New Developments in $d = 4$, $\mathcal{N} = 2$ Superconformal Field Theories

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Recent developments in the understanding of the dynamics of $\mathcal{N} = 2$ superconformal field theory in four dimensions are reviewed.

§1. Introduction

As we all know and love, $\mathcal{N} = 2$ supersymmetric gauge theories have proved to be a most effective playground to better understand various non-perturbative dynamics of strongly-coupled gauge theory, since the seminal work pioneered by Seiberg and Witten.[1),2)] One of the curious byproducts of this line of research was the discovery of a new class of $\mathcal{N} = 2$ superconformal field theories (SCFTs), first by Argyres and Douglas[3)] in 1995 and later elaborated by many groups, Argyres-Plesser-Seiberg-Witten,[4)] Eguchi-Hori-Ito-Yang,[5),6)] and Minahan-Nemeschansky,[7),8)] among others. These SCFTs are characterized by simultaneous existence of mutually non-local massless states. The very fact that the scaling dimensions of operators of such a mysterious system could be determined highlights the power of the approach of Seiberg and Witten, but relatively little has been uncovered for about ten years since then.

Things started to change in the autumn of 2007 by the appearance of the work[9)] by Argyres and Seiberg, in which a new type of S-duality between $\mathcal{N} = 2$ SCFTs was found. It enabled the prediction of the conformal central charges a and c of the $\mathcal{N} = 2$ SCFTs with flavor symmetry $E_{6,7}$. This work was remarkable in that it was the very first determination of the central charges of Argyres-Douglas-type SCFTs. A holographic method to obtain the central charges was soon developed by Aharony and the author,[10)] and then a general method applicable to any $\mathcal{N} = 2$ SCFTs was devised by Shapere and the author[11)] based on the topological twisting and holomorphy. The prediction from the S-duality was beautifully confirmed, and moreover for the $USp(2N)$ gauge theory with N_f fundamental quarks and one antisymmetric hypermultiplet, the central charges were calculated both via the holography,[10)] and via the topological twisting[11)] with perfect agreement. Here we would like to review these recent developments.

§2. Basics of $\mathcal{N} = 2$ SCFTs

Let us start by recalling how the superconformal point can be reached in the $SU(2)$ gauge theory with $N_f = 1$ quark of bare mass m.[4)] The moduli space of vacua is parametrized by $u = \langle \mathrm{tr}\, \phi^2 \rangle$ constructed from the complex adjoint scalar field ϕ, and there are three special values $u = u_{1,2,3}$ where a quark, a monopole or a dyon becomes respectively massless. $u_{1,2,3}$ depends on the bare mass m, which can be carefully chosen so that $u_1 = u_2$. When we set the moduli u to be $u_1 = u_2$, one finds that there are both massless quarks and massless monopoles. The Seiberg-Witten curve close to this point is given by

$$y^2 = x^3 - 2(\delta m)x - \delta u \tag{2·1}$$

with the Seiberg-Witten differential

$$\lambda_{SW} \sim (\delta u)dx/y. \tag{2·2}$$

The scaling of the curve determines the ratio of the dimensions of operators to be

$$D(x) : D(y) : D(\delta m) : D(\delta u) = 2 : 3 : 4 : 6. \tag{2·3}$$

Now the dimension of the differential $D(\lambda)$ should be one because it gives the scaling of the masses of BPS states. Thus we know

$$D(\delta m) = 4/5, \qquad D(\delta u) = 6/5. \tag{2·4}$$

The number of $U(1)$ fields which couples to mutually non-local states is called the rank of the theory. Thus the rank of the SCFT point reviewed above is one.

This construction was soon generalized to the $\mathcal{N}=2$ supersymmetric QCD.[5),6)] It was soon noticed also that the rank 1 SCFTs including the example just discussed can be realized in terms of a D3-brane probing an F-theory 7-brane, and it led to the discovery of the rank 1 SCFTs with flavor symmetry $E_{6,7,8}$.[7),8)] The scaling dimensions of operators were determined in the original papers, and various other properties were soon studied: e.g. behavior of the beta functions near SCFT points,[12),13)] more examples using M/F-theory,[14)] detailed construction of the mass deformations,[15)] and the BPS spectrum.[16),17)]

The subject lay dormant for a while, apart from sporadic works e.g. Refs. 18) and 19) by staunch adherents, despite a great advance in the understanding of $\mathcal{N}=1$ SCFTs initiated by Ref. 20). To understand the situation let us review how the central charges can be calculated.

In two dimensions, the conformal central charge c is proportional to the trace anomaly in a curved background

$$\langle T^\mu_\mu \rangle = -\frac{c}{12} R. \tag{2·5}$$

Similarly, in 4D conformal field theories the trace anomaly depends on two constants, a and c:

$$\langle T^\mu_\mu \rangle = \frac{c}{16\pi^2}(\mathrm{Weyl})^2 - \frac{a}{16\pi^2}(\mathrm{Euler})\ , \tag{2·6}$$

where Weyl is the Weyl tensor and Euler stands for the Euler density. For superconformal theories, the supersymmetry relates the definition above to the anomalous conservation of the R-current, which is given by

$$\partial_\mu R^\mu_{\mathcal{N}=1} = \frac{c-a}{24\pi^2} R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} + \frac{5a-3c}{9\pi^2} F^{\mathcal{N}=1}_{\mu\nu}\tilde{F}^{\mu\nu}_{\mathcal{N}=1} \tag{2·7}$$

for $\mathcal{N}=1$ SCFTs.[21] Here, $F^{\mathcal{N}=1}_{\mu\nu}$ is the field strength of the external gauge field coupling to the $U(1)_{R,\mathcal{N}=1}$ current. This relation allows us to calculate a and c when the $U(1)_R$ current can be uniquely identified in the ultraviolet, with the help of 't Hooft's anomaly matching. When there are extra $U(1)$'s in the ultraviolet which can mix with the $U(1)_R$ symmetry, the a-maximization procedure can be used to fix which linear combination of $U(1)$ symmetries becomes the infrared superconformal R-symmetry,[20] which ignited a great revolution in the study of the $\mathcal{N}=1$ SCFTs. Unfortunately this method is not applicable for most of the $\mathcal{N}=2$ SCFTs, because usually there is no continuous $U(1)$ symmetry at all in the ultraviolet.

§3. New *S*-duality

The breakthrough came from an unexpected direction which was the study of the weak-strong coupling duality of another class of $\mathcal{N}=2$ SCFTs. Consider $SU(N_c)$ gauge theory with $N_f=2N_c$ massless quarks. The one-loop beta function can be easily seen to vanish. Supersymmetry then guarantees that the theory is conformal to all loop order. Study of the Seiberg-Witten curves[22),23] indeed suggests that these theories are conformal even non-perturbatively, and the theory comes in a one-parameter family labeled by the complexified gauge coupling

$$\tau=\frac{\theta}{\pi}+\frac{8\pi i}{g^2}. \tag{3·1}$$

For $N_c=2$ and $N_f=4$, it was uncovered in one of the original papers of Seiberg and Witten[2] that there is a duality group sending

$$\tau\to\tau+1,\qquad \tau\to-\frac{1}{\tau}, \tag{3·2}$$

which is the well-known action of $SL(2,\mathbb{Z})$. The existence of the symmetry $\tau\to\tau+1$ rests on the fact there is no distinction of the doublet and the anti-doublet of $SU(2)$. For $N_c\geq 3$, the duality group becomes instead

$$\tau\to\tau+2,\qquad \tau\to-\frac{1}{\tau}. \tag{3·3}$$

The fundamental regions of two duality group actions are depicted in Fig. 1. The nontrivial strong-weak duality $\tau\to-1/\tau$ exists for $N_c\geq 3$, but it is not enough to eliminate the infinitely strongly coupled point $\tau\to 1$ from the fundamental region.

The main objective of the work[9] was to understand the physics close to the infinitely strongly-coupled point for $N_c=3$. Their proposal is the S-dual description which is an $SU(2)$ gauge theory with $N_f=1$ quark at a weak coupling $\tau'=1/(1-\tau)$, which is also coupled to the strongly-coupled rank-1 SCFT with flavor symmetry E_6. We denote this duality schematically as

$$SU(3)\text{ with }6\times(\mathbf{3}+\bar{\mathbf{3}})\longleftrightarrow SU(2)\text{ with }2\times\mathbf{2}\text{ and }\mathrm{SCFT}_{E_6}. \tag{3·4}$$

The best evidence for this duality comes from the detailed study of the Seiberg-

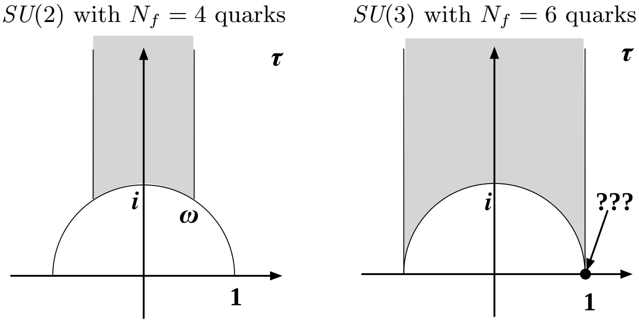

Fig. 1. Fundamental region of the coupling constant τ.

Witten curves.[9),*)] Here we present two simple consistency checks.

First is the matching of the global symmetry. The flavor symmetry on the LHS is the unitary group which rotates 6 flavors, $U(6) \sim U(1) \times SU(6)$. On the RHS, there is an $O(2)$ symmetry which rotates two doublets. The E_6 flavor symmetry at first seems too big to match the $SU(6)$ part, but the duality posits that a subgroup $SU(2) \subset E_6$ is gauged by the gauge group $SU(2)$. Therefore, the flavor symmetry which remains is the maximal commuting subgroup which is $SU(6)$. Thus the global symmetries match.

Second is the matching of the central charge of the global symmetry just determined. The conformal symmetry restricts the two-point functions of conserved currents to be of the form

$$\langle J^a_\mu(x) J^b_\nu(0)\rangle = \frac{3k_G}{4\pi^4}\delta^{ab}\frac{x^2 g_{\mu\nu} - 2x_\mu x_\nu}{x^8} + \cdots . \tag{3·5}$$

k_G is normalized[9)] so that a hypermultiplet in the fundamental representation of $SU(N)$ contributes 2 to it.

If the duality is true, the central charges of two subgroups $SU(2) \times SU(6)$ of E_6 should be the same. The $SU(6)$ central charge can be determined on the LHS of the duality, where we can tune τ to go to the free field limit. There, we have three hypermultiplet in the fundamental of $SU(6)$, which gives $k_{SU(6)} = 6$. The $SU(2)$ central charge can be read off from the RHS, by assuming the $SU(2)$ gauge group factor is conformal. When $SU(2)$ is weakly coupled, the contribution to the one-loop beta function is given by the current two point functions (3·5). We know that the vector multiplet contribution is canceled by $\langle J^a_\mu J^b_\mu\rangle$ from $N_f = 4$ quarks, which have $k = 8$. In this case we only have $N_f = 1$ quarks, which contribute 2 to k. Thus we conclude the coupling of the $SU(2)$ gauge multiplet to the SCFT of type E_6 should contribute $k_{SU(2)} = 8 - 2 = 6$. Therefore, having $k_{E_6} = 6$ explains both the $SU(6)$ flavor central charge and the finiteness of the dual $SU(2)$ gauge group simultaneously. This can also be thought of as the prediction of the E_6 symmetry central charge of this mysterious theory.

*) Recently the agreement of the Higgs branches of both sides of the duality was demonstrated in Ref. 40), which also provided a non-trivial check.

Table I. Central charges calculated from the new S-duality.

G	D_4	E_6	E_7	E_8
k_G	4	6	8	12
$24a$	23	41	59	95
$6c$	7	13	19	31

Assuming that this duality is correct, we can compute a and c of the E_6 SCFT by calculating those of the $SU(3)$ gauge theory and subtracting those of the $SU(2)$ gauge theory (including the hypermultiplet contributions) on the right hand side, because we can tune the coupling constant τ or τ' in either side to the weak coupling region. The contribution of a free hypermultiplet is

$$a = 1/24, \qquad c = 1/12 \ , \tag{3·6}$$

while for a vector multiplet it is

$$a = 5/24, \qquad c = 1/6. \tag{3·7}$$

To get a and c of the E_6 theory, we recall that on the LHS we have 8 vectors and 18 hypers, while we have 3 vectors and 2 hypers on the RHS in addition to the E_6 SCFT. Thus we find

$$a = 41/24, \qquad c = 13/6. \tag{3·8}$$

Another duality was discussed in the work,[9] which is schematically

$$USp(4) \text{ with } 12 \times \mathbf{4} \longleftrightarrow SU(2) \text{ with } \mathrm{SCFT}_{E_7}. \tag{3·9}$$

The global symmetry of both sides is $SO(12)$, and the flavor central charge is seen to be $k = 8$. a and c can be obtained just as before. Later, the rank one isolated SCFT with E_8 flavor symmetry was found[24] as a sector of the S-dual of rank three perturbative gauge theories, e.g.

$$USp(6) \text{ with } \mathbf{14} + 11 \times \mathbf{6} \ \longleftrightarrow \ USp(4) \text{ with } \mathrm{SCFT}_{E_8} \ . \tag{3·10}$$

Again, the central charges a, c, k_G can be calculated in a similar manner.[*]

One amazing aspect of these new dualities is that they are the first purely field-theoretical construction of these strange SCFTs with flavor symmetry $E_{6,7,8}$, which was only known from F-theoretic construction.[7,8] From that perspective, the $SU(2)$ gauge theory with $N_f = 4$ quarks whose global symmetry group is $SO(8) = D_4$ naturally sits in the family. The central charges of these theories are tabulated in Table I for convenience.

[*] The author recently learned that $k_{E_8} = 12$ was already calculated in 1997 in an almost neglected work by Cheung, Ganor and Krogh,[25] both from field theory and from string theory. Their methods are not applicable to a and c, but as for k_G theirs are of the same flavor as the ones explained below in §§4 and 5.

Table II. Properties of F-theory singularities.

G	A_0	A_1	A_2	D_4	E_6	E_7	E_8
n_7	2	3	4	6	8	9	10
D	6/5	4/3	3/2	2	3	4	6

§4. Holographic realization

4.1. *F-theory construction*

The SCFTs for which the central charges were predicted from the S-duality is exactly the one which can be realized using F-theory. In this section we describe[10)] how we can use string theoretical methods to obtain the central charges, thus providing a cross-check of the duality.

A rank-1 SCFT can be realized in F-theory by placing a D3-brane close to a special type of 7-branes, close to which the dilaton is constant so that the gauge coupling of the D3-brane is constant. Such special 7-branes were found[26)] using the Kodaira classification, and are tabulated in Table II. There, G stands for the low-energy gauge symmetry living on the singularity, and n_7 is the tension of the 7-brane measured against that of a D7-brane. As codimension-2 objects, they produce conical singularities in the transverse space, with the total deficit angle proportional to the number of 7-branes n_7. For convenience we parametrize the deficit angle by the change in the periodicity of the angular coordinate around the 7-brane,

$$2\pi \to 2\pi/D. \tag{4·1}$$

D is related to the tension of the 7-brane by

$$D = \frac{12}{12 - n_7} \ . \tag{4·2}$$

Note that twelve (p, q) 7-branes produce a deficit angle of 2π, closing up the space, while the D_4 singularity may be viewed as an orientifold (reflecting the transverse space) together with four D7-branes, so it has a deficit angle of π.

Now let us introduce N D3-branes on top of the 7-brane. The flavor symmetry of the theory comes from the gauge symmetry G of the 7-brane. If we put the theory at the origin of the Coulomb branch, i.e. if we put all of the D3-branes together on top of the singularity, the theory becomes conformal. One can show that the lowest dimension operator u which parameterizes the position on the Coulomb branch has dimension $D(u) = D$, which explains our usage of the symbol D for the deficit angle. There is also a Higgs branch emanating from the origin of the Coulomb branch, which will not be relevant for our discussion here.

The only singularity where the value of the dilaton can be arbitrarily tuned is the D_4 singularity, which in the perturbative region may be viewed as a system of four D7-branes on top of an $O7^-$ orientifold plane thus cancelling the dilaton tadpole. The gauge theory on N D3-branes is an $\mathcal{N} = 2$ $USp(2N)$ gauge theory with four hypermultiplets in the fundamental representation, and one hypermultiplet in the antisymmetric tensor. The hypermultiplets in the fundamental transform as the fundamental of the $D_4 = SO(8)$ flavor symmetry. The trace part of the antisymmetric tensor corresponds to the overall motion of the D3-branes parallel to the 7-brane, which is completely decoupled from the rest of the theory.

The monodromy around other types of 7-branes fixes the dilaton, so these systems are inherently strongly coupled. The theories on one D3-brane near the 7-brane of type A_n correspond to the $\mathcal{N} = 2$ SCFTs found in the works,[3),4)] which were briefly reviewed in §2.

Holographic dual description in the large N limit can be obtained by taking the near-horizon limit of N D3-branes sitting on the 7-brane.[27),28)] This gives type IIB string theory on $AdS_5 \times X_D$, with N units of F_5 flux. Here X_D is the round 5-sphere

$$\{|x|^2 + |y|^2 + |z|^2 = \text{constant}\} \subset \mathbb{C}^3, \tag{4·3}$$

with the phase of z restricted to $[0, 2\pi/D]$ and periodically identified. The 7-brane of type G wraps the locus $z = 0$, which is a round S^3. Massless gauge fields on AdS_5 arise both from the isometries of X_D, and from the gauge fields on the 7-brane. Now, the central charges can be expressed as a linear combination of 't Hooft anomaly coefficients as in (2·7), which in turn can be translated to the Chern-Simons interaction in the AdS_5 side. Therefore, we need to evaluate the Chern-Simons terms in the gravity side to obtain a and c and k_G. We discuss the $\mathcal{O}(N^2)$, $\mathcal{O}(N)$ and $\mathcal{O}(1)$ contributions in this order, and denote the $\mathcal{O}(N^2)$ contribution to a by $a^{(2)}$, etc.

4.2. $\mathcal{O}(N^2)$ *contributions*

The $\mathcal{O}(N^2)$ contributions to the anomalies come only from the gravity in the bulk, since the action of the 7-brane is of order N.[29)] The conformal anomalies a and c were determined[30)] to be

$$a = c = \frac{N^2\pi^3}{4\,\mathrm{vol}(X_5)} \tag{4·4}$$

for general Einstein manifolds X_5, where $\mathrm{vol}(X_5)$ is the volume of X_5, normalized to have a unit radius of curvature. In our case the volume of X_D is that of a 5-sphere divided by D, so we get

$$a^{(2)} = c^{(2)} = \frac{N^2 D}{4} \tag{4·5}$$

to this order. There is obviously no bulk contribution to the Chern-Simons term of the G flavor symmetry, which lives only on the 7-brane. In the D_4 case it is easy to compute all of these central charges in the free field theory limit, leading to the same results at leading order in $1/N$.

4.3. $\mathcal{O}(N)$ *contributions*

In the bulk there are no $\mathcal{O}(N)$ contributions to the central charges and anomalies, since the one-loop corrections in the bulk are $\mathcal{O}(1)$. Thus, the $\mathcal{O}(N)$ contributions to the anomalies come purely from the Chern-Simons interactions on the 7-brane at the singularity, which include terms of the form $C_4 \wedge \mathrm{tr}(R \wedge R)$ and $C_4 \wedge \mathrm{tr}(F \wedge F)$. The dimensional reduction of these terms gives rise to five-dimensional Chern-Simons interactions, since the five dimensional gauge fields involving the isometries include, in addition to the ten dimensional metric, a contribution of the form $C_4 \sim A_R \wedge \omega$ where A_R is the $U(1)_R$ gauge potential on AdS_5 and ω is the volume form of the 3-cycle wrapped by the singularity.

To determine the terms on the 7-brane, let us first recall that the 7-brane of type D_4 can be realized perturbatively as 4 D7-branes put on top of the O7$^-$-plane. The Chern-Simons coupling on the worldvolume to the four-form field C_4 is known.[29] The other types of 7-branes have the dilaton pinned down to the strong coupling region, but their Chern-Simons terms are related to the anomaly inflow and can still be reliably determined. Each constituent (p, q) 7-brane carries the same coupling to $C_4 \wedge \mathrm{tr}(R \wedge R)$ (recall that all (p, q) 7-branes are related by the $SL(2, \mathbb{Z})$ duality symmetry which leaves C_4 invariant), so when we bring n_7 7-branes together the strength of that term is proportional to n_7. As for the coupling $C_4 \wedge \mathrm{tr}(F \wedge F)$, the gauge symmetries on the worldvolume for various types of 7-branes are related by the removal of (p, q) 7-branes one by one which enables flows between the different theories, with a natural embedding of the (simply laced) symmetries

$$A_1 \subset A_2 \subset D_4 \subset E_6 \subset E_7 \subset E_8. \tag{4·6}$$

Therefore, the strength of the coupling does not depend on the type of the 7-brane, as long as we use the same normalization of the root vectors. We conclude that the Chern-Simons terms on the 7-brane worldvolume are of the form

$$\alpha\, n_7 \int C_4 \wedge [\mathrm{tr}(R_T \wedge R_T) - \mathrm{tr}(R_N \wedge R_N)] + \beta \int C_4 \wedge \mathrm{tr}(F \wedge F), \tag{4·7}$$

with constants α and β independent of the type of the 7-brane. Here $R_{T,N}$ are the curvature of the tangent bundle and the normal bundle of the 7-brane, and we normalize the trace of the flavor symmetry so that $\mathrm{tr}(T^a T^b) = \delta^{ab}/2$ independent of the group.

These terms reduce to various Chern-Simons interactions in five dimensions after the integral over the S^3 wrapped by the 7-brane. The $C_4 \wedge \mathrm{tr}(F \wedge F)$ term gives rise to the $U(1)_R G^2$ Chern-Simons term on AdS_5, while the $C_4 \wedge \mathrm{tr}(R_T \wedge R_T)$ term in (4·7) produces $U(1)_R^3$ and $U(1)_R$-gravity-gravity Chern-Simons terms. Finally, the $C_4 \wedge \mathrm{tr}(R_N \wedge R_N)$ term gives $U(1)_R SU(2)_R^2$. Together, the $C_4 \wedge \mathrm{tr}(R \wedge R)$ terms contribute to a and c.

Again, we can easily determine the scaling with N and with the volume of X_D of the terms in the five dimensional action arising from the integration of (4·7) on the 3-sphere and involving the $U(1)_R$ field A_R. On dimensional grounds, the full low-energy 7-brane Lagrangian density is proportional to $R_{AdS}^4 \sim N/\,\mathrm{vol}(X_5)$. On the other hand, the volume of the 3-sphere which the 7-brane wraps is independent of the deficit angle D. Thus, the coefficients of the five dimensional terms we obtain from (4·7) scale as $N/\,\mathrm{vol}(X_5) \propto ND$.

Therefore, we find that the $\mathcal{O}(N)$ correction to the various anomalies scales as

$$k_G^{(1)} \propto ND, \qquad a^{(1)}, c^{(1)} \propto N n_7 D. \tag{4·8}$$

The coefficients can be fixed by a careful computation, or by comparing them to the perturbative case D_4 from the known spectrum of the gauge theory. Thus we obtain the $\mathcal{O}(N)$ contributions

$$k_G^{(1)} = 2ND, \qquad a^{(1)} = \frac{Nn_7D}{24}, \quad c^{(1)} = \frac{Nn_7D}{16}. \tag{4·9}$$

4.4. $\mathcal{O}(1)$ *contributions*

Relatively little is known about the $\mathcal{O}(1)$ corrections to the anomaly coefficients. For the $\mathcal{N}=4$ $SU(N)$ super Yang-Mills, Kaluza-Klein analysis in the dual $AdS_5 \times S^5$ background leads to a and c both proportional to N^2. The $\mathcal{O}(1)$ correction then accounts for the difference between the gauge group $U(N)$ and $SU(N)$. On the stack of N D3-branes in a flat space, we naturally have a $U(N)$ gauge symmetry, where the $U(1)$ part describes the center-of-mass motion of the D3-branes in the transverse $\mathbb{R}^6$. This overall motion decouples from the rest of the dynamics in the near-horizon limit, and it is not the part of the theory dual to type IIB string theory on $AdS_5 \times S^5$. This effect accounts for the extra (-1) in a and c.

Generalization of this effect to our setup is immediate. Here, the degrees of freedom which decouple in the near-horizon limit correspond to the overall motion parallel to the 7-brane, which is described by a free hypermultiplet neutral under the flavor symmetry. Then the $\mathcal{O}(1)$ contributions in this case should be precisely minus those of this free hypermultiplet, namely

$$k_G^{(0)} = 0, \quad a^{(0)} = -1/24, \quad c^{(0)} = -1/12, \tag{4·10}$$

independently of the type of the 7-brane. This precisely agrees with the known result in the D_4 case, and we will see more evidence in the next section that this procedure is correct.

4.5. *Summary*

Combining the results obtained so far, we obtain our final equations

$$\begin{aligned} k_G &= 2ND, \\ a &= \frac{1}{4}N^2D + \frac{1}{2}N(D-1) - \frac{1}{24}, \\ c &= \frac{1}{4}N^2D + \frac{3}{4}N(D-1) - \frac{1}{12} \end{aligned} \tag{4·11}$$

for the central charges. Here, we used the relation (4·2) which relates the deficit angle and the number of 7-branes to rewrite $n_7D = 12(D-1)$.

Based on the arguments above we believe that these formulas are exact, so we can use them even in the case of $N=1$. k_G, a and c which result from this calculation are tabulated in Table III.

We find perfect agreement for $G = E_{6,7,8}$ by comparing it against Table I, which gives us important consistency checks of both the new S-duality and the holographic calculation. Chronologically, the last column of Table III was a prediction from holography when the work[10)] was published, which was later confirmed by the field theoretical analysis.[24)] This exemplifies the power of string theory to understand field theoretical phenomena.

§5. Holomorphy and the central charges

5.1. *General methods*

We have reviewed two calculations of central charges, one based on the S-duality and another based on holography, neither of which is applicable generically. Here we review how central charges can be calculated for arbitrary $\mathcal{N} = 2$ SCFTs.[11)]

We begin by recalling the relation of a and c to the $\mathrm{U}(1)_R$-current in any $\mathcal{N} = 2$ field theory:[31)]

$$\partial_\mu R^\mu_{\mathcal{N}=2} = \frac{c-a}{8\pi^2} R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} + \frac{2a-c}{8\pi^2} F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a \tag{5·1}$$

in the presence of a background metric and a background $SU(2)_R$ gauge field $F^a_{\mu\nu}$.

In backgrounds where the $SU(2)_R$ gauge field is equal to the self-dual part of the curvature

$$F^a_{\mu\nu} t^a_{\rho\sigma} = \frac{1}{2}(R_{\mu\nu\rho\sigma} + \tilde{R}_{\mu\nu\rho\sigma}) \tag{5·2}$$

correlation functions of physical operators depend only on the topology of the background manifold. Substituting this condition into (5·1) and integrating over the 4-manifold with the Euler characteristic χ and the signature σ, we find that the total R-charge of the vacuum is given by

$$\Delta R = 2(2a - c)\chi + 3c\,\sigma. \tag{5·3}$$

Thus to determine a and c, it suffices to be able to compute the dependence of ΔR on the topology of the background. This information is encoded in the path integral measure, which for a topological gauge theory takes the form

$$[d\mu]A^\chi B^\sigma. \tag{5·4}$$

The factor $[d\mu]$ is the measure for the r vector multiplets, which at a generic point in moduli space are the only massless modes. The measure factors A and B depend holomorphically on the Coulomb branch moduli and are associated with the additional massless states that appear on special loci of complex codimension 1 and higher.

The R-charge of the vacuum can then be directly read off from the measure (5·4)

$$\Delta R = \chi R(A) + \sigma R(B) + \frac{\chi + \sigma}{2} r + \frac{\sigma}{4} h. \tag{5·5}$$

Table III. Central charges of rank one SCFTs calculated from holography.

G	A_0	A_1	A_2	D_4	E_6	E_7	E_8
k_G	12/5	8/3	3	4	6	8	12
$24a$	43/5	11	14	23	41	59	95
$6c$	11/5	3	4	7	13	19	31

Here $R(A)$ and $R(B)$ denote the R charges of A and B. The last two terms are the contributions from r free massless vector multiplets and h free massless hypermultiplets which exists at generic points of the moduli space. Comparing with (5·3), we find the following general expressions for a and c:

$$a = \frac{1}{4}R(A) + \frac{1}{6}R(B) + \frac{5}{24}r + \frac{1}{24}h, \qquad c = \frac{1}{3}R(B) + \frac{1}{6}r + \frac{1}{12}h\ . \tag{5·6}$$

Thus, finding a and c is reduced to calculating the R-charge of the functions $A(u)$ and $B(u)$. In general, these functions are believed to take the form[32)–36)]

$$A = \alpha \left[\det \frac{\partial u_i}{\partial a^I}\right]^{1/2}, \qquad B = \beta \Delta^{1/8} \tag{5·7}$$

for generic gauge theories. Here, u_i are gauge- and monodromy-invariant coordinates on the Coulomb branch, a^I are special coordinates, and Δ is the physical discriminant of the Seiberg-Witten curve. α and β are prefactors independent of the u_i which can in principle depend on the mass parameters. The functions in (5·7) are readily computable in the vicinity of many superconformal points of $\mathcal{N} = 2$ gauge theories.

5.2. *Application*

Let us apply this method to the theories we studied using holography in §4.1. We start by placing an O7-plane and N_f parallel D7-branes in a flat 10d spacetime. If probed by a single D3-brane, the gauge group on it is $SU(2)$ and there are N_f hypermultiplets in the doublet of the gauge group. The geometry transverse to the D7-branes can be identified with the u-plane,[37)] where as usual u is a gauge and modular invariant complex coordinate on the Coulomb branch of the gauge theory, identified with $\langle \mathrm{tr}\, \phi^2 \rangle$ in the $|u| \to \infty$ limit, and the relative positions of the D7-branes determine the mass parameters of the hypermultiplets.

When we probe the 7-brane system by N D3-branes, the quantization of open strings gives us the gauge group $USp(2N)$ with N_f massive hypermultiplets in the fundamental $2N$-dimensional representation and one massless hypermultiplet in the antisymmetric tensor. The Coulomb branch can be parameterized by the locations of D3-branes in the u-plane, u_i $(i = 1, \dots, N)$. The D3-branes are indistinguishable, so the coordinates u_i are identified under interchanges $u_i \leftrightarrow u_j$ for each pair $i < j$.[38)]

For $N_f = 4$, this is the D_4 theory in §4.1. For $N_f < 4$, the O7-plane splits into two 7-branes nonperturbatively, which correspond to the monopole and the dyon points of the u-plane. The 7-brane of type A_{N_f-1} in §4.1 is realized by placing N_f 7-branes on top of the monopole point. Thus the tension is given by $n_7 = N_f + 1$. The dimension $D(u)$ of the Coulomb branch operator is then related to the deficit angle (4·2). Probing with multiple D3-branes gives rank-N versions of these SCFTs.

$R(A)$ and $R(B)$ can be extracted by analyzing the singularities in moduli space. First let us recall the Coulomb branch of the theory more fully, for the case $N = 1$. For generic hypermultiplet masses m_a $(a = 1, \dots, N_f)$, there are $2 + N_f$ singularities on the u-plane at $u = u_\alpha(m_1, \dots, m_{N_f})$, $(\alpha = 1, 2, \dots, 2 + N_f)$, which are given by zeros of the discriminant

$$\Delta \equiv \Delta_1(u; m_1, \dots, m_{N_f}). \tag{5·8}$$

As such, the functions $u_\alpha(m_1, \ldots, m_{N_f})$ have monodromies exchanging them, and there are no absolute distinctions among them. Still, if $m_a \gg \Lambda$, there are two zeros with $u \sim \mathcal{O}(\Lambda)$ and N_f zeros at $u \sim m_a^2$. The former are the points where a monopole or a dyon becomes massless, and the latter are where the quarks become massless. For $N > 1$ the Coulomb branch is parameterized by $u_1, \ldots, u_N$, identified under the exchanges $u_i \leftrightarrow u_j$. Monodromy invariant coordinates are given by the k-th symmetric polynomials $u^{(k)}$ of the u_i, in correspondence with the Casimirs $\langle \operatorname{tr} \phi^{2k} \rangle$ of the $USp(2N)$ gauge group.

The factor A is then given by

$$A = \left[\det \frac{\partial u^{(k)}}{\partial a_i} \right]^{1/2} . \tag{5·9}$$

The physical discriminant is

$$\Delta = \prod_{i>j} (u_i - u_j)^6 \prod_{i,\alpha} (u_i - u_\alpha(m_1, \ldots, m_{N_f})) \tag{5·10}$$

$$\equiv \prod_{i>j} (u_i - u_j)^6 \prod_i \Delta_1(u_i; m_1, \ldots, m_{N_f}) , \tag{5·11}$$

where the first factor accounts for the enhancement of a single $U(1)$ vector multiplet to an $\mathcal{N} = 4$ $SU(2)$ multiplet when $u_i = u_j$, i.e., when two D3-branes collide, and the second factor accounts for the appearance of one massless hypermultiplet when $u_i = u_a(m_1, \ldots, m_{N_f})$. It is also easy to see that Δ is, as required, a polynomial in the gauge invariant coordinates $u^{(k)}$ and the masses m_i.

A superconformal point is reached if we tune $m_1, \ldots, m_{N_f}$ so that the N_f "quark" zeros of the discriminant collide with the monopole zero. We shift u by a constant so that the multiple zero is at $u = 0$. Then the discriminant becomes

$$\Delta = \prod_{i>j} (u_i - u_j)^6 \prod_i {u_i}^{1+N_f} . \tag{5·12}$$

Therefore

$$R(\Delta) = 2D(u) \left[(1 + N_f)N + 3N(N-1) \right] . \tag{5·13}$$

$R(A)$ is also easy to determine, because $D(u^{(k)}) = kD(u)$ and all of the a_i behave as dimension-1 operators. Thus we have

$$R(A) = \sum_k (kD(u) - 1) = \frac{1}{2} N(N+1)D(u) - N. \tag{5·14}$$

Finally we need the number r of free vector multiplets and the number h of free hypermultiplets at generic points of the moduli space, which are easily found to be $r = N$ and $h = N - 1$. Combining the data, we have

$$a = \frac{1}{4} DN^2 + \frac{1}{24}(1 + N_f)DN - \frac{1}{24}, \tag{5·15}$$

$$c = \frac{1}{4} DN^2 + \frac{1}{12} \left[(N_f - 2)D + 3 \right] N - \frac{1}{12} , \tag{5·16}$$

where we have abbreviated $D(u)$ as D. Using the relation (4·1), these equations become

$$a = \frac{1}{4}DN^2 + \frac{1}{2}(D-1)N - \frac{1}{24}, \qquad c = \frac{1}{4}DN^2 + \frac{3}{4}(D-1)N - \frac{1}{12}. \tag{5·17}$$

Now, the result (5·17) completely reproduces the central charges (4·11) calculated using the gravity dual. The agreement is indeed quite nontrivial. In the holographic approach, the $\mathcal{O}(N^2)$, $\mathcal{O}(N)$ and $\mathcal{O}(1)$ terms arose as contributions due to classical bulk gravity, branes, and one-loop effects, respectively, whereas in our present approach a and c were calculated nonperturbatively and received contributions from completely different sources, $R(A)$ and $R(B)$. Furthermore, our formula (5·17) also reproduces the central charges of rank-N versions of the E_n theories if we use the corresponding dimensions $D(u)$. These mysterious theories have yet to be realized in a purely field-theoretical language, but our result strongly suggests that their gravitational measure factors A and B should still be given by the general formulas (5·9), (5·11).

5.3. *Flavor symmetry*

Let us calculate the central charge of the flavor symmetry of the $USp(2N)$ theory for $N_f = 2, 3$. The flavor symmetry acting on the hypermultiplets in the fundamental is $U(N_f) = U(1) \times SU(N_f)$ when we take all of the masses to be equal, $m = m_1 = \cdots = m_{N_f}$. We study the response of the gauge theory to the introduction of an external gauge field for the flavor symmetry, for a generic value of m, which can be expressed as an extra factor in the low energy path integral (5·4)

$$[d\mu] A^\chi B^\sigma C^n \tag{5·18}$$

where n is the instanton number of the external $SU(N_f)$ field. Repeating the argument in §5.1, one finds the relation

$$k_G = -R(C). \tag{5·19}$$

We begin with the case $N = 1$. As we have discussed, if the masses are generic and unequal, there are $2 + N_f$ singular points in the u-plane which are given by the zeros of the discriminant $\varDelta_1$. When we take $m = m_1 = \cdots = m_{N_f}$, the discriminant has a zero of order N_f,

$$\varDelta_1(u; m, \ldots, m) = \underline{\varDelta}(u, m)(u - u_q(m))^{N_f}. \tag{5·20}$$

Here $\underline{\varDelta}(u, m)$ is a quadratic polynomial whose zeros give the points where a monopole or a dyon becomes massless. $u = u_q(m)$ is the point where a hypermultiplet in the fundamental representation of $SU(N_f)$ appears. It is important to note that $u_q(m)$ is a polynomial in m and has no monodromy. The physical reason is that there is an $(N_f - 1)$-dimensional Higgs branch emanating from $u = u_q(m)$, so that the singularity there can be clearly differentiated from the monopole and dyon points.

Let us now turn to the rank-N version of the theory, parameterized by $u_1, \ldots, u_N$ with the identification $u_i \leftrightarrow u_j$. When $u_i = u_q(m)$, one free massless hypermultiplet in the fundamental of $U(N_f)$ appears, so $C \sim (u_i - u_q(m))^{-1}$. Thus

$$C = \prod_i (u_i - u_q(m))^{-1}. \tag{5·21}$$

It correctly reproduces the semiclasical behavior when all u_i are large. The superconformal system can then be reached by choosing m so that $u_q(m)$ collides with another zero of the discriminant, which we take to be at $u = 0$. We then have

$$C = \prod_i u_i^{-1} \tag{5·22}$$

which means that

$$k_G = 2ND(u) \tag{5·23}$$

at the superconformal point. It again reproduces the holographic result (4·11).

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References

1) N. Seiberg and E. Witten, Nucl. Phys. B **426** (1994), 19 [Errata; **430** (1994), 485]; hep-th/9407087.
2) N. Seiberg and E. Witten, Nucl. Phys. B **431** (1994), 484; hep-th/9408099.
3) P. C. Argyres and M. R. Douglas, Nucl. Phys. B **448** (1995), 93; hep-th/9505062.
4) P. C. Argyres, M. R. Plesser, N. Seiberg and E. Witten, Nucl. Phys. B **461** (1996), 71; hep-th/9511154.
5) T. Eguchi, K. Hori, K. Ito and S. K. Yang, Nucl. Phys. B **471** (1996), 430; hep-th/9603002.
6) T. Eguchi and K. Hori, hep-th/9607125.
7) J. A. Minahan and D. Nemeschansky, Nucl. Phys. B **482** (1996), 142; hep-th/9608047.
8) J. A. Minahan and D. Nemeschansky, Nucl. Phys. B **489** (1997), 24; hep-th/9610076.
9) P. C. Argyres and N. Seiberg, J. High Energy Phys. **12** (2007), 088; arXiv:0711.0054.
10) O. Aharony and Y. Tachikawa, J. High Energy Phys. **01** (2008), 037; arXiv:0711.4532.
11) A. D. Shapere and Y. Tachikawa, J. High Energy Phys. **90** (2008), 109; arXiv:0804.1957.
12) A. Bilal and F. Ferrari, Nucl. Phys. B **516** (1998), 175; hep-th/9706145.
13) T. Kubota and N. Yokoi, Prog. Theor. Phys. **100** (1998), 423; hep-th/9712054.
14) S. Gukov and A. Kapustin, Nucl. Phys. B **545** (1999), 283; hep-th/9808175.
15) M. Noguchi, S. Terashima and S. K. Yang, Nucl. Phys. B **556** (1999), 115; hep-th/9903215.
16) A. Gustavsson and M. Henningson, Phys. Lett. B **463** (1999), 201; hep-th/9906053.
17) A. D. Shapere and C. Vafa, hep-th/9910182.
18) P. C. Argyres, M. Crescimanno, A. D. Shapere and J. R. Wittig, hep-th/0504070.
19) P. C. Argyres and J. R. Wittig, hep-th/0510226.
20) K. Intriligator and B. Wecht, Nucl. Phys. B **667** (2003), 183; hep-th/0304128.
21) D. Anselmi, D. Z. Freedman, M. T. Grisaru and A. A. Johansen, Nucl. Phys. B **526** (1998), 543; hep-th/9708042.
22) P. C. Argyres, M. R. Plesser and A. D. Shapere, Phys. Rev. Lett. **75** (1995), 1699; hep-th/9505100.
23) A. Hanany and Y. Oz, Nucl. Phys. B **452** (1995), 283; hep-th/9505075.
24) P. C. Argyres and J. R. Wittig, J. High Energy Phys. **01** (2008), 074; arXiv:0712.2028.
25) Y. K. Cheung, O. J. Ganor and M. Krogh, Nucl. Phys. B **523** (1998), 171; hep-th/9710053.
26) D. R. Morrison and C. Vafa, Nucl. Phys. B **473** (1996), 74; hep-th/9602114.
27) A. Fayyazuddin and M. Spałinski, Nucl. Phys. B **535** (1998), 219; hep-th/9805096.
28) O. Aharony, A. Fayyazuddin and J. M. Maldacena, J. High Energy Phys. **07** (1998), 013; hep-th/9806159.
29) O. Aharony, J. Pawelczyk, S. Theisen and S. Yankielowicz, Phys. Rev. D **60** (1999), 066001; hep-th/9901134.
30) M. Henningson and K. Skenderis, J. High Energy Phys. **07** (1998), 023; hep-th/9806087.
31) S. M. Kuzenko and S. Theisen, Class. Quantum Grav. **17** (2000), 665; hep-th/9907107.
32) E. Witten, Selecta Math. **1** (1995), 383; hep-th/9505186.
33) G. W. Moore and E. Witten, Adv. Theor. Math. Phys. **1** (1998), 298; hep-th/9709193.
34) A. Losev, N. Nekrasov and S. L. Shatashvili, Nucl. Phys. B **534** (1998), 549; hep-th/9711108.
35) M. Mariño and G. W. Moore, Commun. Math. Phys. **199** (1998), 25; hep-th/9802185.
36) M. Mariño, G. W. Moore and G. Peradze, Commun. Math. Phys. **205** (1999), 691; hep-th/9812055.
37) T. Banks, M. R. Douglas and N. Seiberg, Phys. Lett. B **387** (1996), 278; hep-th/9605199.
38) M. R. Douglas, D. A. Lowe and J. H. Schwarz, Phys. Lett. B **394** (1997), 297; hep-th/9612062.
39) V. Pestun, arXiv:0712.2824.
40) D. Gaiotto, A. Neitzke and Y. Tachikawa, arXiv:0810.4541.