], Chapter III, (4.4))

$$\sum_\lambda Q_\lambda(X;q)P_\lambda(Y;q) = \prod_{x\in X,\ y\in Y} \frac{1-qxy}{1-xy}.$$

Hence, (cf. [M], Example 7a on p.234)

$$\sum_\lambda Q'_\lambda(X;q)P_\lambda(Y;q) = \prod_{k\geq 0}\prod_{x\in X,\ y\in Y} \frac{1-q^{k+1}xy}{1-q^kxy} = \prod_{x\in X,\ y\in Y}(1-xy)^{-1}$$

$$= \sum_\lambda s_\lambda(x)s_\lambda(y). \tag{1.3}$$

Here we have used the Cauchy identity for Schur functions, [M], Chapter I, (4.3). It remains to remind the definition of the Kostka–Foulkes polynomials:

$$s_\lambda(Y) = \sum_\mu K_{\lambda\mu}(q)P_\mu(Y;q). \tag{1.4}$$

■

1.2. Modified Hall–Littlewood polynomials for partition $\lambda = (1^N)$.

Corollary 1.3. *Let $\mu = (\mu_1,\ldots,\mu_n)$ be a composition, $|\mu| = N$. Then*

$$\sum_\lambda K_{\lambda\mu}K_{\lambda(1^N)}(q) = q^{n(\mu')}\begin{bmatrix} N \\ \mu_1,\ldots,\mu_n\end{bmatrix}_q, \tag{1.5}$$

where $\begin{bmatrix} N \\ \mu_1,\ldots,\mu_n\end{bmatrix}_q = \dfrac{(q;q)_N}{(q;q)_{\mu_1}\ldots(q;q)_{\mu_n}}$ *is the q–analog of gaussian multinomial coefficient, and* $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$.

Proof. First of all we have to compute $Q'_{(1^N)}(X;q)$. For this goal, let us remark that ([M], Chapter III, (2.8))

$$Q_{(1^N)}(X;q) = (q;q)_N e_N(X), \tag{1.6}$$

where $e_m(X)$ is the elementary symmetric function of degree m in the variables X. Hence

$$\sum_\lambda s_\lambda(X)K_{\lambda(1^N)}(q) = Q_{(1^N)}(X/((1-q);q) = (q;q)_N e_N(X/(1-q)).$$

Now let us put $E(X) = \sum_{N\geq 0} e_N(X)t^N = \prod_{x\in X}(1+tx)$. Then we have

$$E\left(X/(1-q)\right) = \prod_{x\in X}(-tx;q)_\infty,$$

and using the Euler identity

$$(x;q)_\infty = \sum_{n=0}^{\infty} \frac{x^n q^{\frac{n(n-1)}{2}}}{(q;q)_n}, \tag{1.7}$$

we obtain the following result

$$E\left(X/(1-q)\right) = \sum_{N\geq 0} \left(\sum_{\mu\vdash N} \frac{q^{n(\mu')}}{(q;q)_\mu} m_\mu(X) \right) t^N, \tag{1.8}$$

where for a composition $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ we set $(q;q)_\mu := \prod_{j=1}^{n} (q;q)_{\mu_j}$. Finally, from (1.6) and (1.8) we obtain immediately that

$$Q'_{(1^N)}(X_n;q) = \sum_\lambda s_\lambda(X_n) K_{\lambda(1^N)}(q) = \sum_{\mu\vdash N} q^{n(\mu')} \begin{bmatrix} N \\ \mu_1, \ldots, \mu_n \end{bmatrix} m_\mu(X_n). \tag{1.9}$$

■

1.3. Hall-Littlewood polynomials and characters of the affine Lie algebra $\widehat{sl}(n)$.

We consider the identity (1.9) as the finitization of the Weyl–Kac–Peterson character formula (WKR–formula for short, see, e.g. [Kac], (12.7.12)) for the level 1 basic representation $L(_0)$ of the affine Lie algebra $\widehat{sl}(n)$. Indeed, the WKP–formula for the character $\mathrm{ch}L(_0)$ may be recovered as an appropriate limit of (1.9). More exactly, let us consider the following form of (1.9):

$$q^{-\frac{(N^2-N)n}{2}} \sum_\lambda \frac{s_\lambda(x_1,\ldots,x_n)}{(x_1\ldots x_n)^N} K_{\lambda(1^{nN})}(q) =$$
$$\sum_{\substack{k \in \mathbf{Z}^n \\ |k|=0,\ k_i \geq -N,\ \forall i}} x_1^{k_1} \cdots x_n^{k_n} q^{\frac{1}{2}k_i^2} \begin{bmatrix} nN \\ k_1+N, \ldots, k_n+N \end{bmatrix}_q . \tag{1.10}$$

First of all, $\lim_{N\to\infty}$ RHS(1.10) $=$

$$\frac{1}{(q;q)_\infty^{n-1}} \sum_{m\in\mathbf{Z}^n,\ |m|=0} x^m q^{\frac{1}{2}\sum m_i^2} = \frac{1}{(q;q)_\infty^{n-1}} \sum_{k\in\mathbf{Z}^{n-1}} z_1^{k_1}\cdots z_{n-1}^{k_{n-1}} q^{\frac{1}{2}kA_{n-1}k^t},$$

where $z_i = \dfrac{x_i}{x_{i-1}}$, $1\leq i\leq n-1$, $x_0 := x_n$. On the other hand,

$$\lim_{N\to\infty} \mathrm{LHS}(1.10) = \sum_{\lambda=(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n)\in\mathbf{Z}^n,\ |\lambda|=0} s_\lambda(x_1,\ldots,x_n) b_\lambda(q), \tag{1.11}$$

where $b_\lambda(q)$ is defined to be

$$b_\lambda(q) := \lim_{N\to\infty} q^{-\frac{n(N^2-N)}{2}} K_{\lambda_N(1^{|\lambda_N|})}(q) = \frac{q^{n(\lambda')}}{(q;q)_\infty^{n-1}} \prod_{1\le i\le j\le n} (1-q^{\lambda_i-\lambda_j-i+j}), \qquad (1.12)$$

and for a given weight λ we set $\lambda_N := \lambda + (N^n)$. The last equality in (1.12) follows from the hook–formula (see, e.g. [M], Example 2 on p.243):

$$K_{\lambda(1^N)}(q) = \frac{q^{n(\lambda')}(q;q)_N}{\prod_{x\in\lambda}(1-q^{h(x)})},$$

where $h(x) := \lambda_i + \lambda'_j - i - j + 1$ is the hook–length corresponding to the box $x = (i,j) \in \lambda$.

Finally, it follows from (1.10)–(1.12) that

$$\sum_\lambda s_\lambda(x_1,\ldots,x_n) b_\lambda(q) = \frac{(x)}{(q;q)_\infty^{n-1}}, \qquad (1.13)$$

summed over all partitions λ such that $l(\lambda) \le n$, and $|\lambda| \equiv 0(\mathrm{mod}\ n)$, and where

$$(x) = \sum_{m=(m_1,\ldots,m_n)\in\mathbf{Z}^n,\ |m|=0} x^m q^{\frac{1}{2}(m_1^2+\cdots+m_n^2)}$$

is the theta–function corresponding to the basic representation $L(_0)$ of $\widehat{sl}(n)$, [Kac], §12.7. It is well-known that the RHS(1.13) is equal to the character of the level 1 basic representation $L(_0)$. Hence, $b_\lambda(q)$ coincide with the branching functions ([Kac], §12.2) for the level 1 basic representation $L(_0)$ of the affine Lie algebra $\widehat{sl}(n)$ (cf. [Ki2]).

For further results concerning a connection between modified Hall–Littlewood functions and characters, and branching functions of the affine Lie algebra $\widehat{sl}(n)$, see [Ki2], [NY], and [HKKOTY].

1.4. Modified Hall–Littlewood polynomials and unipotent flag varieties.

Polynomials $\mathcal{P}_{\lambda\mu}(q) := \sum_\eta K_{\eta\mu} K_{\eta\lambda}(q)$ have the following geometric interpretation due to [HS] and [Sh]. Let V be an n–dimensional vector space over an algebraically closed field k, and let μ, $l(\mu) = r$, be a composition of n. A μ–flag in V is a sequence $F = \{V_1,\ldots,V_r\}$ of subspaces of V such that $V_1 \subset V_2 \subset \cdots \subset V_r = V$, and $\dim V_i = \mu_1 + \cdots + \mu_i$, $1 \le i \le r$. Let $\mathcal{F}_\mu$ denote the set of all μ–flags in V. The group $G := GL(V)$ acts transitively on $\mathcal{F}_\mu$, so that $\mathcal{F}_\mu$ may be identified with G/P, where P is the subgroup which fixes a given flag, and therefore $\mathcal{F}_\mu$ is a non–singular projective algebraic variety, the partial flag variety of V.

Now let $u \in G$ be a unipotent endomorphism of V of type λ, so that λ is a partition of n which describes the Jordan canonical form of u, and let $\mathcal{F}_\mu^\lambda \subset \mathcal{F}_\mu$ be the set of all μ–flags $F \in \mathcal{F}_\mu$ fixed by u. The set $\mathcal{F}_\mu^\lambda$ is a closed subvariety of $\mathcal{F}_\mu$.

It has been shown by N. Shimomura ([Sh], see also [HS]), that

• if $k = \mathbf{C}$ is the field of complex numbers, the variety $\mathcal{F}_\mu^\lambda$ admits a cell decomposition, involving only cells of even real dimensions, and

$$t^{2n(\lambda)}\mathcal{P}_{\lambda\mu}(t^{-2}) := \sum_\eta K_{\eta\mu}\widetilde{K}_{\eta\lambda}(t^2) = \sum_i t^{2i}\mathrm{dim}H_{2i}(\mathcal{F}_\mu^\lambda, \mathbf{Z}) \tag{1.14}$$

is the Poincare polynomial of $\mathcal{F}_\mu^\lambda/\mathbf{C}$, where $\widetilde{K}_{\eta\lambda}(t) := t^{n(\lambda)}K_{\eta\lambda}(t^{-1})$;

• if k contains the finite field of q elements, $\mathbf{F}_q$, the number $\mathcal{F}_\mu^\lambda(q)$ of $\mathbf{F}_q$–rational points of $\mathcal{F}_\mu^\lambda$ is equal to $q^{n(\lambda)}\mathcal{P}_{\lambda\mu}(q^{-1})$.

1.5. Modified Hall–Littlewood polynomials and Demazure characters.

Let g be a symmetrizable Kac–Moody algebra. Recall that for every dominant integral weight , there exists a unique (up to isomorphism) irreducible module $V = V()$ of highest weight . The character of V, denoted by ch$V$, is the formal sum

$$\mathrm{ch}V = \sum_{'} (\dim V_{'})\, e^{'},$$

summed over all weights $'$, where $V_{'}$ is the weight subspace of V of weight $'$, and where $e^{'}$ is a formal exponential. This sum makes sense because each $V_{'}$ is finite–dimensional. For definitions and further details, see [Kac].

Let b be the Borel subalgebra of g and let w be an element of the Weyl group W. The b–module generated by the one dimensional extremal weight subspace $V_{w()}$ is denoted by $V_w()$ and called a Demazure module. They are finite–dimensional subspaces which form a filtration of V which is compatible with the Bruhat order of W, i.e. $V_w() \subseteq V_{w'}()$ whenever $w \leq w'$ with respect to the Bruhat order, $w, w' \in W$, and $\bigcup_{w\in W} V_w() = V()$, see, e.g., [Ka2].

From now let us assume that g $= \widehat{sl}(n)$. Let ${}_i$ and r_i, $0 \leq i \leq n-1$, denote the fundamental weight and simple reflection with respect to the simple root α_i, of $\widehat{sl}(n)$. It is convenient to define ${}_i$ and r_i for all $i \in \mathbf{Z}$ using the agreement ${}_i = {}_{i+n}$, $r_i = r_{i+n}$.

Now we are ready to explain an interpretation of the modified Hall–Littlewood polynomial $Q'_{(l^L)}(X_n; q)$ corresponding to a rectangular partition (l^L) as the character of certain Demazure's module. This result is due to [KMOTU2]:

Let $L \geq 1$ be an integer, and $w := w_{L,n} = r_{Ln-1}r_{Ln-2}\cdots r_{L+2}r_{L+1}r_L$ be an element of the affine Weyl group $W(A_{n-1}^{(1)})$ of type $A_{n-1}^{(1)}$. Then

$$e^{-l_0}\mathrm{ch}V_w(l_L) = q^{-E_0}Q'_{(l^L)}(X_n; q), \tag{1.15}$$

where $E_0 = l\left[\dfrac{L}{n}\right]\left(L - \dfrac{n}{2}\left(\left[\dfrac{L}{n}\right]+1\right)\right)$.

1.6. Modified Hall–Littlewood polynomials and chains of subgroups in a finite abelian

p–group.

Let p be a prime number. It is well–known (see, e.g. [H]) that any abelian group G of order p^n is isomorphic to a direct product of cyclic groups

$$G \approx \mathbf{Z}/p^{\lambda_1}\mathbf{Z} \times \cdots \times \mathbf{Z}/p^{\lambda_l}\mathbf{Z}$$

where $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_l > 0$, $\lambda_1 + \cdots + \lambda_l = n$. The partition λ is called the type of G.

For any partition $\nu \subseteq \lambda$, let us denote by $\alpha_\lambda(\nu; p)$ the number of subgroups of type ν in a finite abelian p–group of type λ.

More generally, for any flag of partitions $\{\nu\} = \{\nu^{(1)} \subseteq \nu^{(2)} \subseteq \cdots \subseteq \nu^{(m)} \subset \lambda\}$ denote by $\alpha_\lambda(\nu^{(1)}, \ldots, \nu^{(m)}; p)$ (or $\alpha_\lambda(\{\nu\}; p)$ for short) the number of chains of subgroups

$$\{e\} \subseteq H^{(1)} \subseteq H^{(2)} \subseteq \cdots \subseteq H^{(m)} \subseteq G$$

in a finite abelian p–group G of type λ such that the type of $H^{(i)}$ is $\nu^{(i)}$.

The problem of counting the number of subgroups of type ν in a finite abelian p–group of type λ has a long history and goes back at least to the beginning of 1900's, see e.g., papers by G.A. Miller [Mi] and by H. Hiller [Hi]. In 1934 G. Birkhoff [Bi] has discovered an interesting connection between the set of subgroups of finite abelian p–group and that of so–called standard matrices of G. Birkhoff. In 1948 three mathematicians, S. Delsarte [De], P. Dyubyuk [Dy], and Yenchien Yeh [Y] published formulae for the number $\alpha_\lambda(\nu; p)$ of subgroups of type ν in a finite abelian p–group of type λ:

$$\alpha_\lambda(\nu; p) = \prod_{j \ge 1} p^{\nu'_{j+1}(\lambda'_j - \nu'_j)} \begin{bmatrix} \lambda'_j - \nu'_{j+1} \\ \nu'_j - \nu'_{j+1} \end{bmatrix}_p, \tag{1.16}$$

where λ' is the conjugate of λ, and ν' is the conjugate of ν.

In order to explain a connection between the numbers $\alpha_\lambda(\nu; p)$ and $\alpha_\lambda(\{\nu\}; p)$ and unrestricted one dimensional sums $\mathcal{P}_{\lambda\mu}(t)$, it is convenient to introduce the following polynomials $p^{n(\lambda)}\alpha_\lambda(\nu; p^{-1})$ and $p^{n(\lambda)}\alpha_\lambda(\{\nu\}; p^{-1})$.

Proposition 1.4. *i) For any partitions $\nu \subseteq \lambda$,*

$$p^{n(\lambda)}\alpha_\lambda(\nu; p^{-1}) = \prod_{j \ge 1} p^{\binom{\lambda'_j - \nu'_j}{2} + \binom{\nu'_j}{2}} \begin{bmatrix} \lambda'_j - \nu'_{j+1} \\ \nu'_j - \nu'_{j+1} \end{bmatrix}_p. \tag{1.17}$$

ii) Let $\{\nu\} = \{0 = \nu^{(0)} \subset \nu^{(1)} \subset \cdots \subset \nu^{(m)} \subset \nu^{(m+1)} = \lambda\}$ be a flag of partitions. Then

$$p^{n(\lambda)}\alpha_\lambda(\{\nu\}; p^{-1}) = p^{c(\nu)} \prod_{i=1}^{m} \prod_{j \ge 1} \begin{bmatrix} (\nu^{(i+1)})'_j - (\nu^{(i)})'_{j+1} \\ (\nu^{(i)})'_j - (\nu^{(i)})'_{j+1} \end{bmatrix}_p, \tag{1.18}$$

where $c(\nu)=\sum_{i=0}^{m}\sum_{j\geq 1}\binom{(\nu^{(i+1)})'_j-(\nu^{(i)})'_j}{2}$.

Proofs of (1.17) and (1.18) easily follow from the formula (1.16).

Definition 1.5 (see, e.g., [Bu1]). *Let* p *be a prime number,* λ *be a partition of* n*, and* $S=\{1\leq a_1<\cdots<a_m<n\}$ *be a subset of* $[1,n-1]$*. Let us denote by* $\alpha_\lambda(S;p)$ *the number of chains of subgroups*

$$\{e\}\subseteq H^{(1)}\subseteq\cdots\subseteq H^{(m)}\subseteq G$$

in a finite abelian p*–group* G *of type* λ*, where each subgroup* $H^{(i)}$ *has order* p^{a_i}*.*

It follows from Definition 1.5, that

$$\alpha_\lambda(S;p)=\sum_{\{\nu\}}\alpha_\lambda(\{\nu\};p), \tag{1.19}$$

summed over all flags of partitions $\{\nu\}=\{0=\nu^{(0)}\subset\nu^{(1)}\subset\cdots\subset\nu^{(m)}\subset\nu^{(m+1)}=\lambda\}$ such that $|\nu^{(k)}|=a_k$, $1\leq k\leq m$.

Proposition 1.6. *Let* λ *be a partition, and* $S=\{1\leq a_1<\cdots<a_m<|\lambda|\}$ *be a subset of* $[1,|\lambda|-1]$*. Then*

$$p^{n(\lambda)}\alpha_\lambda(S;p^{-1})=\mathcal{P}_{\lambda\mu}(p), \tag{1.20}$$

where $\mu:=\mu(S)$ *stands for the composition* $\mu=(a_1,a_2-a_1,\ldots,a_m-a_{m-1},|\lambda|-a_m)$*.*

Proof follows easily from (1.19) and (0.2).

■

Corollary 1.7. *Let* λ *and* S *be as in Proposition 1.6. Then*

$$\alpha_\lambda(S;p)=\sum_{\eta}K_{\eta\mu(S)}\widetilde{K}_{\eta\lambda}(p), \tag{1.21}$$

where $\mu(S)=(a_1,a_2-a_1,\ldots,a_m-a_{m-1},|\lambda|-a_m)$*, and* $\widetilde{K}_{\eta\lambda}(p)=p^{n(\lambda)}K_{\eta\lambda}(p^{-1})$*.*

Below we will give few examples of application of the formula (1.21).

• Follow to [R], [Bu1], [Fi], let us define the generalized p–binomial coefficient $\begin{bmatrix}\lambda'\\k\end{bmatrix}$ to be the number of subgroups of order p^k of a finite abelian group of type λ. If $\lambda=(1^n)$, then $\lambda'=(n)$, and $\begin{bmatrix}\lambda'\\k\end{bmatrix}$ coincides with p–binomial coefficient $\begin{bmatrix}n\\k\end{bmatrix}_p$. Note also that

$$\begin{bmatrix}\lambda'\\k\end{bmatrix}=\sum_{\nu\vdash k}\alpha_\lambda(\nu;p)=\sum_{\nu\vdash k}p^{\overline{c}(\nu)}\prod_{j\geq 1}\begin{bmatrix}\lambda'_j-\nu'_{j+1}\\ \nu'_j-\nu'_{j+1}\end{bmatrix}_p, \tag{1.22}$$

where $\bar{c}(\nu)=\sum_{j\ge 1}\nu'_{j+1}(\lambda'_j-\nu'_j)$.

It follows from formulae (1.22) and (3.3) that

$$\begin{bmatrix}\lambda'\\ k\end{bmatrix}=\begin{bmatrix}\mathbf{L}\\ a\end{bmatrix}_p,$$

where $L_i=\lambda'_i-\lambda'_{i+1}$, $1\le i\le l(\lambda')$, $\mathbf{L}=(L_i)$, $a=\dfrac{n}{2}-k$, and $\begin{bmatrix}\mathbf{L}\\ a\end{bmatrix}_t$ stands for the Schilling–Warnaar t–supernomial coefficient (0.6).

It is easy to see from Corollary 1.7, that

$$\begin{bmatrix}\lambda'\\ k\end{bmatrix}=\alpha_\lambda(\{k\};p)=\sum_\eta K_{\eta\mu(k)}\widetilde{K}_{\eta\lambda}(p),$$

where $\mu(k)=(k,n-k)$. On the other hand, it is clear that the Kostka–Foulkes number $K_{\eta,(k,n-k)}=0$ unless $\eta=(\eta_1,\eta_2)$ and $\eta_1\ge k$; in the later case $K_{\eta,(k,n-k)}=1$. Thus,

$$\begin{bmatrix}\lambda'\\ k\end{bmatrix}=\sum_{l\ge k}\widetilde{K}_{(l,n-l),\lambda}(p),$$

and

$$\begin{bmatrix}\lambda'\\ k\end{bmatrix}-\begin{bmatrix}\lambda'\\ k-1\end{bmatrix}=\widetilde{K}_{(k,n-k),\lambda}(p).$$

This result is due to Lynne Butler [Bu3].

• Let λ be a partition, $|\lambda|=n$, and $S=\{a_1,\ldots,a_m\}$ be a subset of $[1,n-1]$. Follow [St], [Bu1], [Bu2], consider the following polynomial

$$\beta_\lambda(S;p)=\sum_{T\subseteq S}(-1)^{|S-T|}\alpha_\lambda(T;p).$$

It is known, [St], [Bu1], [Bu2], that $\beta_\lambda(S;p)$ is equal to the top (and only non–vanishing) Betti number of a certain simplicial complex ${}_\lambda(S)={}_\lambda(S;p)$; we refer the reader to [St] for definition of the simplicial complex ${}_\lambda(S)$ and further details.

Let us show that polynomial $\beta_\lambda(S;p)$ has nonnegative coefficients. Indeed,

$$\beta_\lambda(S;p)=\sum_{T\subseteq S}(-1)^{|S-T|}\alpha_\lambda(T;p)=\sum_\eta\left(\sum_{T\subseteq S}(-1)^{|S-T|}K_{\eta\mu(T)}\right)\widetilde{K}_{\eta\lambda}(p).$$

Our nearest aim is to show that the number $\left(\sum_{T\subseteq S}(-1)^{|S-T|}K_{\eta\mu(T)}\right)\ge 0$. For this goal let us show that the latter number counts the number of Littlewood–Richardson tableaux of a certain skew shape $b(S)$ and weight η. More precisely, for a given subset $S = \{a_1 < a_2 < \cdots < a_m\}$ of the set $[1, n-1]$, the skew shape $b(S)$ is the *border strip* with a_1 squares in row 1, $a_2 - a_1$ squares in row $2, \ldots, n - a_m$ squares in row $m+1$. For the reader's convenience, let us remind that

• a skew shape is called a border strip if consecutive rows overlap by exactly one square (see, e.g., [M], p.5);

• a skew tableau T is called Littlewood–Richardson tableau, if the word $w(T)$ corresponding to the tableau T is a lattice permutation (see, e.g. [M]. Chapter I, §9).

Proposition 1.8 (R. Stanley [St]). *Let $S = \{a_1 < \cdots < a_m\}$ be a subset of the set $[1, n-1]$, and $b(S)$ stands for border strip with a_1, squares in row 1, $a_2 - a_1$ squares in row $2, \ldots, n - a_m$ squares in row $m+1$. Then*

$$\beta_\lambda(S;p) = \widetilde{K}_{b(S),\lambda}(p),$$

where $\widetilde{K}_{b(S),\lambda}(p)$ stands for the cocharge Kostka–Foulkes polynomial corresponding to the skew shape $b(S)$, see, e.g., [Ki1], [Bu1].

Indeed,

$$\beta_\lambda(S;p) = \sum_\eta \left(\sum_{T \subseteq S} (-1)^{|S-T|} K_{\eta\mu(T)} \right) \widetilde{K}_{\eta\lambda}(p) = \sum_\eta \#|\mathrm{Tab}^0(b(S),\eta)| \widetilde{K}_{\eta\lambda}(p)$$
$$= \widetilde{K}_{b(S),\lambda}(p).$$

To deduce the second equality we used the following formula

$$\sum_{T \subseteq S} (-1)^{|S-T|} K_{\eta\mu(T)} = \#|\mathrm{Tab}^0(b(S),\eta)|, \tag{1.23}$$

where $\mathrm{Tab}^0(b(S),\eta)$ stands for the set of all Littlewood–Richardson tableaux of skew shape $b(S)$ and weight η. The formula (1.23) can be obtained using the results from [KKN].

■

Finally, let us describe (see [Bu1], Definition 1.3.1) the statistic *value*, denoted by v on the set of tabloids. This statistic generates the generalized mahonian statistic VAL on the set of transport matrices, see Section 2.

Definition 1.9 ([Bu1]). *Let T be a tabloid of any shape and weight, and $x \in T$ be an entry of T. Then the value $v(x)$ of the entry x in T is the number of smaller entries in the same column and above x, or in the next column to the right and below x. The value $v(T)$ of T is the sum of the values of the entries in T : $v(T) = \sum_{x \in T} v(x)$.*

Example. Consider

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 1 & 2 \\ \hline 2 & 2 & 1 & 3 \\ \hline 3 & 2 & 2 \\ \cline{1-3} 3 & 2 & 3 \\ \cline{1-3} \end{array} \quad \in \; T(4433,\ 374),$$

Then $v(T) = (0+1+3+2)+(1+0+0+0)+(0+0+2+3)+(0+1) = 13$. Note that the Shimomura statistic $d(T)$ (see Section 2) of the tabloid T is equal to 10, see Example in Subsection 2.3.

Proposition 1.10 ([Bu1]). *Let λ be a partition, $|\lambda| = n$, and $S = \{a_1 < a_2 < \cdots < a_m\}$ be a subset of $[1, n-1]$. Then*

$$\alpha_\lambda(S;p) = \sum_T p^{v(T)},$$

summed over all tabloids T of shape λ and weight $\mu := \mu(S)$.

§2. Generalized mahonian statistics.

2.1. Mahonian statistics on the set $M(\mu)$.

We start with recalling the definition of mahonian statistic on words, [F]. A word is a finite sequence of letters, $w = w_1 \ldots w_N$, where each letter is in the set $\{1, \ldots, n\}$. Let $\mu = (\mu_1, \ldots, \mu_n)$ be a composition, $|\mu| = N$, denote by $M(\mu)$ the set of all words $w = w_1 \ldots w_N$ of weight μ, i.e. μ_i is the number of occurrences of i in the word w. It is well–known (see, e.g. [An]) that the cardinality of the set $M(\mu)$ is equal to the multinomial coefficient $\begin{pmatrix} N \\ \mu_1, \ldots, \mu_n \end{pmatrix}$.

Definition 2.1 ([F]). *A function φ on the set $M(\mu)$ is called mahonian statistic, if*

$$\sum_{w \in M(\mu)} q^{\varphi(w)} = \begin{bmatrix} N \\ \mu_1, \ldots, \mu_n \end{bmatrix}_q .$$

Examples. 1^0 (Inversion number, [Ma]). Let $w \in M(\mu)$ be a word, define the number of inversions for the word w to be

$$INV(w) = \sum_{1 \le i < j \le N} \chi(w_i > w_j).$$

where $\chi(A) = 1$ if A is true and 0 otherwise.

2^0 (Major index, [Ma]). Define the major index of the word w to be

$$MAJ(w) = \sum_{m=1}^{N-1} m\chi(w_m > w_{m+1}).$$

3^0 (Modified major index, [Ki2]). Define the modified major index of the word w to be

$$\widetilde{MAJ}(w) = \sum_{m=1}^{N-1} m\chi(w_m \ge w_{m+1}) - n(\mu'), \quad \text{where} \quad n(\mu') = \sum_{i=1}^{n} \binom{\mu_i}{2}.$$

4^0 (Zeilberger's index, [ZB]). For given $w \in M(\mu)$ let w_{ij} be the subword of w formed by deleting all letters w_m such that $w_m \neq i$ or j. For example, if $w = 2411213144321 \in M(5323)$, then $w_{12} = 21121121$, $w_{13} = 1113131$, $w_{14} = 41111441$, $w_{23} = 22332$, $w_{24} = 242442$, $w_{34} = 43443$.

• Zeilberger's index, or Z–index, of a word w is defined to be the sum of major indices of all 2–letter subwords w_{ij} of w:

$$Z(w) = \sum_{1 \le i < j \le n} MAJ(w_{ij}).$$

• Modified Zeilberger's index, or $\widetilde{Z}$–index, of a word w is defined to be the sum of modified major indices of all 2–letter subwords w_{ij} of w:

$$\widetilde{Z}(w) = \sum_{1 \le i < j \le n} \widetilde{MAJ}(w_{ij}).$$

Next example will require some definitions. First, for a word w let denote $\overline{w}$ the non–decreasing rearrangement of the letters of w. Second, if a and b are positive integers, with $a \le n$, let

$$C[a,b] = \begin{cases} [a+1, a+2, \ldots, b], & \text{if } a \le b; \\ [1, 2, \ldots, b, a+1, a+2, \ldots, n], & \text{if } a > b. \end{cases}$$

5^0 (Denert's index, M. Denert, see e.g., [FZ], [GaW]). Define the Denert index of a word w to be

$$DEN(w) = \sum_{1 \le i < j \le n} \chi(w_i \in C[w_j, \overline{w}_j]).$$

For example, if $w = M(5323)$ as above, then $INV(w) = 29$, $MAJ(w) = 47$, $\widetilde{MAJ}(w) = 42$, $Z(w) = 46$, $\widetilde{Z}(w) = 31$, $DEN(w) = 46$.

Theorem 2.2 ([Ma], [ZB], [Ki2], [FZ]). *The statistics INV, MAJ, $\widetilde{MAJ}$, Z, $\widetilde{Z}$, DEN are mahonian.*

2.2. Dual mahonian statistics.

Now we are going to extend the notion of mahonian statistic to the set of transport matrices. Let us denote by $\mathcal{P}_{\lambda\mu}$ (respectively $\mathcal{R}_{\lambda\mu}$) the set of all matrices of non–negative integers (respectively the set of all (0,1)–matrices) with row sums λ_i and column sums μ_j. It is clear that if $\lambda = (1^N)$ then the both sets $\mathcal{P}_{(1^N)\mu}$ and $\mathcal{R}_{(1^N)\mu}$ can be naturally identified with the set $M(\mu)$.

Definition 2.3. *A function ψ on the set $\mathcal{R}_{\lambda\mu}$ is called dual mahonian statistic if*

$$\sum_{m \in \mathcal{R}_{\lambda\mu}} q^{\psi(m)} = \sum_{\eta} K_{\eta\mu} K_{\eta'\lambda}(q).$$

Let us give a few examples of the dual mahonian statistics.

1^0. The first example is due to A. Zelevinsky, see [M], Chapter III, §6, Example 5, p.244. Let λ and μ be compositions of the same integer n, and $m = (m_{ij}) \in \mathcal{R}_{\lambda\mu}$. For each element $a = m_{ij}$ of the matrix m we denote by $i(a) := i$, and $j(a) := j$ its first and second coordinates. We denote by $\mathrm{supp}(m) = \{m_{ij} \in m \mid m_{ij} \neq 0\}$ the set of all nonzero entries of m. If $a = m_{ij} \in \mathrm{supp}(m)$ we define the height of a to be $\mathrm{ht}(a) = \sum\limits_{1 \le k \le i} m_{kj}$. For each $a \in \mathrm{supp}(m)$ let us define

$$i^+(a) = \begin{cases} i(b), & \text{if } \exists b \in \mathrm{supp}(m) \text{ such that } j(a) = j(b) \text{ and } \mathrm{ht}(b) = \mathrm{ht}(a) + 1; \\ +\infty, & \text{if such } b \text{ doesn't exist.} \end{cases}$$

Follow A. Zelevinsky [*ibid*], for each $m \in \mathcal{R}_{\lambda\mu}$ we define $\widetilde{ZEL}(m) = \sum\limits_{a \in \mathrm{supp}(m)} \widetilde{z}(a)$, where $\widetilde{z}(a)$ is equal to the number of $b \in \mathrm{supp}(m)$ such that

i) $j(b) < j(a)$,

ii) $\mathrm{ht}(b) = \mathrm{ht}(a)$,

iii) $i(a) < i(b) < i^+(a)$.

Theorem 2.4 (A. Zelevinsky).

$$\mathcal{R}_{\lambda\mu}(q) = \sum_{m \in \mathcal{R}_{\lambda\mu}} q^{\widetilde{\mathrm{ZEL}}(m)},$$

in other words, the statistic $\widetilde{\mathrm{ZEL}}$ *is dual mahonian.*

There exists a bijection between the set $\mathcal{R}_{\lambda\mu}$ and that of all column strict tabloids of shape λ' and weight μ. Let ν and μ be compositions of the same integer n. A tabloid of shape ν and weight μ is a filling of the diagram of boxes with row lengths $\nu_1, \nu_2, \ldots, \nu_r$, such that the number i occurs μ_i times, and such that each column is nondecreasing. A tabloid of shape ν and weight μ is called a column strict if each column is strictly decreasing. For example,

$$\begin{array}{ccc} 1 & 2 & \\ 1 & 3 & 1 \\ 2 & & \\ 4 & 3 & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} 1 & 1 & \\ 2 & 2 & 1 \\ 3 & & \\ 4 & 3 & \end{array}$$

are tabloid and column strict tabloid of weight (3221) and shape (2312). We denote by $T(\nu, \mu)$ (respectively, $\widetilde{T}(\nu, \mu)$) the set of all tabloids (respectively, the set of all column strict tabloids) of shape ν and weight μ.

Now we are ready to describe a bijection $\mathcal{R}_{\lambda\mu} \leftrightarrow \widetilde{T}(\lambda', \mu)$ in the case when λ is a partition. Namely, consider a matrix $m = (m_{ij}) \in \mathcal{R}_{\lambda\mu}$. Let us fill the shape λ' by positive integers according to the following rule: if $m_{ij} \neq 0$, put the number i in the box of the shape λ' with coordinates (i,j). As a result we obtain the tabloid T of shape λ' and weight μ. For example, consider $\lambda=(3221)$, $\mu=(2231)$, and

$$m=\left\{\begin{matrix} 1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\\ 1 & 1 & 0 & 1\\ 1 & 0 & 0 & 0\end{matrix}\right\}\in\mathcal{R}_{\lambda,\mu}.$$

The corresponding column strict tabloid is

1	2	1	3
3	3	2	
4			

.

It is easy to see that the correspondence $m\to T$ defines a bijection. Let us continue and define a statistic $\widetilde{ZEL}$ on the set of all column strict tabloids. Namely, for any column strict tabloid T of shape ν, let $\widetilde{ZEL}(T)$ denote the number of pairs $(x,y)\in\nu\times\nu$ such that y lies to the left from x (in the same row) and $T(x)<T(y)<T(x\downarrow)$. We have used here the following notations: if a box x has coordinates (i,j), then $x\downarrow=(i+1,j)$ and $\vec{x}=(i,i+1)$; for any $x\in\nu$, $T(x)$ is the integer located in the box x of the tabloid T; if $x\downarrow$ does not belong to the shape ν, then we put $T(x\downarrow)=+\infty$. It is clear that if a composition ν contains only one part, then $\widetilde{T}(\nu,\mu)=M(\mu)$, and $\widetilde{ZEL}$ coincides with statistic INV.

Theorem 2.5 (A. Zelevinsky).

$$\mathcal{R}_{\lambda\mu}(q)=\sum_{T\in\widetilde{T}(\lambda',\mu)}q^{\widetilde{ZEL}(T)}.$$

2^0. Let λ be a partition and μ be a composition of the same integer n, and $m\in\mathcal{R}_{\lambda\mu}$. There is an explicit one–to–one correspondence, due to Knuth [Kn], between the set of (0,1)–matrices with row sums λ_i and column sums μ_j, and pairs of semistandard tableaux of conjugate shapes and weights λ, μ, (Knuth's dual correspondence):

$$\begin{array}{ccc}\mathcal{R}_{\lambda\mu} & \cong & \coprod_\eta \mathrm{SST}(\eta,\mu)\times\mathrm{SST}(\eta',\lambda)\\[2ex] m & \leftrightarrow & (P,Q).\end{array}$$

Let us define the charge CH of a matrix $m\in\mathcal{R}_{\lambda\mu}$ to be the Lascoux–Schützenberger charge ([LS]) of the corresponding semistandard tableaux Q of weight λ:

$$CH(m)=c(Q).$$

It follows from the results of Lascoux and Schützenberger [LS], and Knuth [Kn], that

$$\mathcal{R}_{\lambda\mu}(q)=\sum_{m\in\mathcal{R}_{\lambda\mu}}q^{CH(m)}.$$

It is an interesting problem to find a bijective proof that the statistics $\widetilde{ZEL}$ and CH have the same distribution on the set $\mathcal{R}_{\lambda\mu}$.

2.3. Generalized mahonian statistics.

Definition 2.6. *A function φ on the set of transport matrices $\mathcal{P}_{\lambda\mu}$ is called generalized mahonian statistic if*

$$\sum_{m\in\mathcal{P}_{\lambda\mu}} q^{\varphi(m)} = q^{E_0}\sum_{\eta} K_{\eta\mu}K_{\eta\lambda}(q),$$

for a certain constant $E_0 := E_{0,\varphi}$.

There is a well–known bijection between sets $\mathcal{P}_{\nu\mu}$ and $T(\nu,\mu)$. To describe this bijection, let $m\in\mathcal{P}_{\nu\mu}$, and $D(\nu)$ be the diagram of the composition ν. To obtain a tabloid, let us fill the first m_{1j} boxes of the j-th row of $D(\nu)$ by the number 1, the next m_{2j} boxes of same row by the number 2, and so on. As a result we obtain the tabloid of shape ν and weight μ. This construction defines the bijection under consideration. To go further, let us recall the Shimomura cells decomposition [Sh] of the fixed point variety $\mathcal{F}_{\mu}^{\lambda}$ of a unipotent u of type λ (λ is a partition) acting on the partial flag variety $\mathcal{F}_{\mu}$. The cells in Shimomura's decomposition are indexed by tabloids of shape λ and weight μ. The dimension $d(T)$ of the cell c_T indexed by $T\in T(\lambda,\mu)$ is computed by algorithm described below ([Sh], [LLT]), and defines the mahonian statistic $\widetilde{d}(T) = n(\lambda)-d(T)$ (=codimension of the cell c_T) on the set $\mathcal{P}_{\lambda\mu}$:

$$\mathcal{P}_{\lambda\mu}(t) = \sum_{T\in T(\lambda,\mu)} t^{\widetilde{d}(T)}.$$

The dimensions $d(T)$ are given by the following algorithm ([LLT], Section 8.1).

1) If $T\in T(\lambda,(n))$ then $d(T)=0$;

2) If $\mu=(\mu_1,\mu_2)$ has exactly two parts, and $T\in T(\lambda,\mu)$, then $d(T)$ is computed as follows. A box x of T is called special if $T(x)$ is the lowest 1 of the column containing x. For a box y such that $T(y)=1$, put $d(y)=0$; if $T(y)=2$, set $d(y)$ equals to the number of nonspecial 1's lying in the row of y, plus the number of special 1's lying in the same row, but from the right side of y. Then $d(T)=\sum d(y)$, summed over all $y\in T$ such that $T(y)=2$.

3) Let $\mu=(\mu_1,\ldots,\mu_k)$ and $\mu^*=(\mu_1,\ldots,\mu_{k-1})$. For $T\in T(\lambda,\mu)$, let T_1 be the tabloid obtained from T by changing the entries k into 2 and all the other ones by 1. Let T_2 be the tabloid of weight μ^* obtained from T by erasing all the entries k, and rearranging the columns in the appropriate order. Then $d(T)=d(T_1)+d(T_2)$.

Example. Consider

$$T = \begin{array}{|c|c|c|c|}\hline 1&2&1&2\\ \hline 2&2&1&3\\ \hline 3&2&2\\ \cline{1-3} 3&2&3\\ \cline{1-3}\end{array} \in T(4433,\ 374),$$

then

$$T_1 = \begin{array}{cccc} 1 & 1 & 1 & \mathbf{1} \\ 1 & 1 & 1 & 2 \\ 2 & 1 & \mathbf{1} & \\ 2 & \mathbf{1} & 2 & \end{array} \qquad T_2 = \begin{array}{cccc} 2 & 1 & \mathbf{1} & 2 \\ 2 & \mathbf{1} & 2 & \\ 2 & 2 & & \\ 2 & & & \end{array}$$

where the special entries are printed in bold type. Thus, $d(T) = d(T_1) + d(T_2) = (3+2+1) + (2+1+1) = 10$.

There is a variant of this construction due to A. Lascoux, B. Leclerc and J.-Y. Thibon [LLT], in which the shape λ is allowed to be an arbitrary composition. Such a variant has already been used by I. Terada [T] in the case of complete flags (i.e. $\mu = (1^N)$).

Let ν be a composition, and $T \in T(\nu, \mu)$. Follow to [LLT], define an integer $e(T)$ by the following rules:

i) for $T \in T(\nu, (n))$, $e(T) = d(T) = 0$;

ii) for $T \in T(\nu, (\mu_1, \mu_2))$, $e(T) = d(T)$;

iii) otherwise $e(T) = e(T_1) + e(T_2)$ where T_1 is defined as above, but this time T_2 is obtained from T by erasing the entries k, without reordering.

Let us define $\widetilde{e}(T) = n(\lambda) - e(T)$.

Proposition 2.7 ([LLT]). Let ν be a composition and $\lambda = \nu^+$ be the corresponding partition. Then

$$\sum_{T \in T(\lambda,\mu)} t^{\widetilde{d}(T)} = \sum_{T \in T(\nu,\mu)} t^{\widetilde{e}(T)} = \mathcal{P}_{\lambda\mu}(t).$$

Example (cf. [LLT], Example 8.4). Take $\lambda = (321)$, $\mu = (42)$ and $\nu = (312)$. The set $T(\lambda, \mu)$ consists of the following tabloids

T	$\begin{array}{ccc} 1 & 1 & 2 \\ 1 & 2 & \\ 1 & & \end{array}$	$\begin{array}{ccc} 1 & 2 & 1 \\ 1 & 2 & \\ 1 & & \end{array}$	$\begin{array}{ccc} 1 & 1 & 2 \\ 1 & 1 & \\ 2 & & \end{array}$	$\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 2 & \\ 2 & & \end{array}$	$\begin{array}{ccc} 1 & 1 & 1 \\ 2 & 1 & \\ 2 & & \end{array}$
$\widetilde{d}(T)$	2	1	2	4	3

The set of tabloids of shape ν and weights μ contains the following ones

T	$\begin{array}{ccc} 1 & 2 & 1 \\ 1 & & 2 \\ 1 & & \end{array}$	$\begin{array}{ccc} 1 & 1 & 2 \\ 1 & & 2 \\ 1 & & \end{array}$	$\begin{array}{ccc} 1 & 2 & 1 \\ 1 & & 1 \\ 2 & & \end{array}$	$\begin{array}{ccc} 1 & 1 & 1 \\ 1 & & 2 \\ 2 & & \end{array}$	$\begin{array}{ccc} 1 & 1 & 1 \\ 2 & & 1 \\ 2 & & \end{array}$
$\widetilde{e}(T)$	1	2	2	4	3

Let us give yet another example of generalized mahonian statistic denoted by VAL. This example is due essentially to Lynne Butler [Bu1].

Let λ be a partition and μ be a composition, $|\lambda| = |\mu|$. On the set $T(\lambda,\mu)$ of tabloids of shape λ and weight μ one can define the statistic *value* v, see [Bu1], Definition 1.3.1, or Subsection 1.6,

Definition 2.8. *Let us define* $\mathrm{VAL}(T) = n(\lambda) - v(T)$.

Example. Take $\lambda = (321)$ and $\mu = (42)$, Consider the set of tabloids $T(\lambda,\mu)$ in the same order as in the previous Example. Then the values of statistic VAL on the set $T(\lambda,\mu)$ are the following 3,4,2,1,2, and

$$\sum_{T\in T(\lambda,\mu)} t^{\mathrm{VAL}(T)} = \mathcal{P}_{\lambda\mu}(t).$$

Proposition 2.9 ([Bu1]). *Let λ be a partition and μ be a composition. Then*

$$\sum_{T\in T(\lambda,\mu)} t^{\mathrm{VAL}(T)} = \mathcal{P}_{\lambda\mu}(t).$$

Problem 1. Find a bijective proof that if λ is a partition, then the Shimomura statistic $\widetilde{d}$, LLT–statistic $\widetilde{e}$, statistic VAL, and the energy function E are equidistribute on the set of transport matrices $\mathcal{P}_{\lambda\mu}$.

§3. Main results.

3.1. Combinatorial formula for modified Hall-Littlewood polynomials.

Theorem 3.1. ([HKKOTY]) *Let λ be a partition, and μ, $l(\mu) = r$, be a composition of the same integer n, then*

$$\mathcal{P}_{\lambda\mu}(t) := \sum_{\eta} K_{\eta\mu}K_{\eta\lambda}(t) = \sum_{\{\nu\}} t^{c(\nu)} \prod_{k=1}^{r-1}\prod_{i\geq 1} \begin{bmatrix} \nu_i^{(k+1)} - \nu_{i+1}^{(k)} \\ \nu_i^{(k)} - \nu_{i+1}^{(k)} \end{bmatrix}_t, \tag{3.1}$$

summed over all flags of partitions $\nu = \{0 = \nu^{(0)} \subset \nu^{(1)} \subset \cdots \subset \nu^{(r)} = \lambda'\}$, such that $|\nu^{(k)}| = \mu_1 + \cdots + \mu_k$, $1 \leq k \leq r$; and

$$c(\nu) = \sum_{k=0}^{r-1}\sum_{i\geq 1} \binom{\nu_i^{(k+1)} - \nu_i^{(k)}}{2}.$$

where for any real number α we put $\binom{\alpha}{2} := \alpha(\alpha-1)2$.

Proof of Theorem 3.1 will be given in Subsection 4.2.

Remark. It is well–known ([Kn]; [M], Chapter I, Section 6) that $\mathcal{P}_{\lambda\mu}(1)$ is equal to the number of matrices of non–negative integers with row sums λ_i and column sums μ_j. This number is equal also to that of pairs of semistandard tableaux of the same shape and weights λ and μ, [Kn].

Examples. 1^0. Let us take a length two composition $\mu = (\mu_1, \mu_2)$, and a partition λ. Let $\lambda' = (\lambda'_1, \ldots, \lambda'_k)$ be the conjugate partition. Then the identity (3.1) takes the following form

$$\sum_{\eta} K_{\eta\mu} K_{\eta\lambda}(t) = \sum_{\nu \vdash \mu_1} t^{c(\nu)} \prod_{i=1}^{k} \begin{bmatrix} \lambda'_i - \nu_{i+1} \\ \nu_i - \nu_{i+1} \end{bmatrix}_t, \tag{3.2}$$

summed over all partitions ν of μ_1, $l(\nu) = k$, and $c(\nu) = \sum_{i=1}^{k} \binom{\lambda'_i - \nu_i}{2} + \sum_{i=1}^{k} \binom{\nu_i}{2}$.

Let us put $L_i = \lambda'_i - \lambda'_{i+1}$ and $j_i = \lambda'_i - \nu_i$, $1 \le i \le k$, $j_{k+1} = 0$. Then we have

$$\sum_{i=1}^{k} j_i = \mu_2, \qquad \begin{bmatrix} \lambda'_i - \nu_{i+1} \\ \nu_i - \nu_{i+1} \end{bmatrix}_t = \begin{bmatrix} L_i + j_{i+1} \\ j_i \end{bmatrix}_t,$$

and

$$\begin{aligned} c(\nu) &= \sum_{i=1}^{k} \binom{\lambda'_i}{2} + \sum_{i=1}^{k} j_i (j_i - \lambda'_i) \\ &= \sum_{i=1}^{k} j_i (j_i - L_i - L_{i+1} - \cdots - L_k) + \frac{1}{2} \sum_{1 \le i,j \le k} \min(i,j) L_i L_j - \frac{1}{2} |\mu|. \end{aligned}$$

Thus, RHS(3.2)$= t^A \begin{bmatrix} \mathbf{L} \\ a \end{bmatrix}_{1/t}$, where
$A = \frac{1}{2} \sum_{1 \le i,j \le k} \min(i,j) L_i L_j - \frac{1}{2} |\mu|$, and $a = -\frac{\mu_1 - \mu_2}{2}$;
$\begin{bmatrix} \mathbf{L} \\ a \end{bmatrix}_t$, stands for the Schilling–Warnaar t–supernomial coefficients (0.5), see [ScW], (2.9).

It follows from the formulae above that

$$\begin{bmatrix} \mathbf{L} \\ a \end{bmatrix}_t = \sum_{\eta} K_{\eta\mu} \widetilde{K}_{\eta\lambda}(t), \tag{3.3}$$

where $\widetilde{K}_{\eta\lambda}(t) = t^{n(\lambda)} K_{\eta\lambda}(t^{-1})$, $\quad \mu = \left(\frac{1}{2} \left(\sum_{i=1}^{k} i L_i \right) - a,\ \frac{1}{2} \left(\sum_{i=1}^{k} i L_i \right) + a \right)$,
and $\lambda'_i = L_i + \cdots + L_k$, $1 \le i \le k$.

Now let us assume additionally that $\lambda'_1 = \cdots = \lambda'_k = N$, or equivalently, $\lambda = (k^N)$. Then the RHS(3.2) can be rewritten in the following form

$$\sum_{\nu \vdash \mu_1} t^{c(\nu)} \begin{bmatrix} N \\ N-\nu_1, \nu_1-\nu_2, \ldots, \nu_{k-1}-\nu_k, \nu_k \end{bmatrix}_t, \tag{3.4}$$

where $c(\nu) = k\binom{N}{2} - N\mu_1 + \sum_{i=1}^{k} \nu_i^2$. The sum in (3.4) is taken over all partitions ν of μ_1 such that $l(\nu) = k$.

If we put $m_i = k(\nu_1 + \cdots + \nu_{k-i}) - (k-i)\mu_1$, $1 \le i \le k-1$, (and, consequently, $k\nu_i = m_{k-i} - m_{k-i+1} + \mu_1$, $1 \le i \le k-1$), then the sum (3.4) coincides with the RHS(2.49), [Ki2], Theorem 14 (in [Ki2] we have used q instead of t).

Let us remark that the sum (3.4) is closely related to the special value $p = 0$ of the Schilling and Warnaar q–multinomial coefficient $\begin{bmatrix} L \\ a \end{bmatrix}_k^{(p)}$ ([Sc], §2, and [W], Definition 1). More precisely, we state that sum (3.4) is equal to $t^{k\binom{N}{2}} \begin{bmatrix} N \\ \mu_1 \end{bmatrix}_k^{(0)} (t^{-1})$. This statement is equivalent (cf. (3.3)) to the following one:

$$\begin{bmatrix} N \\ \mu_1 \end{bmatrix}_k^{(0)} = \sum_{\eta} K_{\eta\mu} \widetilde{K}_{\eta,(k^N)}(q). \tag{3.5}$$

Formulae (3.3) and (3.5) suggest the following definition:

Definition 3.2. *Let λ be a partition and μ be a composition, $|\lambda| = |\mu|$. Define the t–multinomial coefficient $\begin{bmatrix} \lambda \\ \mu \end{bmatrix}^{(0)}$ to be*

$$\begin{bmatrix} \lambda \\ \mu \end{bmatrix}^{(0)} = \sum_{\eta} K_{\eta\mu} \widetilde{K}_{\eta\lambda}(t).$$

Thus, see Corollary 1.7, if $t = p$ is a prime number and $l(\mu) = m+1$, then the t–multinomial coefficient $\begin{bmatrix} \lambda \\ \mu \end{bmatrix}^{(0)}$ counts the number of chains of subgroups

$$\{e\} \subseteq H^{(1)} \subseteq H^{(2)} \subseteq \cdots \subseteq H^{(m)} \subseteq G$$

of a finite abelian p–group G of type λ such that each subgroup $H^{(i)}$ has order $p^{\mu_1 + \cdots + \mu_i}$.

It follows from (3.3) that if a composition $\mu = (\mu_1, \mu_2)$ consists of two parts then the t–multinomial coefficient $\begin{bmatrix} \lambda \\ \mu \end{bmatrix}^{(0)}$ coincides with the t–supernomial coefficient $\begin{bmatrix} \mathbf{L} \\ a \end{bmatrix}_t$, where

$a = -\dfrac{\mu_1 - \mu_2}{2}$, and if $\lambda' = (\lambda'_1, \ldots, \lambda'_k)$ is the conjugate partition, then $\mathbf{L} := (L_1, \ldots, L_k)$ with $L_i = \lambda'_i - \lambda'_{i+1}$, $1 \le i \le k$.

More generally, let B be a crystal (see, e.g., [Ka1], [KMOTU1], [HKKOTY]), and $b \in B$. Define the (unrestricted) t–multinomial coefficient $T^{(b)}(\lambda; \mu)$ to be

$$T^{(b)}(\lambda; \mu) = t^{-E_{\min}} \sum_{p \in \mathcal{P}_\mu(b,\lambda)} t^{E(p)},$$

where $\mathcal{P}_\mu(b, \lambda)$ is the set of paths $p = b \otimes b_1 \otimes \cdots \otimes b_m \in B \otimes B_{(\mu_1)} \otimes \cdots \otimes B_{(\mu_m)}$ such that $wt(b_1) + \cdots + wt(b_m) = \lambda$; $E(p)$ is the energy of a path p (see, e.g., [HKKOTY]).

Similarly, one can define classically restricted and restricted t–multinomial coefficients. We intend to consider the properties (including recurrence relations, bosonic formulae, multinomial analogue of Bailey's lemma, and applications to polynomial identities and q–series) of these t–multinomial coefficients in a separate publication.

2^0. If $\mu = (1^n)$, then (3.1) coincides with the formula for modified Green's polynomials $X^{\lambda}_{(1^n)}(t)$ from [M], Example 4 on p.249.

Let us describe two generalized mahonian statistics on the set $M(\lambda)$. The first one is the Lascoux–Schützenberger charge c defined on the set of dominant weight words w, i.e. $w \in M(\lambda)$, where λ is a partition, see [LS]; [M], Chapter III, §6, p.242. The second one is the LP statistic (see, e.g., [GaW]) which can be defined for arbitrary words.

Definition 3.3. *Let w be a word, define $lp_i(w)$ to be the number of distinct letters to the left of position i and having the same multiplicity as the letter in position i in the truncated word $w_1 \ldots w_i$. Let $LP(w) = \sum_{i \ge 2} lp_i(w)$.*

For example, $LP(3422231413) = 0 + 2 + 0 + 0 + 1 + 1 + 0 + 2 + 0 = 6$. One can show that if ν is a composition, $\lambda = \nu^+$ is the corresponding partition, then

$$\mathcal{P}_{\lambda(1^n)}(q) = \sum_{w \in M(\nu)} q^{LP(w)} = \sum_{w \in M(\lambda)} q^{c(w)}.$$

3^0. If $\lambda = (1^N)$, then the RHS(3.1) coincides with that of (1.5).

4^0. Let μ be a composition of length n, and $\lambda = (2^N)$, so that $|\mu| = 2N$. In this case we have

- $\nu^{(k)} = (\nu_1^{(k)}, \nu_2^{(k)})$, $|\nu^{(k)}| = \mu_1 + \cdots + \mu_k$, $1 \le k \le n$;
- $0 \le \nu_2^{(k-1)} \le \nu_2^{(k)} \le \nu_1^{(k)} \le \nu_1^{(k+1)} \le N$, if $1 \le k \le n-1$, and $\nu^{(n)} = (N, N)$.

If we define $m_i = 2\nu_1^{(i)} - \mu_1 - \cdots - \mu_i \ge 0$, $0 \le i \le n-1$, and

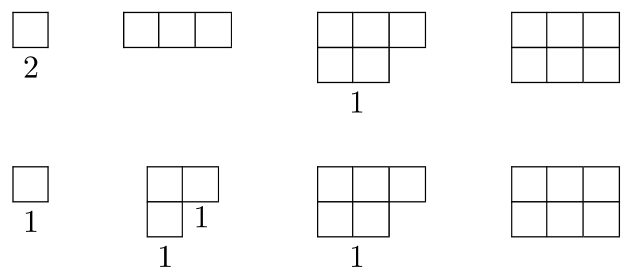
$$\beta_i = \frac{\mu_i + m_{i-1} - m_i}{2} \in \mathbf{Z}_{\ge 0}, \quad 1 \le i \le n, \quad m_0 = m_n = 0,$$

then the RHS(3.1) takes the following form

$$\sum_{m \in \mathbf{Z}_{\ge 0}^{n-1}} t^{c(m)} \begin{bmatrix} N \\ \beta_1, \ldots, \beta_n \end{bmatrix}_t \prod_{k=1}^{n-1} \begin{bmatrix} \beta_{k+1} + m_{k+1} \\ m_k \end{bmatrix}_t, \tag{3.6}$$

summed over all sequences $m = (m_1, \ldots, m_{n-1}) \in \mathbf{Z}_{\geq 0}^{n-1}$ such that $m_i + m_{i-1} + \mu_i \equiv 0 (\mathrm{mod}\ 2)$, $1 \leq i \leq n$, $m_0 = m_n = 0$, and $c(m) = \sum_{i=1}^{n} \binom{\mu_i}{2} + \frac{1}{4} m C_{n-1} m^t$, where C_{n-1} is the Cartan matrix of type A_{n-1}.

It is well–known that the LHS(3.1) does not depend on the permutations of components of the composition μ. Hence, the same is valid for the RHS(3.1) as well. This is not obvious at all because the number of terms in the right hand side sum (3.1) do depends on the composition μ, but not only on the corresponding partition μ^+. For example, let us take $\mu = (1221)$ and $\lambda = (2^3)$. The summands in the RHS(3.1) correspond to the following flags of partitions $\nu = \{\nu^{(1)} \subset \nu^{(2)} \subset \nu^{(3)} \subset \nu^{(4)}\}$:

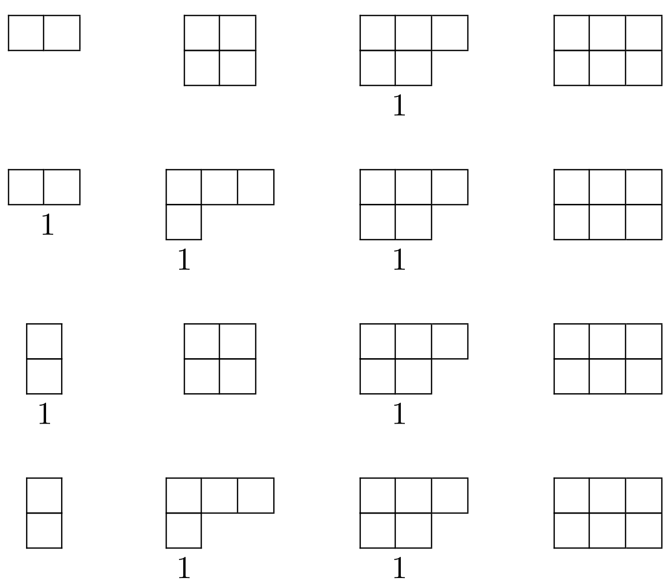


$c(\nu) = 2,$

$c(\nu) = 0.$

Hence, the RHS(3.1)$= 1 + 4t + 7t^2 + 7t^3 + 4t^4 + t^5 + t^2(1 + 2t + 3t^2 + 2t^3 + t^4) = 1 + 4t + 8t^2 + 9t^3 + 7t^4 + 3t^5 + t^6$.

On the other hand, for the partition $\mu^+ = (2211)$ the contribution to the RHS(3.1) is given by the following flags of partitions:



$c(\nu) = 2,$

$c(\nu) = 1,$

$c(\nu) = 0,$

$c(\nu) = 1.$

Hence, the RHS(3.1)$= t^2(1 + t + t^2) + t(1 + 3t + 5t^2 + 5t^3 + 3t^4 + t^5) + (1 + 2t + 2t^2 + t^3) + t(1 + 2t + 2t^2 + t^3) = 1 + 4t + 8t^2 + 9t^3 + 7t^4 + 3t^5 + t^6$.

We see that $\mathcal{P}_{\lambda\mu}(t) = \mathcal{P}_{\lambda\mu^+}(t)$, but the corresponding sums of the products of t–binomial coefficients have different structures.

3.2. New combinatorial formula for the transition matrix $M(e,P)$.

Now we are going to describe the fermionic formula for the following sum

$$\mathcal{R}_{\lambda\mu}(t)=\sum_{\eta}K_{\eta\mu}K_{\eta'\lambda}(t).$$

This sum is the (λ,μ)–entry of the matrix transposed to the transition matrix between elementary and Hall–Littlewood polynomials, namely, if

$$e_{\lambda}=\sum_{\mu}M(e,P)_{\lambda\mu}P_{\mu},\quad \text{then}$$

$$M(e,P)_{\lambda\mu}=\sum_{\nu}K_{\nu\lambda}K_{\nu'\mu}(q)=\mathcal{R}_{\mu\lambda}(q).$$

It is well–known ([Kn]) that $\mathcal{R}_{\lambda\mu}(1)$ counts the number of (0,1)–matrices with row sums λ_i and column sums μ_j. This number is equal also to the number of pairs of semistandard tableaux of conjugate shapes and weights λ and μ, see, e.g. [M], Chapter I, Section 6.

Theorem 3.4. ([HKKOTY]) *Let μ be a composition, $l(\mu)=r$. Then*

$$\mathcal{R}_{\lambda\mu}(t)=\sum_{\{\nu\}}\prod_{k=1}^{r-1}\prod_{i\geq 1}\begin{bmatrix}\nu_i^{(k+1)}-\nu_{i+1}^{(k+1)}\\ \nu_i^{(k)}-\nu_{i+1}^{(k+1)}\end{bmatrix}_t, \tag{3.7}$$

where the sum is taken over all flags of partitions $\nu=\{0=\nu^{(0)}\subset\nu^{(1)}\subset\cdots\subset\nu^{(r)}=\lambda'\}$ such that $\nu^{(k)}/\nu^{(k-1)}$ is a horizontal strip of length μ_k, $1\leq k\leq r$.

Proof of Theorem 3.4 will be given in Subsection 4.1.

Remark. The last condition on the flag ν means that ν defines a semistandard tableau of shape λ' and weight μ. Thus, the number of terms in the RHS(3.7) is equal to that of semistandard tableaux of shape λ' and weight μ.

Examples. 1^0. It is clear that if $\mu=(1^n)$, then

$$\mathcal{R}_{\lambda\mu}(q)=\mathcal{P}_{\lambda\mu}(q)=X^{\lambda}_{(1^n)}(q).$$

2^0. If $\lambda=(1^N)$, and $\mu=(\mu_1,\ldots,\mu_n)$, $|\mu|=N$ then

$$\mathcal{R}_{\lambda\mu}(t)=\begin{bmatrix}N\\ \mu_1,\ldots,\mu_n\end{bmatrix}_t.$$

Indeed, the RHS(3.7) contains only one product

$$\begin{bmatrix}\mu_1+\mu_2\\ \mu_1\end{bmatrix}_t\begin{bmatrix}\mu_1+\mu_2+\mu_3\\ \mu_1+\mu_2\end{bmatrix}_t\cdots\begin{bmatrix}N\\ \mu_1+\cdots+\mu_{n-1}\end{bmatrix}_t.$$

3^0. Let μ be a composition of length n, and $\lambda = (2^{\lambda_2}1^{\lambda_1-\lambda_2})$, so that $\lambda' = (\lambda_1, \lambda_2)$. In this case the following partitions give the contribution to the RHS(3.7):

- $\nu^{(k)} = (\nu_1^{(k)}, \nu_2^{(k)})$, $|\nu^{(k)}| = \mu_1 + \cdots + \mu_k$, $1 \le k \le n$;
- $0 \le \nu_2^{(k)} \le \nu_2^{(k+1)} \le \nu_1^{(k)} \le \nu_1^{(k+1)}$, $1 \le k \le n-1$, $\nu_2^{(1)} = 0$, $\nu^{(n)} = (\lambda_1, \lambda_2)$.

If we define $m_k = \nu_1^{(k+1)} - \nu_1^{(k)}$, $1 \le k \le n-1$, $m_0 = \mu_1$, then the RHS(3.7) takes the following form

$$\sum_{m \in \mathbf{Z}_{\ge 0}^{n-1}} \begin{bmatrix} \lambda_2 \\ \mu_1 - m_1, \mu_2 - m_2, \ldots, \mu_n - m_{n-1} \end{bmatrix}_t \prod_{k=1}^{n-1} \begin{bmatrix} \sum_{i=0}^{k} (2m_i - \mu_{i+1}) \\ m_k \end{bmatrix}_t ,$$

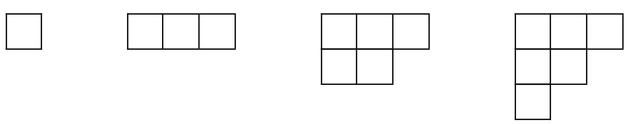
summed over all sequences $m \in \mathbf{Z}_{\ge 0}^{n-1}$, such that $m_1 + \cdots + m_{n-1} = \lambda_1 - \mu_1$.

Finally, let us assume that $n = 3$ and $\lambda_1 = \lambda_2$. Then $m_2 = 0$, $m_1 = \lambda_1 - \mu_1$ and the RHS(3.7) takes the form $(\mu_1 + \mu_2 + \mu_3 = N)$

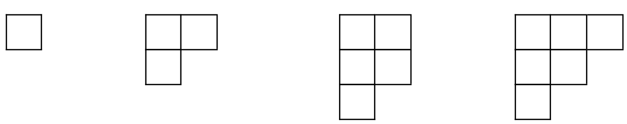
$$\begin{bmatrix} N \\ N - \mu_1, N - \mu_2, N - \mu_3 \end{bmatrix}_t .$$

Hence, the number of (0,1)–matrices of size $N \times 3$ with row sums μ_i, $i = 1, 2, 3$, and column sums $\lambda_i = 2$, $1 \le i \le N$, is equal to $\dfrac{N!}{(N-\mu_1)!(N-\mu_2)!(N-\mu_3)!}$.

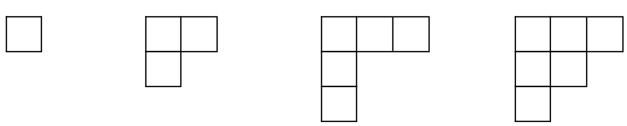
4^0. Consider $\mu = (1221)$ and $\lambda = (321)$. The summands in the RHS(3.7) correspond to the following flags of partitions $\nu = \{\nu^1 \subset \nu^{(2)} \subset \nu^{(3)} \subset \nu^{(4)}\}$:



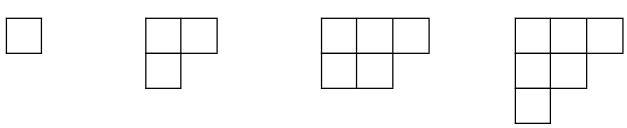
$\begin{matrix} 1 & 2 & 2 \\ 3 & 3 & \\ 4 & & \end{matrix} \qquad \begin{bmatrix} 3-0 \\ 1-0 \end{bmatrix}_t ,$



$\begin{matrix} 1 & 2 & 4 \\ 2 & 3 & \\ 3 & & \end{matrix} \qquad 1,$



$\begin{matrix} 1 & 2 & 3 \\ 2 & 4 & \\ 3 & & \end{matrix} \qquad \begin{bmatrix} 3-1 \\ 2-1 \end{bmatrix}_t ,$



$\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & \\ 4 & & \end{matrix} \qquad \begin{bmatrix} 2-0 \\ 1-0 \end{bmatrix}_t .$

Hence, the RHS(3.7)$= (1+t+t^2)+1+(1+t)+(1+t) = 4+3t+t^2$.

Let us remark that RHS(3.7) does not depend on the permutations of components of the composition μ. This is clear since the LHS(3.7) does. However, the number of summands in the RHS(3.7) do depends on the composition μ, but not only on the corresponding partition μ^+.

§4. Proofs of Theorems 3.1 and 3.4.

Let $f^{\lambda}_{\mu\nu}(t)$ be the structural constants for the Hall–Littlewood functions, i.e.

$$P_\mu(x;t)P_\nu(x;t) = \sum_\lambda f^{\lambda}_{\mu\nu}(t)P_\lambda(x;t). \tag{4.1}$$

It is well–known (see, e.g., [M], Chapter III, §3, p.215, formula (3.2)) that

$$f^{\lambda}_{\mu(1^m)}(t) = \prod_{i\ge 1}\begin{bmatrix} \lambda'_i-\lambda'_{i+1} \\ \lambda'_i-\mu'_i \end{bmatrix}_t, \tag{4.2}$$

and therefore $f^{\lambda}_{\mu(1^m)}(t) = 0$ unless λ–μ is a vertical m–strip.

Now let T be a pure supertableau of shape λ and weight μ, i.e. T is a sequence of partitions $0 = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(r)} = \lambda$, such that each skew diagram $\lambda^{(i)} - \lambda^{(i-1)}$ $(1 \le i \le r)$ is a vertical μ_i–strip. For such tableau T, let us define

$$f_T(t) = \prod_{i\ge 1} f^{\lambda^{(i)}}_{\lambda^{(i-1)}(1^{\mu_i})}(t).$$

Then the RHS(3.7) can be rewritten in the following form $\displaystyle\sum_T f_T(t)$, summed over all pure supertableaux of shape λ and weight μ.

4.1. Proof of Theorem 3.4.

It is well known that the Hall–Littlewood polynomial $P_\lambda(X_n;t)$, when $\lambda = (1^m)$, coincides with the m-th elementary symmetric function in the variables X_n:

$$P_{(1^m)}(X_n;t) = e_m(X_n),$$

see e.g., [M], Chapter III, (2.8).

Using (4.1) and (4.2) we can write

$$e_m(x)P_\nu(x;t) = \sum_\lambda f^{\lambda}_{\nu(1^m)}(t)P_\lambda(x;t), \tag{4.3}$$

and more generally using induction,

$$e_{\mu_1}(x)\ldots e_{\mu_r}(x)P_\nu(x;t) = \sum_\lambda R^{(\nu)}_{\lambda\mu}P_\lambda(x;t), \tag{4.4}$$

where

$$R^{(\nu)}_{\lambda\mu}(t)=\sum f_T(t), \tag{4.5}$$

summed over all pure supertableaux T of skew shape $\lambda-\nu$ and weight μ; in other words, the sum in (4.5) is taken over all sequences of partitions $\nu=\lambda^{(0)}\subset\lambda^{(1)}\subset\cdots\subset\lambda^{(r)}=\lambda$, such that each skew diagram $\lambda^{(i)}-\lambda^{(i-1)}$ $(1\le i\le r)$ is a vertical μ_i–strip, and

$$f_T(t)=\prod_{i=1}^{r} f^{\lambda^{(i)}}_{\lambda^{(i-1)}(1^{\mu_i})}(t).$$

To finish the proof of Theorem 3.4 we need the following formulae (see, e.g., [M], Table 1 on p.101 and Table on p.241):

$$e_{\mu_1}(x)\cdots e_{\mu_r}(x)=\sum_{\eta} K_{\eta'\mu}s_\eta(x),$$

$$s_\lambda(x)=\sum_{\mu} K_{\lambda\mu}(t)P_\mu(x;t). \tag{4.6}$$

Thus, we have

$$e_{\mu_1}(x)\cdots e_{\mu_r}(x)P_\nu(x;t)=\sum_{\lambda}\left(\sum_{\eta,\beta}K_{\eta'\mu}K_{\eta\beta}(t)f^{\lambda}_{\nu\beta}(t)\right)P_\lambda(x;t),$$

and consequently,

$$R^{(\nu)}_{\lambda\mu}(t)=\sum_{\eta,\beta}K_{\eta'\mu}K_{\eta\beta}(t)f^{\lambda}_{\nu\beta}(t). \tag{4.7}$$

Finally, if we take $\nu=\emptyset$ in (4.7), then $f^{\lambda}_{\emptyset\beta}(t)=\delta_{\lambda\beta}$, and formula (3.5) follows.

■

4.2. Proof of Theorem 3.1.

Proof of Theorem 3.1 is similar to that of Theorem 3.4 and based on the following

Lemma 4.1. *Let μ be a partition, $l(\mu)\le n$, and*

$$h_k(X_n)P_\mu(X_n;t)=\sum_{\lambda} g^{\lambda}_{\mu}(t)P_\lambda(X_n;t), \tag{4.7}$$

where $h_k(X_n)$ denotes the complete homogeneous symmetric function of degree k in the variables $X_n=(x_1,\dots,x_n)$. Then

$$g^{\lambda}_{\mu}(t)=t^{\sum_{i\ge1}\binom{\lambda'_i-\mu'_i}{2}}\prod_{i\ge1}\begin{bmatrix}\lambda'_i-\mu'_{i+1}\\ \lambda'_i-\mu'_i\end{bmatrix}_t, \tag{4.8}$$

and therefore $g_\mu^\lambda(t)=0$ *unless* $\mu\subset\lambda$, $|\lambda/\mu|=k$.

Let us postpone the proof of Lemma 4.1 to the end of this subsection and show first how using the formula (4.8) one can deduce the formula (3.1) from Theorem 3.1.

To do this we will need the formula (4.6) and the following one (see, e.g., [M], Table 1 on p.101):

$$h_{\mu_1}(x)\dots h_{\mu_r}(x)=\sum_\eta K_{\eta\mu}s_\eta(x).$$

Thus, we have

$$h_{\mu_1}(x)\dots h_{\mu_r}(x)P_\nu(x;t)=\sum_\lambda\left(\sum_{\eta,\beta}K_{\eta\mu}K_{\eta\beta}(t)f_{\nu\beta}^\lambda(t)\right)P_\lambda(x;t). \tag{4.9}$$

On the other hand, we can compute the LHS(4.9) using Lemma 4.1. Namely,

$$\mathrm{LHS}(4.9)=\sum_\lambda\mathcal{P}_{\lambda\mu}^{(\nu)}(t)P_\lambda(x;t), \tag{4.10}$$

where

$$\mathcal{P}_{\lambda\mu}^{(\nu)}(t)=\sum_\pi g_\pi(t), \tag{4.11}$$

summed over all reverse plain partitions π of skew shape $\lambda-\nu$ and weight μ; in other words, the sum in (4.11) is taken over all sequences of partitions $\nu=\lambda^{(0)}\subset\lambda^{(1)}\subset\dots\subset\lambda^{(r)}=\lambda$ such that $|\lambda^{(i)}/\lambda^{(i-1)}|=\mu_i$, $1\le i\le r$, and

$$g_\pi(t)=\prod_{i=1}^r g_{\lambda^{(i-1)}}^{\lambda^{(i)}}(t).$$

Thus, it follows from (4.9)–(4.11) that

$$\mathcal{P}_{\lambda\mu}^{(\nu)}(t)=\sum_{\eta,\beta}K_{\eta\mu}K_{\eta\beta}(t)f_{\nu\beta}^\lambda(t). \tag{4.12}$$

Finally, if we take $\nu=\emptyset$ in (4.12), then $f_{\emptyset\beta}^\lambda(t)=\delta_{\lambda\beta}$, and formula (3.1) follows.

Proof of Lemma 4.1. We will prove (4.8) by induction on the number $|\lambda/\mu|$. It is clear that if $|\lambda/\mu|=1$, then $g_\mu^\lambda=f_{\mu,(1)}^\lambda=\mathrm{RHS}(4.8)$. Because of the relation $\sum_{r=0}^k(-1)^re_rh_{k-r}=\delta_{k,0}$, it is enough to prove that if $\nu\subset\lambda$, $|\lambda\setminus\nu|>0$, then

$$\sum_\mu(-1)^{|\mu-\nu|}f_{\nu,(1^{|\mu-\nu|})}^\mu g_\mu^\lambda=0, \tag{4.13}$$

summed over all partitions μ such that $\nu \subset \mu \subset \lambda$. Now, using (4.8) and (4.2), we can write

$$\begin{aligned} \mathrm{RHS}(4.13) &= \sum_{\nu\subset\mu\subset\lambda} (-1)^{\sum(\mu_i'-\nu_i')} t^{\sum\binom{\lambda_i'-\mu_i'}{2}} \prod_{i\geq 1} \begin{bmatrix} \lambda_i'-\mu_{i+1}' \\ \lambda_i'-\mu_i' \end{bmatrix}_t \begin{bmatrix} \mu_i'-\mu_{i+1}' \\ \mu_i'-\nu_i' \end{bmatrix}_t \\ &= \prod_i {}_i(t), \end{aligned} \tag{4.14}$$

where

$$ {}_i(t) = \sum_{\nu_i'\leq\mu_i'\leq\lambda_i'} (-1)^{\mu_i'-\nu_i'} t^{\binom{\lambda_i'-\mu_i'}{2}} \frac{(t;t)_{\lambda_{i-1}'-\mu_i'}}{(t;t)_{\lambda_i'-\mu_i'}(t;t)_{\mu_i'-\nu_i'}(t;t)_{\nu_{i-1}'-\mu_i'}},$$

$\lambda_0' = \nu_0' = 0$, and by definition $(t;t)_m = 0$, if $m < 0$.

Consider at first ${}_1(t)$. We have

$$\begin{aligned} (-1)^{\lambda_1'-\nu_1'}(t)_{\lambda_1'-\nu_1'} {}_1(t) &= \sum_{\nu_1'\leq\mu_1'\leq\lambda_1'} (-1)^{\lambda_1'-\mu_1'} t^{\binom{\lambda_1'-\mu_1'}{2}} \begin{bmatrix} \lambda_1'-\nu_1' \\ \lambda_1'-\mu_1' \end{bmatrix}_t \\ &= \sum_{m\geq 0} (-1)^m t^{\binom{m}{2}} \begin{bmatrix} \lambda_1'-\nu_1' \\ m \end{bmatrix}_t = \delta_{\lambda_1',\nu_1'}. \end{aligned}$$

The last equality follows from the q–binomial theorem

$$\sum_{m=0}^{N} (-z)^m q^{\binom{m}{2}} \begin{bmatrix} N \\ m \end{bmatrix}_q = (z,q)_N := \prod_{i=1}^{N} (1-q^{i-1}z).$$

Thus, if the product (4.14) does not equal to zero, then $\lambda_1' = \nu_1'$, and

$$(-1)^{\lambda_2'-\nu_2'}(t)_{\lambda_2'-\nu_2'} {}_2(t) = \sum_{m\geq 0} (-1)^m t^{\binom{m}{2}} \begin{bmatrix} \lambda_2'-\nu_2' \\ m \end{bmatrix}_t = \delta_{\lambda_2',\nu_2'}.$$

Repeating these arguments we see that the product (4.14) does not equal to zero only if $\lambda = \nu$. But this is a contradiction with our assumption $|\lambda/\nu| > 0$. This proves (4.13) and (by induction) Lemma 4.1.

■

Remark. The similar proofs of Theorems 3.1 and 3.4 can be found in [HKKOTY].

It seems the formula (4.8) is new. The formula (4.12) in the case $\nu = \emptyset$, probably, goes back to R. Stanley, unpublished; see, e.g., [Bu2], Lemma 3.1.

Corollary 4.2. *Let λ and μ be partitions, $|\mu| = n$, and $f^{\lambda}_{\nu\mu}(t)$ be the structural constants for the Hall–Littlewood functions, see [M], Chapter III, or Section 4, (4.1). Then*

$$\sum_{\nu} t^{n(\nu)} f^{\lambda}_{\nu\mu}(t) = t^{\sum_{i\ge1}\binom{\lambda'_i-\mu'_i}{2}} \prod_{i\ge1} \begin{bmatrix} \lambda'_i-\mu'_{i+1} \\ \lambda'_i-\mu'_i \end{bmatrix}_t . \tag{4.15}$$

Proof. It follows from Lemma 4.1 that the RHS(4.15)$= g^{\lambda}_{\mu}(t)$. Hence,

$$\sum_{\lambda} g^{\lambda}_{\mu}(t) P_{\lambda} = h_n P_{\mu} = \sum_{\nu} K_{(n)\nu}(t) P_{\nu} P_{\mu} = \sum_{\lambda} \left(\sum_{\nu} K_{(n)\nu}(t) f^{\lambda}_{\nu\mu}(t) \right) P_{\lambda},$$

and consequently, $g^{\lambda}_{\mu}(t) = \sum_{\nu} K_{(n)\nu}(t) f^{\lambda}_{\nu\mu}(t)$. The identity (4.15) follows from a simple observation that $K_{(n)\nu}(t) = t^{n(\nu)}$.

■

If $\mu = (1^n)$, then the RHS(4.15)$= t^{n(\lambda)-n(\mu)} \begin{bmatrix} \lambda'_1 \\ n \end{bmatrix}_{t^{-1}}$, and identity (4.15) is reduced to that in [M], Chapter III, Example 1.

Exercise. Let $\mu = (\mu_1, \ldots, \mu_s)$ and $\nu = (\nu_1, \ldots, \nu_r)$ be compositions. For each partition η, $l(\eta) \le n$, denote by $K_{\eta,\mu|\nu}$ the multiplicity of the highest weight irreducible representation $V^{(n)}_{\eta}$ of the general linear group $gl(n)$ in the tensor product

$$V^{(n)}_{\mu_1} \otimes \cdots \otimes V^{(n)}_{\mu_s} \otimes V^{(n)}_{(1^{\nu_1})} \otimes \cdots \otimes V^{(n)}_{(1^{\nu_r})}.$$

Let λ be a partition, $l(\lambda) \le n$. Find a fermionic formulae for the following sum

$$\sum_{\eta} K_{\eta,\mu|\nu} K_{\eta\lambda}(q),$$

which generalizes (3.1) and (3.7).

Conjecture 4.3. *Let λ, μ, ν be partitions. Define a family of polynomials $g^{\lambda}_{\mu;\nu}(t)$ via decomposition*

$$s_{\nu}(x) P_{\mu}(x;t) = \sum_{\lambda} g^{\lambda}_{\mu;\nu}(t) P_{\lambda}(x;t).$$

Then $g^{\lambda}_{\mu;\nu}(t)$ is a polynomial with nonnegative integer coefficients.

Problem 2. Find a combinatorial formula for polynomials $g^{\lambda}_{\mu;\nu}(t)$.

The answer on this problem is known when either $\nu = (1^N)$, see, e.g., [M], p.215, or $\nu = (n)$, see Lemma 4.1.

§5. Polynomials $\mathcal{P}_{\lambda\mu}(t)$ and their interpretations.

In this Section we summarize the known interpretations and some properties of polynomials $\mathcal{P}_{\lambda\mu}(t)$. The main reason for this is the following: we suppose that all generalizations of polynomials $\mathcal{P}_{\lambda\mu}(t)$ considered in the coming sections, should have properties similar to (5.2)-(5.10).

Polynomials $\mathcal{P}_{\lambda\mu}(t)$ admit the following interpretations:

• Transition coefficients between modified Hall-Littlewood polynomials and monomial symmetric functions

$$Q'_\lambda(X_n;t) = \sum_\mu \mathcal{P}_{\lambda\mu}(t) m_\mu(X_n). \tag{5.2}$$

• Inhomogeneous unrestricted one dimensional sum with "special boundary conditions":

$$\mathcal{P}_{\lambda\mu}(t) = t^{n(\mu')} \sum_{m \in \mathcal{P}_{\lambda\mu}} t^{E(m)}, \tag{5.3}$$

summed over the set $\mathcal{P}_{\lambda\mu}$ of all transport matrices m of type $(\lambda;\mu)$, i.e. the set of all matrices of non–negative integers with row sums λ_i and column sums μ_j; $E(m)$ stands for the value of energy function $E(p)$ of the path p which corresponds to the transport matrix m under a natural identification of the set of paths $\mathcal{P}_\mu(b_{\max},\lambda)$ (see, e.g., [KMOTU2], or Subsection 3.1, Example 1^0) with that of transport matrices $\mathcal{P}_{\lambda\mu}$.

Problem 3. Find a combinatorial rule for computation of the energy function $E(m)$ of a transport matrix $m \in \mathcal{P}_{\lambda\mu}$.

• Generating function of a generalized mahonian statistic φ on the set of transport matrices $\mathcal{P}_{\lambda\mu}$:

$$\mathcal{P}_{\lambda\mu}(t) = t^{n(\mu')} \sum_{m \in \mathcal{P}_{\lambda\mu}} t^{\varphi(m)}.$$

For examples of generalized mahonian statistics, see Section 2.6.

Problem 4. It is natural to ask: are there exist combinatorial analogues of statistics INV, MAJ, $\widetilde{MAJ}$, Z, $\widetilde{Z}$ and DEN (see Subsection 2.1), and LP (see Definition 3.3) on the set of transport matrices $\mathcal{P}_{\lambda\mu}$ with generating function $\mathcal{P}_{\lambda\mu}(t)$?

• The Poincare polynomial of the partial flag variety $\mathcal{F}_\mu^\lambda/\mathbf{C}$:

$$\mathcal{P}_{\lambda\mu}(t) = \sum_{i \ge 0} t^{n(\lambda)-i} \dim H_{2i}(\mathcal{F}_\mu^\lambda;\mathbf{Z}). \tag{5.4}$$

• The number of $\mathbf{F}_q$–rational points of the partial flag variety $\mathcal{F}_\mu^\lambda/\mathbf{F}_q$:

$$q^{n(\lambda)}\mathcal{P}_{\lambda\mu}(q^{-1}) = \mathcal{F}_\mu^\lambda(\mathbf{F}_q). \tag{5.5}$$

• The number of chains of subgroups

$$\{e\} \subseteq H^{(1)} \subseteq H^{(2)} \subseteq \cdots \subseteq H^{(m)} \subseteq G$$

in a finite abelian p–group G of type λ, such that each subgroup $H^{(i)}$ has order $p^{\mu_1+\cdots+\mu_i}$:

$$\alpha_\lambda(S;p) = p^{n(\lambda)}\mathcal{P}_{\lambda\mu}(p^{-1}), \tag{5.6}$$

where $S := S(\mu) = (\mu_1, \mu_1+\mu_2, \ldots, \mu_1+\mu_2+\cdots+\mu_m)$, and $l(\mu) = m+1$.

• String function of affine Demazure's module $V_w(l_L)$ corresponding to the element $w = r_{Ln-1}r_{Ln-2}\ldots r_{L+2}r_{L+1}r_L$ of the affine Weyl group $W(A_{n-1}^{(1)})$:

$$t^{E_0}\mathcal{P}_{(l^L)\mu}(t) = \sum_{n\ge 0}\dim V_w(l_L)_{\mu-n\delta}t^n, \tag{5.7}$$

for some known constant E_0; see [KMOTU2], or Subsection 1.6.

• Generalized t–supernomial and t–multinomial coefficients $\begin{bmatrix}\lambda\\ \mu\end{bmatrix}^{(0)}$ and $T^{(0)}(\lambda;\mu)$:

$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix}^{(0)} = \sum_\eta K_{\eta\mu}\widetilde{K}_{\eta\lambda}(t) = t^{n(\lambda)}\sum_\eta K_{\eta\mu}K_{\eta\lambda}(t^{-1}), \tag{5.8}$$

$$T^{(0)}(\lambda;\mu) = t^{-E_{\min}}\mathcal{P}_{\lambda\mu}(t), \tag{5.9}$$

for some known constant $E_{\min}$.

As it was shown in Subsection 3.1, the coefficients (5.8) and (5.9) are a natural generalization of those introduced by A. Schilling and S.O. Warnaar in the case $l(\mu) = 2$, see [Ki2], [Sc], [ScW], [W].

• "Fermionic expression". Let λ be a partition and μ be a composition, $l(\mu) = n$, then

$$\mathcal{P}_{\lambda\mu}(t) = \sum_{\{\nu\}} t^{c(\{\nu\})}\prod_{k=1}^{n-1}\prod_{i\ge 1}\begin{bmatrix}(\nu^{(k+1)})'_i - (\nu^{(k)})'_{i+1}\\ (\nu^{(k)})'_i - (\nu^{(k)})'_{i+1}\end{bmatrix}, \tag{5.10}$$

summed over all flags of partitions $\nu = \{0 = \nu^{(0)} \subset \nu^{(1)} \subset \cdots \subset \nu^{(n)} = \lambda\}$, such that $|\nu^{(k)}| = \mu_1+\cdots+\mu_k$, $1 \le k \le n$, and

$$c(\{\nu\}) = \sum_{k=0}^{n-1}\sum_{i\ge 1}\binom{(\nu^{(k+1)})'_i - (\nu^{(k)})'_i}{2}.$$

See [HKKOTY] and Sections 3 and 4, where further details and applications of the fermionic formula (5.10) can be found.

• Truncated form, or finitization of characters and branching functions of (some) integrable representations of the affine Lie algebra of type $A_{n-1}^{(1)}$, and more generally, for Kac–Moody algebras, W–algebras, ...

The observation that certain special limits of polynomials $\mathcal{P}_{\lambda\mu}(t)$ and Kostka–Foulkes polynomials may play an important role in the representation theory of affine Lie algebras originally was made in [Ki2]. It was observed in [Ki2] that the character formula for the level 1 vacuum representation $V({}_0)$ of the affine Lie algebra of type $A_{n-1}^{(1)}$ (see, e.g., [Kac], Chapter 13) can be obtained as an appropriate limit $N \to \infty$ of the modified Hall–Littlewood polynomials $Q'_{(1^N)}(X_n;q)$. The proof was based on the following formula

$$\mathcal{P}_{(1^N)\mu}(q) = q^{n(\mu')} \begin{bmatrix} N \\ \mu_1, \ldots, \mu_n \end{bmatrix}_q , \tag{5.11}$$

see [Ki2], (2.28), or Subsection 1.2, (1.5).

The latter observation about a connection between the character $\mathrm{ch}(V({}_0))$ and modified Hall-Littlewood polynomials $Q'_{(1^N)}(X_n;q)$ immediately implies that the level 1 branching functions $b_\lambda^0(q)$ can be obtained as an appropriate limit $\lambda_N \to \infty$ of the "normalized" Kostka–Foulkes polynomials $q^{-A_N} K_{\lambda_N,(1^N)}(q)$. We refer the reader to [Kac], Chapter 12, for definitions and basic properties of branching functions $b_\lambda(q)$ corresponding to an integrable representation $V()$ of affine Lie algebra.

It was conjectured in [Ki2], Conjecture 4, that the similar result should be valid for the branching functions $b_\lambda(q)$ corresponding to the integrable highest weight irreducible representation $V()$ of the affine Lie algebra $\widehat{sl}(n)$. This conjecture has been proved in [Ki2] in the following cases: $\widehat{sl}(n)$ and $\ = {}_0$, $\widehat{sl}(2)$ and $\ = l_0$, and $\widehat{sl}(n)$ and $\ = 2_0$. It had not been long before A. Nakayashiki and Y. Yamada [NY] proved this conjecture in the case $\widehat{sl}(n)$ and $\ = l_i$, $0 \le i \le n-1$. See also [KKN] for another proof of the result of Nakayashiki and Yamada in the case $i = 0$. The general case has been investigated in [HKKOTY]. It happened that in general the so–called thermodynamical Bethe ansatz limit of Kostka–Foulkes polynomials gives the branching function of a certain reducible integrable representation of $\widehat{sl}(n)$, see details in [HKKOTY].

Problem 5. Find an interpretation of the branching functions $b_\lambda(q)$ of the integrable highest weight irreducible representation $V()$ of the affine Lie algebra $\widehat{sl}(n)$ as the thermodynamical Bethe ansatz type limit of a certain family of the Kostka–Foulkes type polynomials.

§6. Generalizations of polynomials $\mathcal{P}_{\lambda\mu}(t)$ and $K_{\lambda\mu}(t)$.

In this Section we summarize possible generalizations of polynomials $\mathcal{P}_{\lambda\mu}(t)$ and Kostka–Foulkes polynomials $K_{\lambda\mu}(t)$, their properties, and some special cases. Let us remind that

$$\mathcal{P}_{\lambda\mu}(t) = \sum_\eta K_{\eta\mu} K_{\eta\lambda}(t), \tag{6.1}$$

where λ is a partition, μ is a composition; summation in (6.1) runs over all partitions η; $K_{\eta\lambda}(t)$ is the Kostka–Foulkes polynomial (see, e.g., [M], Chapter III, Section 6), and $K_{\eta\mu} := K_{\eta\mu}(1)$ is the Kostka number which is equal to the number of semistandard Young tableaux of shape η and content μ.

6.1. Crystal Kostka polynomials.

First let us recall the result of A. Nakayashiki and Y. Yamada [NY] that the Kostka–Foulkes polynomial $K_{\lambda\mu}(t)$ coincides with the classically restricted one dimensional sum with special boundary conditions. For another proof, see, e.g., [KKN]; cf. [KMOTU2], [HKKOTY].

Let $n \geq 2$ be a natural integer which is fixed throughout this subsection.

Definition 6.1. *Let $R = \{R_1, \ldots, R_p\}$ be a sequence of partitions, μ be a partition such that $|\mu| = |R_1| + \cdots + |R_p|$. Define the polynomial $C\mathcal{P}_{R\mu}(t)$ to be the weight μ unrestricted one dimensional sum corresponding to the tensor product of crystals $B_{R_1} \otimes \cdots \otimes B_{R_p}$, and boundary condition $b_{T_{\min}}$, $T_{\min} \in STY(\lambda, \lambda)$, where B_{R_i} is the crystal (see, e.g., [Ka1]) corresponding to the irreducible highest weight R_i representation V_{R_i} of the Lie algebra $sl(n)$.*

Definition 6.2. *The crystal Kostka polynomial $CK_{\lambda R}(q)$ corresponding to a set of partitions $R = \{R_1, \ldots, R_p\}$ is defined to be the weight λ classically restricted one dimensional sum corresponding to the tensor product of crystals $B_{R_1} \otimes \cdots \otimes B_{R_p}$, and boundary condition $b_{T_{\min}}$.*

We refer the reader to [LS], [DLT] and [Ki1], where definition and basic properties of Kostka–Foulkes polynomials can be found, and to [HKMOTU2] and [HKKOTY] for definitions of unrestricted, classically restricted and restricted one dimensional sums.

Let us remark that

$$CK_{\lambda R}(1) = \mathrm{Mult}[V_\lambda \ : \ V_{R_1} \otimes \cdots \otimes V_{R_p}], \tag{6.2}$$

i.e. $CK_{\lambda R}(1)$ is equal to the multiplicity of the highest weight λ irreducible representation V_λ of $sl(n)$ in the tensor product $V_{R_1} \otimes \cdots \otimes V_{R_p}$. Thus, the crystal Kostka polynomial $CK_{\lambda R}(q)$ may be considered as a q–analog of the tensor product multiplicity (6.2).

6.2. Fusion Kostka polynomials.

The problem of finding a "natural" q–analog of the tensor product multiplicities has a long story. To our knowledge, there exists at least three natural algebraic ways to define a q–analog of the tensor product multiplicity (6.2). The first one is based on the so–called fusion rules for the tensor product of "restricted" representations of the quantized universal enveloping algebra $U_q(sl(n))$ when q is a root of unity, see, e.g., [GoW]; [Kac], Exercises 13.34-13.36; and [BKMW], where a combinatorial description of the fusion rules for representations of $sl(3)$ and $sl(4)$ are given. We denote by $FK_{\lambda R}(q)$, and call it *fusion* Kostka polynomial, a q–analog of the tensor product multiplicity (6.2) which corresponds to the fusion rules.

Let us explain informally the meaning of the fusion Kostka polynomials $FK_{\lambda R}(q)$. Let $\mathcal{F}_r(n)$ be the fusion algebra corresponding to the quantized universal enveloping algebra $U_q(sl(n))$, when $q = \exp(2\pi i/r+n)$. Each finite dimensional $sl(n)$–module V defines an element $[V]$ of the fusion algebra $\mathcal{F}_r(n)$. This algebra is generated by the so–called "restricted" representations V_λ, which correspond to partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda_1 - \lambda_n \le r$. It is well–known that the fusion algebra is commutative and associative. We denote by $\widehat{\otimes}$ the product in the algebra $\mathcal{F}_r(n)$. This product depends on r and n. Let $R = (R_1, \ldots, R_p)$ be a sequence of partitions, denote by $\mathrm{Mult}^{(r)}(V_\lambda \ : \ V_{R_1} \widehat{\otimes} \cdots \widehat{\otimes} V_{R_p})$ the coefficient of $[V_\lambda]$ in the decomposition of the product $[V_{R_1}] \widehat{\otimes} \cdots \widehat{\otimes} [V_{R_p}]$ in the fusion algebra $\mathcal{F}_r(n)$:

$$[V_{R_1}] \widehat{\otimes} \cdots \widehat{\otimes} [V_{R_p}] = \sum_\lambda \mathrm{Mult}^{(r)} \left(V_\lambda \ : \ V_{R_1} \widehat{\otimes} \cdots \widehat{\otimes} V_{R_p} \right) [V_\lambda].$$

Definition 6.3. *The fusion Kostka polynomial $FK_{\lambda R}(q)$ is defined to be*

$$FK_{\lambda R}(q) = \sum_{r \ge 0} \left(\mathrm{Mult}^{(r+1)}(V_\lambda \ : \ V_{R_1} \widehat{\otimes} \cdots \otimes V_{R_p}) - \mathrm{Mult}^{(r)}(V_\lambda \ : \ V_{R_1} \widehat{\otimes} \cdots \widehat{\otimes} V_{R_p}) \right) q^r. \tag{6.3}$$

It is well–known that $FK_{\lambda R}(q)$ is a polynomial with nonnegative integer coefficients, and

$$FK_{\lambda R}(1) = \mathrm{Mult}(V_\lambda \ : \ V_{R_1} \otimes \cdots \otimes V_{R_p}).$$

Thus, if all partitions R_i have only one part μ_i, i.e. $R_i = (\mu_i)$, then $FK_{\lambda R}(1) = K_{\lambda\mu}(1)$ is equal to the number $STY(\lambda, \mu)$ of semistandard Young tableaux of shape λ and weight μ.

Problem 6. Give a combinatorial definition of a statistic on the set $STY(\lambda, \mu)$ which has the generating function $FK_{\lambda R}(t)$.

Let us give few illustrative examples of the fusion Kostka polynomials for Lie algebras $sl(5)$ and $sl(3)$.

- Algebra $sl(3)$:

i) if $\lambda = (433)$, $R = \{(1)^{\otimes 10}\}$, then

$$FK_{\lambda R}(q) = q + 54q^2 + 115q^3 + 40q^4;$$

ii) if $\lambda = (422)$, $R = \{(1)^{\otimes 8}\}$, then

$$FK_{\lambda R}(q) = 13q^2 + 30q^3 + 13q^4.$$

- Algebra $sl(5)$:

i) if $\lambda = (65432)$, $R = \{(4321), (4321)\}$, then

$$FK_{\lambda R}(q) = 4q^4 + 10q^5 + 2q^6;$$

ii) if $\lambda = (98653)$, $R = \{(6531), (6532)\}$, then

$$FK_{\lambda R}(q) = 4q^6 + 16q^7 + 13q^8 + 2q^9.$$

Problem 7. Let us introduce the *fusion* modified Hall–Littlewood polynomials

$$FQ'_R(X_n;t) = \sum_{\eta} FK_{\eta R}(t) s_{\eta}(X_n). \tag{6.4}$$

where $s_{\eta}(X_n)$ stands for the Schur function corresponding to a partition η.

Find algebraic, combinatorial, and geometric interpretations of the fusion modified Hall–Littlewood polynomials $FQ'_n(X_n;t)$.

6.3. Ribbon Kostka polynomials.

The second way to define a q–analog of the tensor product multiplicity (6.2) is due to A. Lascoux, B. Leclerc and J.-Y. Thibon, [LLT], and based on the using of ribbon tableaux. We refer the reader to [LLT], Sections 4 and 6, for definitions of a p–ribbon tableau T, spin $s(T)$ of a p–ribbon tableau, and "p–ribbon version" $\widetilde{Q}_{\lambda}^{(p)}(X_n;t)$ of modified Hall–Littlewood polynomials. Here we are only reminding that if λ is a partition with empty p–core, then by definition

$$\widetilde{Q}_{\lambda}^{(p)}(X_n;t) = \sum_{T \in \mathrm{Tab}_p(\lambda, \leq n)} t^{\bar{s}(T)} x^{w(T)}, \tag{6.5}$$

summed over the set $\mathrm{Tab}_p(\lambda, \leq n)$ of all p–ribbon tableaux of shape λ filled by numbers not exceeding n; $\bar{s}(T) = s(T) - \min\{s(T) \mid T \in \mathrm{Tab}_p(\lambda, \leq n)\}$ is a normalized spin of the p–ribbon tableau T, cf. [LLT], (25). It is known, [LLT], Theorem 6.1, that $\widetilde{Q}_{\lambda}^{(p)}(X_n;t)$ is a symmetric polynomial. Let us define the ribbon polynomials $\mathcal{P}_{\lambda\mu}^{(p)}(t)$ and the ribbon Kostka polynomials $K_{\lambda\mu}^{(p)}(t)$ via decompositions (cf. [LLT]):

$$\widetilde{Q}_{\lambda}^{(p)}(X_n;t) = \sum_{\mu} \mathcal{P}_{\lambda\mu}^{(p)}(t) m_{\mu}(X_n), \tag{6.6}$$

$$\widetilde{Q}_{\lambda}^{(p)}(X_n;t) = \sum_{\mu} K_{\lambda\mu}^{(p)}(t) s_{\mu}(X_n). \tag{6.7}$$

Remark. The functions $\widetilde{Q}_{\lambda}^{(p)}(X_n;t)$ were introduced and studied by A. Lascoux, B. Leclerc and J.-Y. Thibon in [LLT], and denoted in [LLT] by $G_{\lambda}^{(p)}(X_n;t)$. We denote these functions by $\widetilde{Q}_{\lambda}^{(p)}(X_n;t)$, and call the *ribbon* modified Hall–Littlewood polynomials in order to underline a certain similarity with modified Hall–Littlewood polynomials $Q'_{\lambda}(X_n;t)$. In fact, it was proved in [LLT], Theorem 6.6, that if λ is a partition, and $L \geq l(\lambda)$, then

$$\widetilde{Q}_{L\lambda}^{(L)}(X_n;t) = Q'_{\lambda}(X_n;t),$$

where $L\lambda = (L\lambda_1, L\lambda_2, \ldots, \lambda L_n)$.

It is well–known and goes back to D. Littlewood, cf. [SW], that

$$K_{\lambda\mu}^{(p)}(1) = \mathrm{Mult}[V_{\mu} \ : \ V_{\lambda^{(1)}} \otimes \cdots \otimes V_{\lambda^{(p)}}], \tag{6.8}$$

and $\mathcal{P}^{(p)}_{\lambda\mu}(1)$ is equal to the number of weight μ unrestricted paths corresponding to the tensor product of crystals $B_{\lambda^{(1)}} \otimes \cdots \otimes B_{\lambda^{(p)}}$, where $(\lambda^{(1)}, \ldots, \lambda^{(p)})$ is the p–quotient of partition λ (see, e.g., [M], Chapter I, Example 8, for definitions of p–core and p–quotient of a partition λ).

Now we are going to formulate two conjectures about connections between polynomials $C\mathcal{P}_{R\mu}(t)$ and $CK_{\lambda R}(t)$, see Definitions 6.1 and 6.2, and the ribbon polynomials $\mathcal{P}^{(p)}_{\lambda\mu}(t)$ and $K^{(p)}_{\lambda\mu}(t)$. Namely, let $R = \{R_1 \ldots, R_p\}$ be a sequence of partitions. According to the result of D. Littlewood there exists the unique partition with the following properties (see, e.g., [M], Chapter I, Example 8, and [SW]):

$$i) \quad p\text{−core}() = \emptyset; \tag{6.9}$$

$$ii) \quad p\text{−quotient}() = (R_1, \ldots, R_p). \tag{6.10}$$

Conjecture 6.4. *Let $C\mathcal{P}_{R\mu}(t)$ be the weight μ unrestricted one dimensional sum corresponding to the tensor product of crystals $B_{R_1} \otimes \cdots \otimes B_{R_p}$, and boundary condition $b_{T_{\min}}$; let be the unique partition which satisfies the conditions (6.9) and (6.10). Then*

$$C\mathcal{P}_{R\mu}(t) = t^{E_0} \mathcal{P}^{(p)}_{\mu}(t),$$

for a certain constant E_0.

Conjecture 6.5. *Let $CK_{\lambda R}(t)$ be the weight λ classically restricted one dimensional sum corresponding to the tensor product of crystals $B_{R_1} \otimes \cdots \otimes B_{R_p}$ and boundary condition $b_{T_{\min}}$; let be the unique partition which satisfies the conditions (6.9) and (6.10). Then*

$$CK_{\lambda R}(t) = q^{E_0} K^{(p)}_{\lambda}(t),$$

for a certain constant E_0.

6.4. Generalized Kostka polynomials.

The third way to define a q–analog, denoted by $K_{\lambda R}(q)$, of the tensor product multiplicity (6.2), in the case R is a sequence of rectangular partitions, is due to M. Shimozono and J. Weyman, see, e.g., [KS]. By definition the polynomials $K_{\lambda R}(q)$ are the Poincare polynomials of isotypic components of Euler characteristics of certain $\mathbf{C}[gl_n]$–modules supported in nilpotent conjugacy class closures.

To give precise definitions, we need little more notations. Our exposition follows to [KS]. Let $\eta = (\eta_1, \eta_2, \ldots, \eta_p)$ be a sequence of positive integers that sum to n. Denote by Roots_η the set of ordered pairs (i, j) such that $1 \le i \le \eta_1 + \cdots + \eta_r < j \le n$ for some r. For example, if $\eta = (1^n)$, then $\mathrm{Roots}_\eta = \{(i, j) \mid 1 \le i < j \le n\}$.

Let $X_n = (x_1, \ldots, x_n)$ be the set of independent variables. For any sequence of integer numbers $\gamma = (\gamma_1, \ldots, \gamma_n)$ we put $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$. The symmetric group S_n acts on polynomials in $X_n = (x_1, \ldots, x_n)$ by permuting variables. Define the operators J and π by

$$J(f) = \sum_{w \in S_n} (-1)^{l(w)} w(x^\delta f), \tag{6.11}$$

$$\pi(f) = J(1)^{-1} J(f), \tag{6.12}$$

where $J(1) = \prod_{i<j} (x_i - x_j)$ is the Vandermond determinant, $\delta = (n-1, n-2, \ldots, 1, 0)$.

For the dominant (weakly decreasing) integral weight $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$, the character $s_\lambda(X_n)$ of the highest weight λ $gl(n)$ module V_λ is given by the Laurent polynomial $s_\lambda(X_n) = \pi(x^\lambda)$. When λ is a partition (that is $\lambda_n \geq 0$), s_λ is the Schur function.

Let $B_\eta(X_n; q)$, $H_{\gamma\eta}(X_n; q)$, and $K_{\lambda,\gamma,\eta}(q)$ be the formal power series defined by

$$B_\eta(X_n; q) = \prod_{(i,j) \in \mathrm{Roots}_\eta} (1 - q x_i / x_j)^{-1}, \tag{6.13}$$

$$H_{\gamma\eta}(X_n; q) = \pi\left(x^\gamma B_\eta(X_n; q)\right) = \sum_\lambda s_\lambda(X_n) K_{\lambda,\gamma,\eta}(q), \tag{6.14}$$

where λ runs over the dominant integral weights in $\mathbf{Z}^n$. It is known (M. Shimozono and J. Weyman) that the coefficients $K_{\lambda,\gamma,\eta}(q)$ are in fact polynomials with integer coefficients. It is not true in general that the polynomials $K_{\lambda,\gamma,\eta}(q)$ have nonnegative coefficients.

Now we are going to introduce the generalized Kostka polynomials $K_{\lambda R}(q)$. Namely, let $R = (R_1, \ldots, R_p)$ be a sequence of partitions. Denote by $\eta = (\eta_1, \ldots, \eta_p)$ the sequence of lengths $\eta_i = l(R_i)$ of partitions R_i. Let $n = |\eta|$, and $\gamma(R) \in \mathbf{Z}^n_{\geq 0}$ denotes the composition obtained by concatenating the parts of the R_i in order.

Definition 6.6. *The generalized Kostka polynomial $K_{\lambda R}(q)$ corresponding to a partition λ and sequence of partitions R is defined by the following formula*

$$K_{\lambda R}(q) = K_{\lambda,\gamma(R),\eta}(q). \tag{6.15}$$

It is known (M. Shimozono and J. Weyman) that

$$K_{\lambda R}(1) = \mathrm{Mult}(V_\lambda \ : \ V_{R_1} \otimes \cdots \otimes V_{R_p}),$$

i.e. $K_{\lambda R}(1)$ is equal to the multiplicity of the highest weight λ irreducible representation V_λ of the Lie algebra $gl(n)$ in the tensor product $V_{R_1} \otimes \cdots \otimes V_{R_p}$. The generalized Kostka polynomials are a far generalization of the Kostka–Foulkes polynomials $K_{\lambda\mu}(q)$, the generalized exponents polynomials $F_q(V_\lambda)$ introduced by B. Kostant and studied by R. Gupta, W. Hessenlink, S. Kato, J. Weyman, A.Broer, ..., see, e.g., [G], [DLT]. More precisely:

1. Let R_i be the single row (μ_i) for all i, where $\mu=(\mu_1,\mu_2,\ldots)$ is a partition of length at most n. Then

$$K_{\lambda R}(q)=K_{\lambda\mu}(q), \tag{6.16}$$

where $K_{\lambda\mu}(q)$ is the Kostka–Foulkes polynomial. The proof of (6.16) follows from the following well–known identity:

$$\pi\left(x^{\mu}\prod_{1\le i<j\le n}\left(1-q\frac{x_i}{x_j}\right)^{-1}\right)=\sum_{k\ge 0}e_n(X_n)^{-k}Q'_{\mu+(k^n)}(X_n;q). \tag{6.17}$$

When $\mu=0$, the LHS(6.17)$=\prod_{1\le i,j\le n}(1-qx_i/x_j)^{-1}=\sum_{k\ge 0}q^k\mathrm{ch}(\mathcal{H}^k)$, where $\mathcal{H}=\bigoplus_{k\ge 0}\mathcal{H}^k$ is the graded module of harmonic polynomials. Follow [Gu], the generalized exponents polynomial $F_q(V)$ of a finite–dimensional $gl(n)$–module V is defined to be $F_q(V)=\sum_{k\ge 0}\langle V,\mathcal{H}^k\rangle q^k$. It is follows immediately from (6.17) with $\mu=0$, that $F_q(V_\lambda)=0$, if $|\lambda|\not\equiv 0(\mathrm{mod}\ n)$, and $F_q(V_\lambda)=K_{\lambda(l^n)}(q)$, if $|\lambda|=ln$. The last equality originally was proved by W. Hessenlink, and "elementary" algebraic proof may be found in [DLT].

2. Let R_i be the single column (1^{η_i}) for all i. Then

$$K_{\lambda R}(q)=\widetilde{K}_{\lambda'\eta^+}(q),$$

is the cocharge Kostka–Foulkes polynomial, where λ' is the conjugate of the partition λ and η^+ is the partition obtaining by sorting the parts of η into weakly decreasing order.

3. (M. Shimozono and J. Weyman). Let k be a positive integer and R_i be the rectangle with k columns and η_i rows, $1\le i\le n$. Then $K_{\lambda R}(q)$ is the Poincare polynomial of the isotypic component of the irreducible $GL(n)$–module of highest weight $(\lambda_1-k,\lambda_2-k,\ldots,\lambda_n-k)$ in the coordinate ring of the Zariski closure of the nilpotent conjugacy class which corresponds to the set of nilpotent matrices with the Jordan canonical form of type $(\eta^+)'$.

As it was mentioned, the generalized Kostka polynomials $K_{\lambda R}(q)$ may have negative coefficients for general λ and R. Nevertheless, for the so–called dominant sequence of partitions R, one expect

Conjecture 6.7 (A. Broer, [KS]) *Let R be a dominant sequence of partitions. Then*

$$K_{\lambda R}(t)\in\mathbf{N}[t].$$

Recall that a sequence of partitions $R=(R_1,\ldots,R_p)$ is called *dominant*, if for all $1\le i\le p$, the last part of R_i is at least as large as the first part of R_{i+1}.

6.5. Summary.

In the previous Subsections we gave definitions of four families of polynomials which may be considered as the "natural" q–analogues of the tensor product multiplicities, namely,

- fusion Kostka polynomials $FK_{\lambda R}(t)$,
- crystal Kostka polynomials $CK_{\lambda R}(t)$,
- ribbon Kostka polynomials $K_{\mu}^{(p)}(t)$,
- generalized Kostka polynomials $K_{\lambda R}(t)$,

where R is a sequence of partitions, λ and μ are partitions, and is a partition without p–core.

It is natural to ask: what are the relations between these four families of polynomials?

First of all, for each sequence of partitions $R = (R_1, \ldots, R_p)$ denote by $:= (R)$ the unique partition which has no p–core, and has R as its p–quotient. It is known that

- $CK_{\lambda R}(1) = FK_{\lambda R}(1) = K^{(p)}_{(R)\lambda}(1) = K_{\lambda R}(1) =$RHS(6.2),
- $CK_{\lambda R}(t)$, $FK_{\lambda R}(t)$ are polynomials with nonnegative coefficients by definition.

It was conjectured in [LLT] that the ribbon Kostka polynomials $K_{\mu}^{(p)}(t)$ have nonnegative coefficients. This conjecture was proved in [CL] in the case $p = 2$. As for the generalized Kostka polynomials $K_{\lambda R}(t)$, they do may have negative coefficients in general. For example, take $\lambda = (2,2)$ and $R = ((1),(3))$, then $K_{\lambda R}(t) = t - 1$.

It seems a very difficult problem to characterize all sequences of partitions $R = (R_1, \ldots, R_p)$ such that $K_{\lambda R}(t) \in \mathbf{N}[t]$ for all partitions λ. But even if it happens that the generalized Kostka polynomial $K_{\lambda R}(t)$ do has nonnegative coefficients for some λ and R, even in this case, $K_{\lambda R}(t) \neq K_{(R)\lambda}$ in general. For example, take $\lambda = (521)$ and $R = ((31),(1),(1),(2))$. In this case $(R) = (32111)$, and $K_{\lambda R}(t) = t^5 + 3t^6 + 2t^7 + t^8$, but $K_{(R)\lambda}(t) = 2q^3 + 3q^4 + 2q^5$.

Summarizing, it seems that there are no simple connection between the ribbon and generalized Kostka polynomials in general. Nevertheless, for the so–called dominant sequences of rectangular partitions R, one can conjectured (see, e.g., [KS]) that the generalized and ribbon Kostka polynomials coincide. Recall that a sequence of partitions $R = (R_1, \ldots, R_p)$ is called *dominant*, if for all $1 \le i \le p-1$, the last part of R_i is at least as large as the first part of R_{i+1}.

Conjecture 6.8 ([KS]). *Let $R = (R_1, \ldots, R_p)$ be a dominant sequence of rectangular partitions, and $= (R)$ be the unique partition with empty p–core and p–quotient $(R_1, \ldots, R_p)$. Then*

$$K_{\lambda R}(t) = K_{\lambda}^{(p)}(t). \tag{6.18}$$

More generally, let λ be a partition with empty p–core and p–quotient $(\lambda^{(1)}, \ldots, \lambda^{(p)})$. Partition λ is called *p–dominant*, if there exists a permutation $s \in S_p$ such that $R = (\lambda^{(s(1))}, \ldots, \lambda^{(s(p))})$ is the dominant sequence of partitions.

Conjecture 6.9. *Let λ be a p–dominant partition, and $R := R(\lambda)$ be the dominant sequence of partitions obtained by rearrangement of the p–quotient of λ. Assume that all partitions in the sequence R have rectangular form, then for any partition μ*

$$K_{\lambda\mu}^{(p)}(t) = K_{\mu R}(t). \tag{6.19}$$

Problem 8. Let λ be a p–dominant partition, and $R := R(\lambda)$ be the dominant rearrangement of the p–quotient of λ. For which partition μ, the p–ribbon Kostka polynomial $K_{\lambda\mu}^{(p)}(t)$ coincides with generalized Kostka polynomial $K_{\mu R}(t)$?

As for the fusion Kostka polynomials $FK_{\lambda R}(t)$, their connection with the corresponding crystal, ribbon or generalized Kostka polynomials is unclear.

Finally, let us consider few examples which illustrate the difference between the ribbon, fusion and generalized Kostka polynomials.

Examples. *i*) Let $R = (R_1, R_2)$ be a dominant sequence of partitions. One can show that in this case

$$K_{\lambda R}(q) = \mathrm{Mult}(V_\lambda : V_{R_1} \otimes V_{R_2}) q^{E_0}, \tag{6.20}$$

for a certain constant $E_0 := E(\lambda R)$. However, the corresponding fusion and ribbon Kostka polynomials contain "in general" more than one term.

ii) Take $p = 4$ and $\lambda = (8,8,8,4,4)$. Then 4–quotient $\lambda = ((2,1),(1),(2),(2))$ and 4–core $(\lambda) = \emptyset$. Hence, we see that λ is the 4–dominant partition, and $R := R(\lambda) = ((2),(2),(2,1),(1))$. One can check that if $\mu = (4211)$, then

$$K_{\lambda\mu}^{(4)}(q) = K_{\mu R(\lambda)}(q) = q^2 + 2q^3 + 3q^4 + q^5.$$

If we take the same p and λ, but take $\mu = (4,2,2)$, then

$$K_{\lambda\mu}^{(4)}(q) = K_{\mu R(\lambda)}(q) = 2q^3 + 2q^4 + 2q^5.$$

However, if we take $p = 4$, $\lambda = (8,8,8,4,4)$, $\mu = (4,2,2)$, but take $R = ((2,1),(1),(2),(2))$ $= p$–quotient (λ), then $K_{\mu R}(q) = q^6 + 3q^7 + q^8 + q^9 \ne K_{\lambda\mu}^{(4)}(q)$.

iii) Take $p = 2$ and $\lambda = (11,9,9,7,7,5,2)$, then 2–core$(\lambda) = \emptyset$ and 2–quotient of λ is equal to $((4,3,2,1),(6,5,4))$; hence, λ is the 2–dominant partition, and $R := R(\lambda) = ((6,5,4),(4,3,2,1))$. Now let us take $\mu = (8,6,6,3,2)$, then

$$\begin{aligned} K_{\mu R}(q) &= 7q^5, \\ FK_{\mu R}(q) &= 2q^6 + 4q^7 + q^8, \\ K_{\lambda\mu}^{(2)}(q) &= 3q^4 + 4q^5 = K_{(R)\mu}^{(2)}(q). \end{aligned}$$

These examples show that for a general p–dominant partition λ with the dominant rearrangement of the p–quotient $R := R(\lambda)$, the ribbon and generalized Kostka polynomials $K_{\lambda\mu}^{(p)}(t)$ and $K_{\mu R}(t)$ give unequivalent q–analogues of the tensor product multiplicities.

§7. Fermionic formulae.

By fermionic formulae for polynomial (or series) $f(t) \in \mathbf{N}[t]$ we roughly mean such expression for $f(t)$ which is free of signs, admits a quasi–particle interpretation, has an origin in the Bethe ansatz, etc. Thanks to the absence of signs, fermionic formulae are suitable for studying the limiting behavior and serve as a key to establish various formulae for the characters related to the affine Lie algebras and Virasoro algebra, see [Ki2], [HKKOTY] for examples illustrating this thesis.

7.1. Multinomial fermionic formulae for one dimensional sums.

The starting point of our investigation is a simple observation that the number of transport matrices $\mathcal{P}_{\lambda\mu}(1)$ of type $(\lambda;\mu)$ is equal to the coefficient of x^{μ} in the product

$$h_{\lambda_1}(X_n)\dots h_{\lambda_p}(X_n),$$

where $\lambda=(\lambda_1,\dots,\lambda_p)$, $p\le n$, $l(\mu)\le n$, and $h_k(X_n)$ denotes the complete homogeneous symmetric function of degree k in the variables $X_n=(x_1,\dots,x_n)$.

More generally, the number $\mathcal{P}_{R\mu}(1)$ of weight μ unrestricted paths corresponding to the tensor product of crystals $B_{R_1}\otimes\cdots\otimes B_{R_p}$ is equal to the coefficient of x^{μ} in the product of Schur functions $s_{R_1}\cdots s_{R_p}$. This is clear. On the other hand it is well–known and goes back to D. Littlewood (see, e.g., [CL], [LLT], [SW]) that the latter coefficient is equal also to the number $|\mathrm{Tab}_p(,\mu)|$ of p–ribbon tableaux of shape and weight μ, where is the unique partition which satisfies the conditions (6.10) and (6.11).

Problem 9 (cf. Conjecture 6.4). To construct a bijection ψ between the sets $\mathcal{P}_{R\mu}(1)$ and $\mathrm{Tab}_p(,\mu)$ which transforms the energy function E on the set $\mathcal{P}_{R\mu}(1)$ to the modified spin function $\bar{s}$ on that $\mathrm{Tab}_p(,\mu)$.

Problem 10. Let $\mathcal{P}^0_{R\mu}(1)$ be the set of weight μ classically restricted paths corresponding to the tensor product of crystals $B_{R_1}\otimes\cdots\otimes B_{R_p}$, see, e.g., [KMOTU2], or [HKKOTY]. It is clear that $\mathcal{P}^0_{R\mu}(1)\subset\mathcal{P}_{R\mu}(1)$. Find characterization of the subset $\mathrm{Tab}^0_p(,\mu)\subset\mathrm{Tab}_p(,\mu)$ which corresponds to that $\mathcal{P}^0_{R\mu}(1)$ under the above bijection ψ.

In the case $p=2$ the set $\mathrm{Tab}^0_2(,\mu)$ was characterized by C. Carre and B. Leclerc [CL] as the set of Yamanouchi domino tableaux. A weight preserving bijection between the set of domino tableaux of a fixed shape and that of ordinary tableaux of a related fixed shape, which maps Yamanouchi domino tableaux to ordinary Yamanouchi tableaux, was constructed by M. van Leeuwen [Le]. The question whether or not the bijection constructed by Leeuwen transforms the spin of a domino tableau to the value of the energy function for corresponding path is still open.

Let us continue and note that there exists yet another way to describe the coefficient of x^{μ} in the product of Schur functions $s_{R_1}\cdots s_{R_p}$ which is based on a combinatorial formula for Schur functions, see, e.g., [M], Chapter I, (5.11). We consider here only the so–called "homogeneous case" $R_1=\cdots=R_p:=\lambda$. The general case can be treated similarly. The starting point for obtaining the multinomial fermionic formulae is the following combinatorial formula for the Schur functions mentioned above:

let λ be a partition, $l(\lambda) \le n$, then

$$s_\lambda(X_n) = \sum_T x^{w(T)},$$

where the sum runs over the set $STY(\lambda, \le n)$ of all semistandard Young tableaux of shape λ filled by numbers not exceeding n, and $w(T)$ is the weight of a tableau T.

Let us define the multinomial coefficient $\binom{L}{\mu}_\lambda$ via decomposition

$$(s_\lambda(X_n))^L = \sum_\mu \binom{L}{\mu}_\lambda x^\mu, \tag{7.1}$$

where the sum is taken over the set of all compositions μ such that $l(\mu) \le n$ and $|\mu| = L|\lambda|$;

$$\binom{L}{\mu}_\lambda := \sum_{\{k_T\}} \binom{L}{\{k_T\}}, \tag{7.2}$$

summed over all sequences of nonnegative integers $\{k_T\}$ parameterized by the set of semistandard Young tableaux $STY(\lambda, \le n)$, such that $\sum_{T \in STY(\lambda, \le n)} w(T) k_T = \mu$;

$\binom{L}{\{k_T\}} := \dfrac{L!}{\prod_T (k_T)!}$ stands for the gaussian multinomial coefficient.

The multinomial coefficients $\binom{L}{\mu}_\lambda$ can be characterized by the following properties:

- $\binom{0}{\mu}_\lambda = \delta_{0,\mu}$ (initial data);
- $\binom{L+1}{\mu}_\lambda = \sum_{T \in STY(\lambda, \le n)} \binom{L}{\mu - w(T)}_\lambda$ (recurrence relations), $\quad$ (7.3)

where we assume that the gaussian multinomial coefficient $\binom{L}{m_1, \ldots, m_n}$ is equal to 0, if $m_i < 0$ for some i.

It is natural to ask: what is a q–analog of the multinomial coefficient $\binom{L}{\mu}_\lambda$, and what are the q–analogues of relations (7.1), (7.2), and (7.3)? The answers on these questions are either well–known or conjectured.

More precisely, for each semistandard tableau $T \in STY(\lambda, \le n)$ let us denote by $\begin{bmatrix} L \\ \mu \end{bmatrix}_\lambda^{(T)}$ the weight μ unrestricted one dimensional sum with boundary condition $b_T \in B_\lambda$. Let $H : B_\lambda \times B_\lambda \to \mathbf{Z}$ stands for the local energy function corresponding to the crystal B_λ, see, e.g., [Ka1], [Ka2]. In the sequel we will identify the sets B_λ and $STY(\lambda, \le n)$.

It is well–known, see, e.g., [KMOTU2], that one dimensional sums $\begin{bmatrix} L \\ \mu \end{bmatrix}_\lambda^{(T)}$ satisfy the following conditions:

- $\left[\begin{matrix} L \\ \mu \end{matrix}\right]_{\lambda}^{(T)} |_{q=1} = \begin{pmatrix} L \\ \mu - w(T) \end{pmatrix}_{\lambda}$,
- $\left[\begin{matrix} L \\ \mu \end{matrix}\right]_{\lambda}^{(T)} = \delta_{0\mu}$ (initial datum),
- let $T_0 \in STY(\lambda, \leq n)$, then

$$\left[\begin{matrix} L+1 \\ \mu \end{matrix}\right]_{\lambda}^{(T_0)} = \sum_{T \in STY(\lambda, \leq n)} q^{H(T_0,T)} \left[\begin{matrix} L \\ \mu \end{matrix}\right]_{\lambda}^{(T)}. \tag{7.4}$$

(recurrence relations)

For q–analogue of (7.2) and (7.3), we are making the following conjectures:

Conjecture 7.1. *There exists a quadratic form $Q : B_\lambda \times B_\lambda \to \mathbf{Z}$, and a set of linear forms $l_T : B_\lambda \to \mathbf{Z}$, $T \in B_\lambda$, such that*

$$\left[\begin{matrix} L \\ \mu \end{matrix}\right]_{\lambda}^{(T_0)} = \sum_{\{k_T\}} q^{\sum_{T,T'} Q(T,T')k_T k_{T'} + \sum_T l_{T_0}(T) k_T} \left[\begin{matrix} L \\ \{k_T\} \end{matrix}\right]_q, \tag{7.5}$$

summed over all sequences of nonnegative integers $\{k_T\}$, $T \in STY(\lambda, \leq n)$, such that $\sum\limits_T w(T) k_T = \mu$; $\left[\begin{matrix} L \\ \{k_T\} \end{matrix}\right]_q = \dfrac{(q;q)_L}{\prod_T (q;q)_{k_T}}$ stands for a q–analog of the gaussian multinomial coefficient.

Remark. The answer to this conjecture is known or conjectured in the case when partition $\lambda = (l)$ consists of one part and for some special values of $T \in STY((l), \leq n)$.

Conjecture 7.2. *Let $\lambda = (\lambda_1, \ldots, \lambda_s)$ be a partition. For each integer $k \geq 1$, denote by $k\lambda$ the following partition $(k\lambda_1, \ldots, k\lambda_s)$. Then*

$$q^{Ln(\lambda')} \mathcal{P}_{L\lambda,\mu}^{(L)}(q) = \left[\begin{matrix} L \\ \mu \end{matrix}\right]_{\lambda}^{(T_{\max})}, \tag{7.6}$$

where $T_{\max}$ denotes the unique maximal with respect to the lexicographic order element in the set $STY(\lambda, \leq n)$.

In other words, let $H_\lambda^{(L)}(X_n; t)$ be the H–function defined in [LLT], Section 6; see also [CL] and [KLLT]. Then

$$H_\lambda^{(L)}(X_n; t) = q^{-Ln(\lambda')} \sum_{\mu} \left[\begin{matrix} L \\ \mu \end{matrix}\right]_{\lambda}^{(T_{\max})} m_\mu(X_n). \tag{7.7}$$

Remark. If partition $\lambda = (l)$ consists of one part, the multinomial coefficients $\left[\begin{matrix} L \\ \mu \end{matrix}\right]_{\lambda}^{(T)}$ coincide with those introduced by A. Schilling and S.O. Warnaar after changing q to q^{-1} and multiplication on some power of q, [Ki2], [Sc], [ScW], [W].

7.2. Rigged configurations polynomials.

In the previous subsection we explained an origin of a (conjectural) multinomial fermionic formulae for polynomials $\mathcal{P}^{(L)}_{L\lambda,\mu}(t)$. In this subsection we are going to to present yet another example of a (conjectural) fermionic formula for generalized Kostka polynomials corresponding to a collection of rectangles $R=(R_1,\ldots,R_p)$, where $R_a=(\eta_a^{\mu_a})$ for all $1\le a\le p$. Our approach is using the so–called rigged configurations polynomials.

Definition 7.3. *Let λ be a partition such that $|\lambda|=\sum_a \mu_a\eta_a$. We define the rigged configurations polynomial $RC_{\lambda R}(q)$ to be*

$$RC_{\lambda R}(q)=\sum_{\{\nu\}} q^{c(\{\nu\})}\prod_{k\ge1}\prod_{i\ge1}\begin{bmatrix} P_i^{(k)}(\nu)+m_i(\nu^{(k)}) \\ m_i(\nu^{(k)})\end{bmatrix}_q, \tag{7.8}$$

where sum runs over all sequences of partitions $\nu=\{\nu^{(1)},\nu^{(2)},\ldots\}$ such that

- $|\nu^{(k)}|=\sum_{j\ge k+1}\lambda_j-\sum_{a=1}^{p}\mu_a\theta(\eta_a-k)$;
- $P_i^{(k)}(\nu):=\sum_{a=1}^{p}\min(i,\mu_a)\delta_{k,\eta_a}+Q_i(\nu^{(k-1)})-2Q_i(\nu^{(k)})+Q_i(\nu^{(k+1)})\ge 0$ for $i,k\ge1$.

We have used the following notations:

i) for any partition λ, $Q_j(\lambda)=\sum_{i\le j}\lambda'_i=\sum_{i\ge1}\min(j,\lambda_i)$;
ii) if $x\in\mathbf{R}$, then $\theta(x)=1$, if $x\ge0$, and $\theta(x)=0$, if $x<0$;
iii) $\nu^{(0)}=\emptyset$;
iv) $m_i(\lambda)=\lambda'_i-\lambda'_{i+1}$ is the number of parts equal to i of the partition λ;
v) $c(\{\nu\})=\sum_{k\ge1}\sum_{i\ge1}\binom{A_{ik}}{2}$, where $A_{ik}=(\nu^{(k-1)})'_i-(\nu^{(k)})'_i+\sum_{a=1}^{p}\theta(\eta_a-k)\theta(\mu_a-i)$.

Conjecture 7.4 (A.N. Kirillov, M. Shimozono, [KS]). *Let λ and R be as above, and be the unique partition which satisfies the conditions (6.10) and (6.11). Assume that $\mu_1\ge\mu_2\cdots\ge\mu_p$, then*

$$K_\lambda^{(p)}(q)=RC_{\lambda R}(q).$$

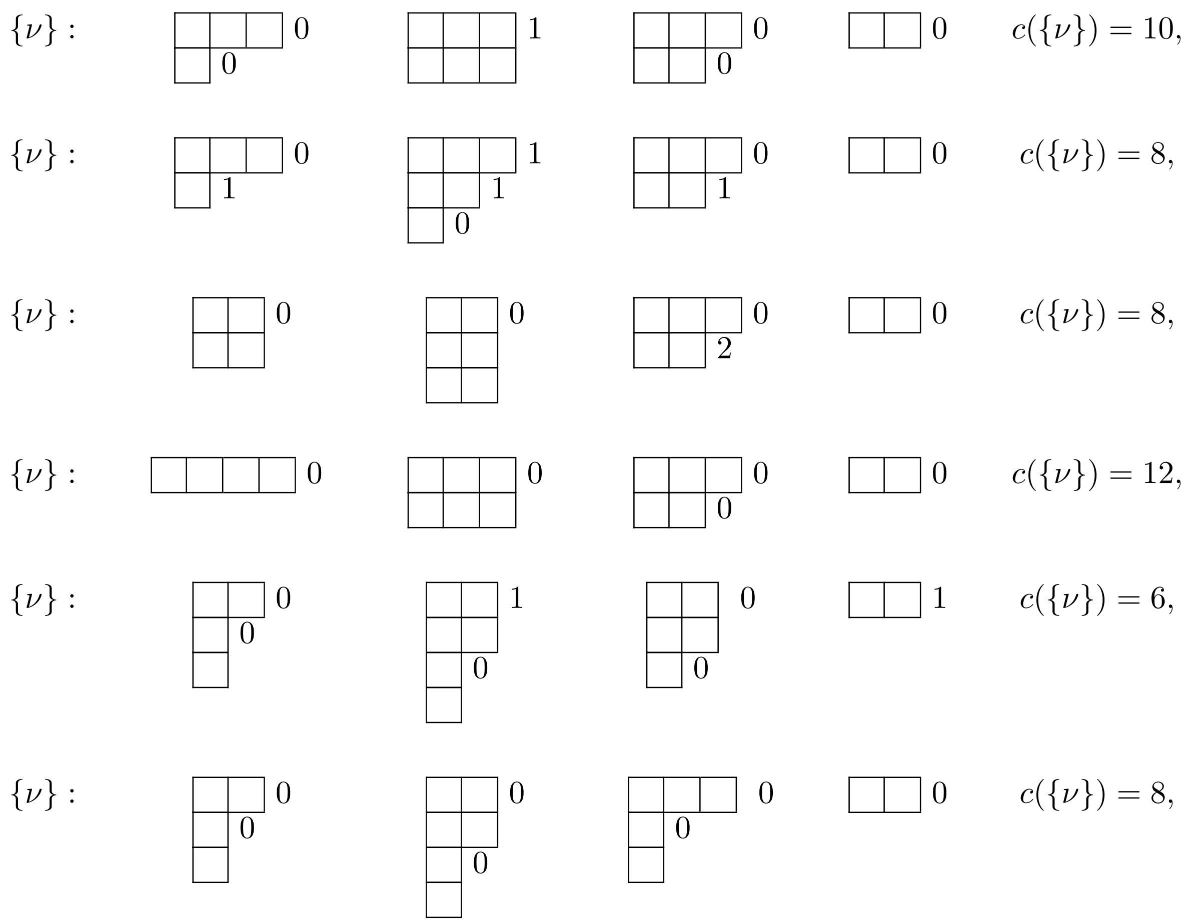
It is known (A.N. Kirillov) that

$$RC_{\lambda R}(1)=\mathrm{Mult}[V_\lambda:V_{R_1}\otimes\cdots\otimes V_{R_p}], \tag{7.9}$$

where for each $1\le a\le p$, $R_a=(\eta_a^{\mu_a})$ is a rectangular partition. A combinatorial proof of (7.9) is based on the construction of rigged configurations bijection (A.N. Kirillov).

Let us illustrate the formula (7.8) by simple example:

Example. Take $\lambda = (44332)$ and $R = \big((2^3), (2^2), (2^2), (1), (1)\big)$. Then $|\nu^{(1)}| = 4$, $|\nu^{(2)}| = 6$, $|\nu^{(3)}| = 5$, and $|\nu^{(4)}| = 2$. It is not hard to check that there exist 6 configurations. They are:



Thus, the rigged configurations polynomial $RC_{\lambda R}(q)$ is equal to

$$q^{10}\begin{bmatrix}3\\1\end{bmatrix} + q^8\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} + q^8\begin{bmatrix}3\\1\end{bmatrix} + q^{12} + q^6\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix} + q^8$$

$$= q^6 + 2q^7 + 5q^8 + 6q^9 + 8q^{10} + 5q^{11} + 3q^{12}.$$

§8. Two parameter deformation of one dimensional sums.

Following [GH], we define the modified Macdonald polynomials in *infinite number of variables* by

$$\widetilde{P}_\lambda(x;q,t) = P_\lambda\left(\frac{x}{1-t};q,t\right), \qquad \widetilde{J}_\lambda(x;q,t) = J_\lambda\left(\frac{x}{1-t};q,t\right) \tag{8.1}$$

in the λ–ring notation. Let us explain briefly the λ–ring notation in the context of symmetric functions. Given a symmetric function $f(x) = f(x_1, x_2, \ldots)$ in infinite set of variables $x = (x_1, x_2, \ldots)$, the symbol $f\left(\dfrac{x}{1-t}\right)$ in the λ–ring notation stands for the symmetric function $f(\widetilde{x})$ obtained by the transformation of variables $\widetilde{x} = (x_i t^j)_{i \ge 1, j \ge 0}$. In infinite number of variables, the symmetric function $f(x)$ can be written uniquely in the form $f(x) = \varphi(p_1(x), p_2(x), \ldots)$ as a polynomial of the power series $p_k(x) = \displaystyle\sum_{j=1}^{\infty} x_j^k$, $k = 1, 2, \ldots$. Then the symbol $f\left(\dfrac{x}{1-t}\right)$ represents the symmetric function

$$f\left(\frac{x}{1-t}\right) = \varphi\left(\frac{p_1(x)}{1-t}, \frac{p_2(x)}{1-t^2}, \ldots\right),$$

obtained by the transformation $p_k(x) \to p_k(x)/(1-t^k)$, $k = 1, 2, \ldots$. When we consider the modified Macdonald polynomials in n variables $X_n = (x_1, \ldots, x_n)$, each function $\widetilde{P}_\lambda(X_n; q, t)$ and $\widetilde{J}_\lambda(X_n; q, t)$ should be understood as the one obtained from the corresponding symmetric function in infinite number of variables by setting $x_{n+1} = x_{n+2} = \cdots = 0$.

An advantage of modified Macdonald polynomials is that they have nice transformation coefficients with classical Schur functions $s_\lambda(x)$:

$$\widetilde{J}_\lambda(x; q, t) = \sum_\lambda K_{\lambda\mu}(q, t) s_\lambda(x), \tag{8.2}$$

where $K_{\lambda\mu}(q, t)$ are the double Kostka coefficients. It is well–known that $K_{\lambda\mu}[q, t] \in \mathbf{Z}[q, t]$ for all λ and μ, see, e.g., [KN].

Our next aim is to construct two parameter deformation of polynomials $\mathcal{P}_{\lambda\mu}(t)$ using the modified Macdonald polynomials $\widetilde{J}_\lambda(X_n; q, t)$ instead of modified Hall–Littlewood polynomials $Q'_\lambda(X_n; t)$. To this end, let us consider, follow [KN], a family of polynomials $B_{\lambda\mu}(q, t)$ via decomposition

$$\widetilde{J}_\lambda(x; q, t) = \sum_\mu B_{\lambda\mu}(q, t) m_\mu(x).$$

It is clear that

$$B_{\lambda\mu}(q, t) = \sum_\eta K_{\eta\mu} K_{\eta\lambda}(q, t).$$

Let us formulate some basic properties of polynomials $B_{\lambda\mu}(q, t)$. For further details and proofs, see [M], Chapter VI, and [KN], Section 8.

Let λ and μ be a partitions of a given natural number N, then

- $B_{\lambda\mu}(1, 1) = \dbinom{N}{\mu_1, \mu_2, \ldots} = \mathcal{P}_{1^N \mu}(1)$;

- $B_{\lambda\mu}(0,t) = \mathcal{P}_{\lambda\mu}(t)$;
- $B_{(N)\mu}(q,t) = q^{n(\mu')} \begin{bmatrix} N \\ \mu_1, \mu_2, \ldots \end{bmatrix}_q = \mathcal{P}_{1^N \mu}(q)$;
- $B_{\lambda'\mu}(q,t) = q^{n(\lambda')} t^{n(\lambda)} B_{\lambda\mu}(t^{-1}, q^{-1})$ (duality).

It follows from duality that

$$B_{\lambda\mu}(q,t) = q^{n(\lambda')}(\widetilde{\mathcal{R}}_{\lambda\mu}(t) + o(q^{-1})),$$

where $\widetilde{\mathcal{R}}_{\lambda\mu}(t) = \sum_{\eta} K_{\eta\mu} \widetilde{K}_{\eta'\lambda}(t)$, and $\widetilde{K}_{\eta'\lambda}(t) = t^{n(\lambda)} K_{\eta'\lambda}(t^{-1})$.

The properties of polynomials $B_{\lambda\mu}(q,t)$ mentioned above show that they can be considered as a natural two parameter deformation of the gaussian multinomial coefficients.

Problem 11. Find a "path realization" of polynomials $B_{\lambda\mu}(q,t)$.

Problem 12 ("Parabolic modified Macdonald polynomials"). Find two parameter deformation of polynomials $\widetilde{Q}^{(p)}_{\lambda}(X_n;t)$ with nice combinatorial, algebraic and geometric properties.

At the end of this Section we give an example of polynomials $B_{\lambda\mu}(q,t)$.

Example. Take $\lambda = (2^3)$, $\mu = (2^2 1^2)$, then

$$\begin{aligned} B_{\lambda\mu}(q,t) &= \sum_{\eta} K_{\eta\mu} K_{\eta\lambda}(q,t) = 1 + 4t + 8t^2 + 9t^3 + 7t^4 + 3t^5 + t^6 \\ &+ q \begin{bmatrix} 3 \\ 1 \end{bmatrix}_t (1 + 5t + 9t^2 + 7t^3 + 3t^4) + q^2 \begin{bmatrix} 3 \\ 2 \end{bmatrix}_t (2t + 6t^2 + 7t^3 + 4t^4) \\ &+ q^3(t^2 + 3t^3 + 5t^4 + 4t^5 + 2t^6). \end{aligned}$$

Using the fermionic formulae (3.1) and (3.7), one can check that

$$\mathcal{P}_{\lambda\mu}(t) = B_{\lambda\mu}(0,t) = 1 + 4t + 8t^2 + 9t^3 + 7t^4 + 3t^5 + t^6,$$

and

$$\mathcal{R}_{\lambda\mu}(t) = q^3 t^6 B_{\lambda\mu}(q^{-1}, t^{-1})|_{q=0} = \sum_{\eta} K_{\eta\mu} K_{\eta'\lambda}(t) = 2 + 4t + 5t^2 + 3t^3 + t^4.$$

References.

[An] Andrews G., *The theory of partitions,* Addison–Wesley Publishing Company, 1976.

[BKMW] Begin L., Kirillov A.N., Mathieu P. and Walton M.A., *Berenstein–Zelevinski triangles, elementary couplings and fusion rules,* Lett. in Math. Phys., 1993, v.28, p.257-268.

[Bi] Birkhoff G., *Subgroups of abelian groups,* Proc. London Math. Soc. (2), 1934-5, v.38, p.385-401.

[Bu1] Butler L., *Subgroup lattices and symmetric functions,* Memoirs of AMS, 1994, v. 112, n. 539.

[Bu2] Butler L., *Generalized flags in finite abelian p–groups,* Discrete Appl. Math., 1991, v.112, p.67-81.

[Bu3] Butler L., *A unimodality result in the enumeration of subgroups of a finite abelian group,* Proc. Amer. Math. Soc., 1987, v.101(4), p.771-775.

[CL] Carré C. and Leclerc B., *Splitting the square of a Schur function into its symmetric and antisymmetric parts,* J. of Alg. Combin., 1995, v.4, p.201-231.

[DLT] Désarmenien J., Leclerc B. and Thibon J.-Y., *Hall–Littlewood functions and Kostka–Foulkes polynomials in representation theory,* Sëminaire Lotharingien de Combinatoire, 1994, v.32.

[De] Delsarte S., *Fonctions de Möbius sur les groupes abelian finis,* Annals of Math., 1948, v.49, p.600-609.

[Dy] Dyubyuk P., *On the number of subgroups of a finite abelian group,* Izv. Akad. Nauk USSR, Ser. Mat., 1948, v.12, p.371-328.

[Fi] Fishel S., *Nonnegativity results for generalized q–binomial coefficients,* PhD thesis, the University of Minnesota, 1993.

[F] Foata D., *Distribution eulériennes et mahoniennes sur le group des permutations,* in *Higher Combinatorics,* M. Aigner ed., Amsterdam, D. Reidel, 1977, p.27-49.

[FZ] Foata D. and Zeilberger D., *Denert's permutation statistic is indeed Euler–Mahonian,* Studies in Applied Math., 1990, v.83, p.31-59.

[GaW] Galovich J. and White D., *Recursive statistics on words,* Discrete Math., 1996, v.157, p.169-191.

[GH] Garsia A. and Haiman M., *A graded representation model for Macdonald's polynomials*, Proc. Nat. Acad. Sci. USA, 1993, v.90, p.3607-3610.

[GoW] Goodman F. and Wenzl H., *Littlewood–Richardson coefficients for Hecke algebras at roots of unity,* Adv. Math., 1990, v.82, p.244-265.

[Gu] Gupta R., *Generalized exponent via Hall–Littlewood symmetric functions,* Bull. Amer. Math. Soc., 1987, v.16, p.287-291.

[HKKOTY] Hatayama G, Kirillov A.N., Kuniba A., Okado M., Takagi T. and Yamada Y., *Character formulae of $\widehat{sl_n}$–modules and inhomogeneous paths*, Preprint math.QA/9802085, 1998, 42p.

[Hi] Hilton H., *On subgroups of a finite abelian group,* Proc. london Math. Soc (2), 1907, v.5, p.1-5.

[HS] Hotta R. and Shimomura N., *The fixed point subvarieties of unipotent transformations on generalized flag varieties and Green functions,* Math. Ann., 1979, v.241, p.193-208.

[H] Huppert B., Endliche Gruppen, vol.1, Spring–Verlag, 1967.

[Kac] Kac V.G., *Infinite dimensional Lie algebras,* Cambridge University Press, 1990.

[Ka1] Kashiwara M., *On crystal bases of the q–analogue of universal enveloping algebras,* Duke Math. J., 1991, v.63, p.465-516.

[Ka2] Kashiwara M., *The crystal base and Littlemann's refined Demazure character formula,* Duke Math. J., 1993, v.71, p.839-858.

[Ki1] Kirillov A.N., *On the Kostka–Green–Foulkes polynomials and the Clebsch–Gordan numbers,* Journ. Geom. and Phys., 1988, v.5, n.3, p.365-389.

[Ki2] Kirillov A.N., *Dilogarithm identities,* Progress of Theor. Phys. Suppl., 1995, v.118, p.61-142.

[KKN] Kirillov A.N., Kuniba A. and Nakanishi T., *Skew Young diagram method in spectral decomposition of integrable lattice models II: Higher levels,* Preprint q-alg/9711009, 1997, 27p.

[KLLT] Kirillov A,N., Lascoux A., Leclerc B. and Thibon J.-Y., *Séries génératrices pour les tableaux de dominos,* C.R. Acad. Sci. Paris, 1994, t.318, Serie I, p.395-400.

[KN] Kirillov A.N. and Noumi M., *Affine Hecke algebras and raising operators for Macdonald polynomials,* to appear in Duke Math. Journ.; q-alg/9605004, 1996, 35p.

[KS] Kirillov A.N. and Shimozono M., *A generalization of the Kostka–Foulkes polynomials,* Preprint math.QA/9803062, 1998, 37p.

[Kn] Knuth D.E., *Permutations, matrices and generalized Young tableaux,* Pacific J. Math., 1970, v.34, p.709-727.

[KMOTU1] Kuniba A., Misra K.C., Okado M., Takagi T. and Uchiyama J., *Crystals for Demazure modules of classical affine Lie algebras,* Preprint q-alg/9707014, 29p.

[KMOTU2] Kuniba A., Misra K.C., Okado M., Takagi T. and Uchiyama J., *Characters of Demazure modules and solvable lattice models,* Nuclear Phys., 1998, v.B510[PM], p.555-576.

[LLT] Lascoux A., Leclerc B. and Thibon J.-Y., *Ribbon tableaux, Hall–Littlewood functions, quantum affine algebras, and unipotent varieties,* J. Math. Phys., 1997, v.38, p.1041-1068.

[LS] Lascoux A. and Schützenberger M.-P., *Sur une conjecture de H.O. Foulkes,* C.R. Acad. Sci. Paris, 1978, t.286A, p.323-324.

[Le] Leeuwen M. van, *Some bijective correspondence involving domino tableaux,* Preprint MAS–R9708, 1997, 28p.

[M] Macdonald I., *Symmetric functions and Hall polynomials,* 2nd ed., Oxford, 1995.

[Ma] MacMahon P.A., *Combinatorial Analysis,* I, II, Cambridge University Press, 1915, 1916 (reprinted by Chelsea, New York, 1960).

[Mi] Miller G., *On subgroups of an abelian group,* Annals of Math., 1904, v.6, p.1-6.

[NY] Nakayashiki A. and Yamada Y., *Kostka polynomials and energy functions in solvable lattice models,* Preprint q-alg/9512027; to appear in Selecta Mathematica.

[R] Regonati F., *Sui numeri dei sottogruppi di dato ordine dei p–gruppi abeliani finiti,* Instit. Lombardo Rend. Sci., 1988, v.A122, p.369-380.

[Sc] Schilling A., *Multinomials and polynomial bosonic forms for the branching functions of the* $\widehat{su}(2)_M \times \widehat{su}(2)_N/\widehat{su}(2)_{M+N}$ *conformal coset models,* Nucl. Phys. B, 1996, v.467, p.247-271.

[ScW] Schilling A. and Warnaar S.O., *Supernomial coefficients, polynomial identities and q–series,* Preprint q-alg/9701007, 34p.

[Sh] Shimomura N., *A theorem of the fixed point set of a unipotent transformation of the flag manifold,* J. Math. Soc. Japan, 1980, v.32, p.55-64.

[St] Stanley R., *Supersolvable lattices,* Algebra Universalis, 1972, v.2, p.197-217.

[SW] Stanton D. and White D., *A Schensted algorithm for rim hook tableaux,* J. Comb. Theory, Ser. A, 1985, v.40, p.211-247.

[T] Terada I., *A generalization of the length –* Maj *symmetry and the variety of N–stable flags,* Preprint, 1993.

[W] Warnaar S.O., *The Andrews–Gordon identities and q–multinomial coefficients,* Comm. Math. Phys., 1997, v.184, p.203-232.

[Y] Yeh Y., *On prime power abelian groups,* Bull. Amer. Math. Soc., 1948, v.54, p.323-327.

[ZB] Zeilberger D. and Bressoud D., *A proof of Andrew's q–Dyson conjecture,* Discrete Math., 1985, v.54, p.201-224.