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New construction of type 2 degenerate central Fubini polynomials with their certain properties

Sunil Kumar Sharma[1*], Waseem A. Khan[2], Serkan Araci[3] and Sameh S. Ahmed[4]

*Correspondence:
s.sharma@mu.edu.sa
[1]College of Computer and Information Sciences, Majmaah University, Majmaah 11952, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

Kim *et al.* (Proc. Jangjeon Math. Soc. 21(4):589–598, 2018) have studied the central Fubini polynomials associated with central factorial numbers of the second kind. Motivated by their work, we introduce degenerate version of the central Fubini polynomials. We show that these polynomials can be represented by the fermionic p-adic integral on $\mathbb{Z}_p$. From the fermionic p-adic integral equations, we derive some new properties related to degenerate central factorial numbers of the second kind and degenerate Euler numbers of the second kind.

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1 Introduction

Let p be chosen as a fixed odd prime number. We make use of the following notations: $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}_p$, $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{R}$ denote the ring of p-adic integers, the field of p-adic rational numbers, the completion of an algebraic closure of $\mathbb{Q}_p$, the set of natural number, the set of natural numbers containing zero and the set of real numbers, respectively. We say that $|\cdot|_p$ is normalized if $|p|_p = p^{-1}$. Let $C(\mathbb{Z}_p)$ be the space of $\mathbb{C}_p$-valued continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic integral of a function f was originally constructed by Kim [12] as follows:

$$\begin{aligned} I_{-1}(f) &:= \int_{\mathbb{Z}_p} f(x)\,d\mu_{-1}(x) \qquad (1) \\ &= \lim_{n\to\infty}\sum_{a=0}^{p^n-1} f(a)\mu_{-1}\left(a+p^n\mathbb{Z}_p\right) \\ &= \lim_{n\to\infty}\sum_{a=0}^{p^n-1} f(a)(-1)^a. \end{aligned}$$

Here I_{-1} is used symbolically due to its fermionic distribution μ_{-1}. It follows from (1) that

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0) \quad \text{(see [12])} \tag{2}$$

with the assumption $f_1(x) := f(x+1)$. For more information about the applications of p-adic integrals, one can refer to [4, 8, 9, 14, 17, 22] and the references cited therein.

Very recently, Kim *et al.* [20] showed that the Fubini polynomials can be represented by the fermionic p-adic integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p} \bigl(x\bigl(1-e^t\bigr)\bigr)^y \, d\mu_{-1}(y) = \frac{2}{1-x(e^t-1)} = 2\sum_{n=0}^{\infty} F_n(x)\frac{t^n}{n!}. \tag{3}$$

For more information about the Fubini polynomials, one can look at the references [5, 10, 24]. The Stirling numbers of the second kind are defined by (see [11])

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \tag{4}$$

where

$$(x)_n = \begin{cases} x(x-1)\cdots(x-n+1), & \text{when } n \geq 1, \\ 1, & \text{when } n = 0, \end{cases}$$

or equivalently by

$$\sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!} = \frac{(e^t-1)^k}{k!} \quad (k \geq 0). \tag{5}$$

Let $\lambda \neq 0$ be any real numbers. Carlitz [1] introduced the degenerate Bernoulli polynomials by means of the following generating function:

$$\sum_{n=0}^{\infty} \beta_n(x;\lambda)\frac{t^n}{n!} = \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}. \tag{6}$$

When $x = 0$ in (6), $\beta_n(\lambda) =: \beta_n(0;\lambda)$ are the degenerate Bernoulli numbers. It is clear from (6) that

$$\lim_{\lambda\to 0} \beta_n(x;\lambda) := B_n(x) \quad (n \in \mathbb{N}_0),$$

where $B_n(x)$ are the Bernoulli polynomials given by

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt}, \quad \textit{cf.}\ [1].$$

Parallel to (6), the degenerate Euler polynomials are defined by means of the following generating function:

$$\sum_{n=0}^{\infty} E_n(x;\lambda)\frac{t^n}{n!} = \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} \quad (0 \neq \lambda \in \mathbb{R}) \text{ (see [1])}. \tag{7}$$

When $x = 0$ in (9), $E_n(\lambda) =: E_n(0;\lambda)$ are the degenerate Euler numbers.

For $0 \neq \lambda \in \mathbb{R}$, it is worth noting from [16, 21] that

$$\begin{aligned} e_\lambda^x(t) &:= (1+\lambda t)^{\frac{x}{\lambda}} \\ &= \sum_{n=0}^{\infty} \binom{\frac{x}{\lambda}}{n} \lambda^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \end{aligned} \tag{8}$$

where

$$(x)_{0,\lambda} = 1, \qquad (x)_{n,\lambda} := x(x-\lambda)(x-2\lambda)\cdots\bigl(x-(n-1)\lambda\bigr) \quad (n \geq 1) \text{ see [19].}$$

The degenerate Euler numbers of the second kind are given by

$$\operatorname{sech}_\lambda(t) = \frac{2}{e_\lambda(t) + e_\lambda^{-1}(t)} = \sum_{n=0}^{\infty} E_{n,\lambda}^* \frac{t^n}{n!}. \tag{9}$$

It is obvious that

$$\lim_{\lambda \to 0} E_{n,\lambda}^* := E_n^*,$$

where E_n^* are the Euler numbers of the second kind given by

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n^* \frac{t^n}{n!}, \quad \textit{cf.} \text{ [3].}$$

By (5) and (6), the degenerate Stirling polynomials of the second kind are defined by the generating function

$$\sum_{n=k}^{\infty} S_{2,\lambda}^{(x)}(n,k) \frac{t^n}{n!} = \frac{(e_\lambda(t)-1)^k}{k!} e_\lambda^x(t), \tag{10}$$

where $x \in \mathbb{R}$, and k is a nonnegative integer (see [13]). In the case $x = 0$, $S_{2,\lambda}^{(0)}(n,k) := S_{2,\lambda}(n,k)$ are the degenerate Stirling numbers of the second kind, *cf.* [1–4, 6, 7].

Since

$$\lim_{\lambda \to 0} e_\lambda(t) = e^t$$

we have

$$\lim_{\lambda \to 0} S_{2,\lambda}^{(x)}(n,k) := S_2^{(x)}(n,k),$$

where $S_2^{(x)}(n,k)$ are the Stirling polynomials of the second kind given by

$$\sum_{n=k}^{\infty} S_2^{(x)}(n,k) \frac{t^n}{n!} = \frac{e^{xt}(e^t-1)^k}{k!}.$$

The central factorial numbers of the second kind, denoted by $T(j,k)$ with the conditions $j \geq 0$ and $k \geq 0$, are defined by

$$x^j = \sum_{k=0}^{j} T(j,k)x^{[k]} \quad \text{(see [11])}, \tag{11}$$

where

$$x^{[k]} = \begin{cases} x(x+\frac{k}{2}-1)(x+\frac{k}{2}-2)\cdots(x-\frac{k}{2}+1), & \text{when } k \geq 1, \\ 1, & \text{when } k = 0 \end{cases}$$

or equivalently by

$$\frac{1}{k!}\bigl(e^{\frac{z}{2}} - e^{-\frac{z}{2}}\bigr)^k = \sum_{j=k}^{\infty} T(j,k)\frac{z^j}{j!} \quad \text{(see [11, 23])}. \tag{12}$$

Kim–Kim [15] introduced the degenerate central factorial polynomials of the second kind as follows:

$$\sum_{j=k}^{\infty} T_\lambda(j,k|x)\frac{z^j}{j!} = \frac{1}{k!}\bigl(e_\lambda^{\frac{1}{2}}(z) - e_\lambda^{\frac{-1}{2}}(z)\bigr)^k e_\lambda^x(t), \tag{13}$$

where k is a nonnegative integer. The case $x = 0$ yields $T_\lambda(j,k) =: T_\lambda(j,k|0)$ that are the degenerate central factorial numbers of the second kind.

This paper is organized as follows. In Sect. 2, we consider the generating function of type 2 degenerate central Fubini polynomials and give some properties of these numbers and polynomials. In Sect. 3, we introduce degenerate central Fubini numbers and polynomials and derive some properties of these polynomials by using p-adic fermionic integrals on $\mathbb{Z}_p$. In Sect. 4, we introduce type 2 degenerate central Fubini polynomials of two variables and construct some properties of these polynomials. Also, these polynomials are closely related to degenerate central factorial numbers of the second kind and degenerate Euler numbers of the second kind.

2 On type 2 degenerate central Fubini polynomials

In this section, we assume that $\lambda \neq 0$ is any real number. We begin with giving type 2 degenerate central Fubini polynomials as follows:

$$\sum_{n=0}^{\infty} F_{n,\lambda}^{(C)}(x)\frac{t^n}{n!} = \frac{1}{1 - x(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t))}. \tag{14}$$

Note that

$$\lim_{\lambda\to 0}\sum_{n=0}^{\infty} F_{n,\lambda}^{(C)}(x)\frac{t^n}{n!} := \sum_{n=0}^{\infty} F_n^{(C)}(x)\frac{t^n}{n!} = \frac{1}{1 - x(e^{\frac{t}{2}} - e^{-\frac{t}{2}})} \quad \text{(see [11])}.$$

By (14), one may see that

$$\begin{aligned}\sum_{n=0}^{\infty} F_{n,\lambda}^{(C)}(x)\frac{t^n}{n!} &= \frac{1}{1-x(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))} \\ &= \sum_{k=0}^{\infty} x^k \big(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)\big)^k \\ &= \sum_{k=0}^{\infty} x^k k! \sum_{n=k}^{\infty} T_\lambda(n,k)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} x^k k! T_\lambda(n,k)\right)\frac{t^n}{n!}.\end{aligned}$$

Thus, we state the following theorem.

Theorem 2.1 *Let n be a nonnegative integer. Then the following holds*:

$$F_{n,\lambda}^{(C)}(x) = \sum_{k=0}^{n} x^k k! T_\lambda(n,k). \tag{15}$$

The degenerate ordered Bell numbers are defined by the generating function to be

$$\frac{1}{1-2\sinh_{2\lambda}(\frac{t}{2})} = \sum_{n=0}^{\infty} b_{n,\lambda}\frac{t^n}{n!}, \tag{16}$$

where

$$\sinh_\lambda(t) = \frac{e_\lambda(t)-e_\lambda^{-1}(t)}{2}, \quad \textit{cf.}\ [3].$$

Corollary 2.1 *Taking* $x = 1$ *in* (15) *gives*

$$F_{n,\lambda}^{(C)}(1) := b_{n,\lambda} = \sum_{k=0}^{n} k! T_\lambda(n,k) \quad (n \geq 0).$$

By (14), we have

$$\begin{aligned}\sum_{n=0}^{\infty} F_{n,\lambda}^{(C)}(x)\frac{t^n}{n!} &= \frac{1}{1-xe_\lambda^{-\frac{1}{2}}(t)(e_\lambda(t)-1)} \\ &= \sum_{k=0}^{\infty} x^k e_\lambda^{-\frac{k}{2}}(t)\big(e_\lambda(t)-1\big)^k \\ &= \sum_{k=0}^{\infty} x^k k! e_\lambda^{-\frac{k}{2}}(t)\frac{(e_\lambda(t)-1)^k}{k!} \\ &= \sum_{k=0}^{\infty} x^k k! \sum_{n=k}^{\infty} S_{2,\lambda}^{(\frac{-k}{2})}(n,k)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} x^k k! S_{2,\lambda}^{(\frac{-k}{2})}(n,k)\right)\frac{t^n}{n!}.\end{aligned} \tag{17}$$

Thus, we state the following theorem.

Theorem 2.2 *Let n be a nonnegative integer. Then the following relation holds true*:

$$F_{n,\lambda}^{(C)}(x) = \sum_{k=0}^{n} x^k k! S_{2,\lambda}^{(\frac{-k}{2})}(n,k).$$

By (9), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} E_{n,\lambda}^{*} \frac{t^n}{n!} &= \frac{2}{e_\lambda(t) + e_\lambda^{-1}(t)} \\
&= \frac{2}{2 + (e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t))^2} \\
&= \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k \left(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t)\right)^{2k} \\
&= \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k (2k)! \sum_{n=2k}^{\infty} T_\lambda(n,2k) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{[\frac{n}{2}]} \left(-\frac{1}{2}\right)^k (2k)! T_\lambda(n,2k) \right) \frac{t^n}{n!}.
\end{aligned}$$

Thus we arrive at the following theorem.

Theorem 2.3 *Let n be a nonnegative integer. Then the following relation between* $E_{n,\lambda}^{*}$ *and* $T_\lambda(n,2k)$ *holds true*:

$$E_{n,\lambda}^{*} = \sum_{k=0}^{[\frac{n}{2}]} \left(-\frac{1}{2}\right)^k (2k)! T_\lambda(n,2k).$$

The following computations based on (14) show that

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{n,\lambda}^{(C)}(x) \frac{t^n}{n!} &= \frac{1}{1 - x(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t))} \\
&= \sum_{k=0}^{\infty} x^k \left(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t)\right)^k \\
&= \sum_{k=0}^{\infty} x^k \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} e_\lambda^{(l-\frac{k}{2})}(t) \\
&= \sum_{k=0}^{\infty} x^k \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \sum_{n=0}^{\infty} \left(l - \frac{k}{2}\right)_{n,\lambda} \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} x^k \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left(l - \frac{k}{2}\right)_{n,\lambda} \right) \frac{t^n}{n!}.
\end{aligned}$$

Thus, we obtain the following theorem.

Theorem 2.4 *Let n be a nonnegative integer. Then the following explicit summation formula holds true*:

$$F_{n,\lambda}^{(C)}(x) = \sum_{k=0}^{\infty} x^k \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left(l - \frac{k}{2}\right)_{n,\lambda}.$$

3 On type 2 degenerate central Fubini polynomials by the fermionic p-adic integral on $\mathbb{Z}_p$

In this section, let us assume that $\lambda \in \mathbb{C}_p$ and $t \in \mathbb{C}_p$ with the condition $|\lambda t|_p < p^{-\frac{1}{p-1}}$. By (3) and (14), it becomes

$$\begin{aligned} \int_{\mathbb{Z}_p} \left(x\left(e_\lambda^{\frac{-1}{2}}(t) - e_\lambda^{\frac{1}{2}}(t)\right)\right)^y d\mu_{-1}(y) &= \frac{2}{1 - x(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t))} \\ &= 2\sum_{n=0}^{\infty} F_{n,\lambda}^{(C)}(x) \frac{t^n}{n!}. \end{aligned} \tag{18}$$

From (18), we have

$$\begin{aligned} 2\sum_{n=0}^{\infty} F_{n,\lambda}^{(C)}(x) \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} \left(x\left(e_\lambda^{\frac{-1}{2}}(t) - e_\lambda^{\frac{1}{2}}(t)\right)\right)^y d\mu_{-1}(y) \\ &= \int_{\mathbb{Z}_p} \left(-x\left(e_{-\lambda}^{\frac{-1}{2}}(-t) - e_{-\lambda}^{\frac{1}{2}}(-t)\right)\right)^y d\mu_{-1}(y) \\ &= 2\sum_{n=0}^{\infty} (-1)^n F_{n,-\lambda}^{(C)}(-x) \frac{t^n}{n!}. \end{aligned}$$

Thus, we get the following theorem.

Theorem 3.1 *Let n be a nonnegative integer. The following symmetric relation holds true*:

$$F_{n,\lambda}^{(C)}(x) = (-1)^n F_{n,-\lambda}^{(C)}(-x).$$

By (2) and (14), we have

$$\int_{\mathbb{Z}_p} \left(x\left(e_\lambda^{-\frac{1}{2}}(t) - e_\lambda^{\frac{1}{2}}(t)\right)\right)^{y+1} d\mu_{-1}(y) + \int_{\mathbb{Z}_p} \left(x\left(e_\lambda^{-\frac{1}{2}}(t) - e_\lambda^{\frac{1}{2}}(t)\right)\right)^y d\mu_{-1}(y) = 2. \tag{19}$$

By (19), we get

$$x\left(e_\lambda^{-\frac{1}{2}}(t) - e_\lambda^{\frac{1}{2}}(t)\right) \frac{2}{1 - x(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t))} + \frac{2}{1 - x(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t))} = 2. \tag{20}$$

It follows from (20) that

$$1 = x\left(\sum_{n=0}^{\infty}\left(\left(-\frac{1}{2}\right)_{n,\lambda} - \left(\frac{1}{2}\right)_{n,\lambda}\right)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} F_{n,\lambda}^{(C)}(x) \frac{t^n}{n!}\right) + \sum_{n=0}^{\infty} F_{n,\lambda}^{(C)}(x) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left[x\sum_{m=0}^{n}\binom{n}{m}F_{m,\lambda}^{(C)}(x)\left(\left(-\frac{1}{2}\right)_{n-m,\lambda}-\left(\frac{1}{2}\right)_{n-m,\lambda}\right)+F_{n,\lambda}^{(C)}(x)\right]\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left[F_{n,\lambda}^{(C)}(x)+x\sum_{m=0}^{n}\binom{n}{m}F_{m,\lambda}^{(C)}(x)\left(\left(-\frac{1}{2}\right)_{n-m,\lambda}-\left(\frac{1}{2}\right)_{n-m,\lambda}\right)\right]\frac{t^n}{n!}.$$

Thus we state the following theorem.

Theorem 3.2 *For $n > 0$, we have*

$$F_{n,\lambda}^{(C)}(x) = -x\sum_{m=0}^{n}\binom{n}{m}F_{m,\lambda}^{(C)}(x)\left(\left(-\frac{1}{2}\right)_{n-m,\lambda}-\left(\frac{1}{2}\right)_{n-m,\lambda}\right). \tag{21}$$

For $n \in \mathbb{N}$, by (21), we get

$$F_{n,\lambda}^{(C)}(x) = x\sum_{m=0}^{n}\binom{n}{m}F_{m,\lambda}^{(C)}(x)\left(\left(\frac{1}{2}\right)_{n-m,\lambda}-\left(-\frac{1}{2}\right)_{n-m,\lambda}\right)$$
$$= x\sum_{m=0}^{n-1}\binom{n}{m}F_{m,\lambda}^{(C)}(x)\left(\left(\frac{1}{2}\right)_{n-m,\lambda}-\left(-\frac{1}{2}\right)_{n-m,\lambda}\right)$$
$$= x\sum_{m=0}^{n-2}\binom{n}{m}F_{m,\lambda}^{(C)}(x)\left(\left(\frac{1}{2}\right)_{n-m,\lambda}-\left(-\frac{1}{2}\right)_{n-m,\lambda}\right)+xnF_{n-1,\lambda}^{(C)}(x).$$

Thus we get the following corollary.

Corollary 3.1 *Let n be a positive integer satisfying with $n \geq 2$. Then the following equation holds true*:

$$F_{n,\lambda}^{(C)}(x) - xnF_{n-1,\lambda}^{(C)}(x) = x\sum_{m=0}^{n-2}\binom{n}{m}F_{m,\lambda}^{(C)}(x)\left(\left(\frac{1}{2}\right)_{n-m,\lambda}-\left(-\frac{1}{2}\right)_{n-m,\lambda}\right).$$

4 On type 2 degenerate central Fubini polynomials of two variable

In this section, we assume that $\lambda \neq 0$ is any real number. We are now in a position to state the type 2 degenerate central Fubini polynomials of two variable as follows:

$$\sum_{n=0}^{\infty}F_{n,\lambda}^{(C)}(x;y)\frac{t^n}{n!} = \frac{1}{1-y(e_{\lambda}^{\frac{1}{2}}(t)-e_{\lambda}^{-\frac{1}{2}}(t))}e_{\lambda}^{x}(t). \tag{22}$$

When $x = 0$, $F_{n,\lambda}^{(C)}(0;y) = F_{n,\lambda}^{(C)}(y)$, $F_{n,\lambda}^{(C)}(0;1) = F_{n,\lambda}^{(C)}(1)$ are called the central Fubini polynomials and the central Fubini numbers, respectively.

Since

$$\lim_{\lambda\to 0}e_{\lambda}^{x}(t) = e^{xt} = \sum_{n=0}^{\infty}\frac{x^n t^n}{n!},$$

it is not difficult to show that

$$\lim_{\lambda\to 0}\sum_{n=0}^{\infty}F_{n,\lambda}^{(C)}(x;y)\frac{t^n}{n!} = \frac{1}{1-y(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}e^{xt},$$

which is the generating function of the central Fubini polynomials of two variables; see [18].

From (22), we have

$$\begin{aligned}\sum_{n=0}^{\infty} F_{n,\lambda}^{(C)}(x;y)\frac{t^n}{n!} &= \frac{1}{1-y(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))} e_\lambda^x(t) \\ &= \left(\sum_{k=0}^{\infty} F_{k,\lambda}^{(C)}(y)\frac{t^k}{k!}\right)\left(\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_{k,\lambda}^{(C)}(y)(x)_{n-k,\lambda}\right)\frac{t^n}{n!}.\end{aligned}$$

Thus, we obtain the following theorem.

Theorem 4.1 *Let n be a nonnegative integer. Then the following identity holds*:

$$F_{n,\lambda}^{(C)}(x;y) = \sum_{k=0}^{n}\binom{n}{k}F_{k,\lambda}^{(C)}(y)(x)_{n-k,\lambda}.$$

Changing t to $\frac{e^{\lambda t}-1}{\lambda}$ in (22) gives

$$\begin{aligned}\frac{1}{1-y(e^{\frac{t}{2}}-e^{-\frac{t}{2}})}e^{xt} &= \sum_{k=0}^{\infty} F_{k,\lambda}^{(C)}(x;y)\lambda^{-k}\frac{(e^{\lambda t}-1)^k}{k!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}F_{k,\lambda}^{(C)}(x;y)\lambda^{n-k}S_2(n,k)\right)\frac{t^n}{n!}.\end{aligned} \tag{23}$$

By (5) and (23), we have the following theorem.

Theorem 4.2 *Let n be a nonnegative integer. Then the following identity holds*:

$$F_{n}^{(C)}(x;y) = \sum_{k=0}^{n}\lambda^{n-k}F_{k,\lambda}^{(C)}(x;y)S_2(n,k).$$

By (22), we see that

$$\begin{aligned}e_\lambda^x(t) &= \left(1-y\left(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t)\right)\right)\sum_{n=0}^{\infty}F_{n,\lambda}^{(C)}(x;y)\frac{t^n}{n!} \\ &= \left(\sum_{n=0}^{\infty}F_{n,\lambda}^{(C)}(x;y)-\sum_{k=0}^{n}\binom{n}{k}\left(yF_{n-k,\lambda}^{(C)}(x;y)\left(\left(\frac{1}{2}\right)_{k,\lambda}-\left(-\frac{1}{2}\right)_{k,\lambda}\right)\right)\right)\frac{t^n}{n!}.\end{aligned} \tag{24}$$

By (8) and (24), we obtain the following theorem.

Theorem 4.3 *Let n be a nonnegative integer. The following formula holds true*:

$$(x)_{n,\lambda} = F_{n,\lambda}^{(C)}(x;y) - y\sum_{k=0}^{n}\binom{n}{k}F_{n-k,\lambda}^{(C)}(x;y)\left(\left(\frac{1}{2}\right)_{k,\lambda}-\left(-\frac{1}{2}\right)_{k,\lambda}\right).$$

Let us observe that

$$\begin{aligned}
&\frac{e_\lambda^{x_1}(t)}{1-y_1(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))}\frac{e_\lambda^{x_2}(t)}{1-y_2(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))} \\
&\quad=\frac{y_1}{y_1-y_2}\frac{e_\lambda^{x_1+x_2}(t)}{1-y_1(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))}-\frac{y_2}{y_1-y_2}\frac{e_\lambda^{x_1+x_2}(t)}{1-y_2(e_\lambda^{\frac{1}{2}}(t)-e_\lambda^{-\frac{1}{2}}(t))}.
\end{aligned}\tag{25}$$

By (22) and (25), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\binom{n}{k}F_{n-k,\lambda}^{(C)}(x_1;y_1)F_{k,\lambda}^{(C)}(x_2;y_2)\frac{t^n}{n!} \\
&\quad=\sum_{n=0}^{\infty}\left(\frac{y_1F_{n,\lambda}^{(C)}(x_1+x_2;y_1)-y_2F_{n,\lambda}^{(C)}(x_1+x_2;y_2)}{y_1-y_2}\right)\frac{t^n}{n!}.
\end{aligned}\tag{26}$$

Therefore, by (26), we obtain the following theorem.

Theorem 4.4 *Let n be a nonnegative integer with $y_1 \neq y_2$. The following formula holds true*:

$$\sum_{k=0}^{n}\binom{n}{k}F_{n-k,\lambda}^{(C)}(x_1;y_1)F_{k,\lambda}^{(C)}(x_2;y_2)=\frac{y_1F_{n,\lambda}^{(C)}(x_1+x_2;y_1)-y_2F_{n,\lambda}^{(C)}(x_1+x_2;y_2)}{y_1-y_2}.$$

Remark 4.1 Taking $x_1 = x_2 = 0$ in Theorem 4.4 reduces to

$$\sum_{k=0}^{n}\binom{n}{k}F_{n-k,\lambda}^{(C)}(y_1)F_{k,\lambda}^{(C)}(y_2)=\frac{y_1F_{n,\lambda}^{(C)}(y_1)-y_2F_{n,\lambda}^{(C)}(y_2)}{y_1-y_2}.$$

5 Conclusion

In the present paper, we have considered type 2 degenerate central Fubini and type 2 degenerate central Fubini polynomials of two variables. We investigated some properties, identities and recurrence relations for these polynomials by making use of generating functions and p-adic fermionic integrals on $\mathbb{Z}_p$. In addition, we have obtained some results related to degenerate central factorial numbers of the second kind and degenerate Euler numbers of the second kind.

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Author details
[1]College of Computer and Information Sciences, Majmaah University, Majmaah 11952, Saudi Arabia. [2]Department of Mathematics and Natural Sciences, Prince Mohammad Bin Fahd University, P.O. Box: 1664, Al Khobar 31952, Saudi Arabia. [3]Department of Economics, Faculty of Economics, Administrative and Social Sciences, Hasan Kalyoncu University, TR-27410 Gaziantep, Turkey. [4]Department of Civil and Environmental Engineering, College of Engineering, Majmaah University, Majmaah 11952, Saudi Arabia.

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