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NEW DIRECTIONS IN DESCRIPTIVE SET THEORY

ALEXANDER S. KECHRIS

§1. I will start with a quick definition of descriptive set theory: It is the study of the structure of definable sets and functions in separable completely metrizable spaces. Such spaces are usually called *Polish spaces*. Typical examples are $\mathbb{R}^n$, $\mathbb{C}^n$, (separable) Hilbert space and more generally all separable Banach spaces, the *Cantor space* $2^{\mathbb{N}}$, the *Baire space* $\mathbb{N}^{\mathbb{N}}$, the infinite symmetric group S_∞, the unitary group (of the Hilbert space), the group of measure preserving transformations of the unit interval, etc.

In this theory sets are classified in hierarchies according to the complexity of their definitions and the structure of sets in each level of these hierarchies is systematically analyzed. In the beginning we have the *Borel* sets in Polish spaces, obtained by starting with the open sets and closing under the operations of complementation and countable unions, and the corresponding *Borel hierarchy* ($\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Delta}^0_\alpha$ sets). After this come the *projective sets*, obtained by starting with the Borel sets and closing under the operations of complementation and projection, and the corresponding *projective hierarchy* ($\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, $\mathbf{\Delta}^1_n$ sets).

There are also transfinite extensions of the projective hierarchy and even much more complex definable sets studied in descriptive set theory, but I will restrict myself here to Borel and projective sets, in fact just those at the first level of the projective hierarchy, i.e., the *Borel* ($\mathbf{\Delta}^1_1$), *analytic* ($\mathbf{\Sigma}^1_1$) and *coanalytic* ($\mathbf{\Pi}^1_1$) sets.

Over the last one hundred years a great deal has been learned about the structure of definable sets in Polish spaces, and a very extensive theory has been developed (see, e.g., Moschovakis [25], Kechris [20]). My goal here is not to review these developments, which I could not possibly hope to do in a reasonably short article, but rather to discuss some new directions into which descriptive set theory has been moving over the last decade or so.

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§**2.** Although Polish spaces and their definable subsets include a great many of the spaces studied in mathematics, there are also spaces of major importance which cannot be realized in any reasonable way as subsets of Polish spaces and are therefore genuinely new objects of study. These spaces have the form of quotients X/E for some Polish space X and a definable equivalence relation E on X. Here are some typical examples:

(i) The orbit spaces of definable actions of Polish groups (i.e., topological groups whose topology is Polish). Here if $(g, x) \mapsto g \cdot x$ is an action of the group G on the set X, the *orbit space* of the action is the set of all orbits $\{ G \cdot x : x \in X \}$. This is X/E, where E is the corresponding *orbit equivalence relation*: $x E y \iff G \cdot x = G \cdot y$. For instance, these include the orbit space of an irrational rotation on the unit circle $\mathbb{T}$, the so-called *dual* of a second countable locally compact group G (e.g., a Lie group), i.e., the space of all irreducible unitary representations of G modulo unitary equivalence, or the space of all isomorphism classes of countable structures of a given (countable) language L.

(ii) The "moduli space" of Riemann surfaces, i.e., the space of equivalence classes of Riemann surfaces modulo conformal equivalence.

(iii) The space of measure classes of probability Borel measures on a Polish space, i.e., the quotient space of measures under the *measure equivalence relation*: $\mu \sim \nu$ if and only if μ, ν are absolutely continuous with respect to each other, that is, have the same null sets.

(iv) The set of Turing (arithmetical, etc.) degrees of subsets of $\mathbb{N}$.

It has long been recognized in diverse areas of mathematics that in many important cases such quotient spaces X/E cannot be viewed as reasonable subsets of Polish spaces and therefore the usual methods of topology, geometry, measure theory, etc., are not directly applicable for their study. Thus they are often referred to as *singular spaces*. Instead, the very popular these days, *non-commutative* counterparts of these classical mathematical disciplines have been developed to provide the tools for their study. See Connes [2] for an illuminating discussion of these issues.

The goal of a lot of recent work in descriptive set theory has been the development of the descriptive set theory of these singular spaces. This essentially amounts to the study of definable equivalence relations on Polish spaces and the closely related study of definable actions of Polish groups on such spaces.

§**3.** To start with, a basic set theoretic question arising in this study is the problem of the *definable cardinality* of singular spaces. According to the classical Cantor cardinality theory, these spaces turn out in practice to be quite often equinumerous with the continuum $\mathbb{R}$. However in these cases one makes essential use of the Axiom of Choice in establishing a bijection with the reals. It is much more interesting and relevant in this context, however, to ask whether there is actually a definable such correspondence. In fact, it has long been understood that this is not the case. Indeed in many interesting cases these spaces have definable cardinality strictly greater than that of the continuum, in the sense that there is a definable embedding of the reals into them but not vice-versa. See, for example, the discussion in Connes [2], p. 74. A standard example is the classical *Vitali equivalence relation* E_0 on $\mathbb{R}$, defined as follows:

$$x \, E_0 \, y \iff x - y \in \mathbb{Q}.$$

Then it is easy to see that there is a Borel function $f : \mathbb{R} \to \mathbb{R}$ with

$$x \neq y \Longrightarrow \neg f(x) \, E \, f(y)$$

but there is no Borel function $g : \mathbb{R} \to \mathbb{R}$ with

$$x \, E \, y \iff g(x) = g(y),$$

so that there is a definable embedding of $\mathbb{R}$ into $\mathbb{R}/E_0 = \mathbb{R}/\mathbb{Q}$ but not vice-versa.

Let me formalize these ideas: Suppose E, F are equivalence relations on Polish spaces X, Y respectively. A *Borel reduction* of E into F is a Borel map $f : X \to Y$ such that

$$(*) \qquad x \, E \, y \iff f(x) \, F \, f(y)$$

If such an f exists, we say that E *Borel reduces* to F and write

$$E \leq_B F.$$

(Other, more complex, notions of definability can be used instead of Borel but, by and large, the theory is quite analogous, and this is the most interesting and natural context anyway.)

Since $(*)$ above simply means that there is an embedding from X/E to Y/F with a Borel lifting, we think of it as saying that X/E Borel embeds in Y/F or that X/E has *Borel cardinality* at most that of Y/F. We take this as the basic notion of definable cardinality theory, playing the role of injection in classical Cantor cardinality theory. We also let

$$E \sim_B F \iff E \leq_B F \,\&\, F \leq_B E,$$
$$E <_B F \iff E \leq_B F \,\&\, F \not\leq_B E.$$

So $E \sim_B F$ intuitively means that X/E, Y/F have the same Borel cardinality and $E <_B F$ means that X/E has (strictly) smaller Borel cardinality than that of Y/F.

Identifying the Polish space X with the equality relation $=_X$ on X, we can say that

$$\mathbb{R} <_B E_0,$$

i.e., that $\mathbb{R}/\mathbb{Q}$ has bigger Borel cardinality than the continuum (although it is obvious that classically $\mathbb{R}$ and $\mathbb{R}/\mathbb{Q}$ are equinumerous).

One basic ingredient of current work in this area is the study and classification of Borel cardinalities under $\leq_B$. Although in classical Cantor theory quite often the cardinality of such a quotient space is just that of the continuum, it turns out that there is a rich and intricate structure of Borel cardinalities, which unveils many new and interesting phenomena.

§4. Before I discuss specific results in this theory, it will be important to reinterpret these concepts in a different way, which actually motivates philosophically a lot of work in this area, since it deals with what appears to be a very interesting foundational problem.

Mathematicians frequently deal with problems of classification of objects up to some notion of equivalence by invariants. Quite often these objects can be represented by elements of some Polish space X and the equivalence by a definable equivalence relation E on X. A *complete classification* of X up to E therefore consists of finding a set of invariants I and a map $c : X \to I$ such that

$$x \, E \, y \iff c(x) = c(y).$$

For this to have any meaning both I, c must be *explicit* or *definable* too and as simple and concrete as possible. For example, taking $c(x) = [x]_E =$ the equivalence class of x, or using the Axiom of Choice to select a point $f(C)$ for each $C \in X/E$ and letting $c(x) = f([x]_E)$ is clearly not an illuminating choice of invariants. What constitutes an interesting and useful complete classification is hard to define precisely, and varies from the very concrete, e.g., classification of finitely generated abelian groups up to isomorphism by invariants which are finite lists of integers, or Ornstein's classification of Bernoulli automorphisms up to conjugacy by the entropy, which is a real number, to somewhat more abstract and set theoretic, e.g., the Ulm classification of countable abelian p-groups up to isomorphism, where the invariants are essentially countable transfinite sequences from $\mathbb{N} \cup \{\infty\}$. However, the preceding ideas can be used to develop a mathematical framework for measuring the complexity of classification problems and understanding the nature of their complete invariants.

Suppose two classification problems are represented by equivalence relations E, F on Polish spaces X, Y respectively. Then $E \leq_B F$ simply means that any complete invariants for F work as well for E (after an appropriate composition by a Borel function), so in some sense E has a classification problem which is at most as difficult as that of F. In particular, $E \sim_B F$ means that E and F have, in some sense, equivalent classification problems, and $E <_B F$ means that E has a (strictly) simpler classification problem than F.

To cite a classical example, if we denote by E the unitary equivalence of normal operators on Hilbert space and by F the measure equivalence relation on any uncountable Polish space, then the Spectral Theorem implies that $E \sim_B F$.

§5. In describing the emerging picture of the hierarchy of complexity of classification problems, I will concentrate on analytic equivalence relations, which contain the vast majority of concrete examples occuring in practice. Among them, the most interesting subclasses are the *Borel equivalence relations* and the *orbit equivalence relations* induced by Borel actions of Polish groups on Polish spaces. (Typical examples of Polish groups are the Lie groups, the infinite symmetric group S_∞, the unitary group, the group of measure preserving transformations on $[0, 1]$, the homeomorphism group of a compact metric space, etc.) These orbit equivalence relations are analytic but not always Borel. For example, the isomorphism relation on countable graphs, with say standard universe $\mathbb{N}$, is the orbit equivalence relation of a Borel action of S_∞ and is not Borel.

I will mostly in fact concentrate in this article on Borel equivalence relations. The theory of Polish groups and their actions is a whole subject in itself and its study also involves quite different issues not necessarily related to the classification of their orbit spaces. I will not attempt to discuss this here except in connection with certain aspects of the theory of Borel equivalence relations that will come up later. Suffice it to say that some fundamental early work has been done by Glimm and Effros in the 1960s and Vaught, Burgess, Miller, Sami in the 1970s, and, in particular, the key Vaught transform was then introduced. An account of recent developments can be found in Becker-Kechris [1].

§6. So I will start by looking at the picture of the hierarchy of classification problems of Borel equivalence relations, i.e., the structure of Borel equivalence relations under the partial (pre)order $\leq_B$. In the beginning things are simple enough. Denoting by X also the equality relation $=_X$ on X, we have that the following is an initial segment of $\leq_B$:

$$1 <_B 2 <_B 3 <_B \cdots <_B \mathbb{N},$$

and $\mathbb{N} <_B E$ for any Borel equivalence relation E not in this list. Next we have the *Silver Dichotomy*: (Silver [27]) If E is a Borel (even $\mathbf{\Pi}^1_1$) equivalence relation, then exactly one of the following holds:

(i) $E \leq_B \mathbb{N}$

or

(ii) $\mathbb{R} \leq_B E$.

This simply says that either E has countably many classes or perfectly many classes, i.e., there is a perfect set of E-inequivalent elements.

Thus

$$1 <_B 2 <_B 3 <_B \cdots <_B \mathbb{N} <_B \mathbb{R}$$

is an initial segment of $\leq_B$ and $\mathbb{R} <_B E$ for any Borel equivalence relation E not in this list.

A well-known problem here is to find out if this dichotomy is also true for the orbit equivalence relation induced by a Borel action of any given Polish group G. This is the *Topological Vaught Conjecture*, TVC(G), for G, proposed by D. Miller in the 1970s, generalizing the famous *Vaught Conjecture*, which asserts that a first order theory (in a countable language) has either countably many or continuum many models up to isomorphism, and its stronger version (also referred to as the Vaught Conjecture) that an $L_{\omega_1\omega}$-theory has either countably many or perfectly many countable models, up to isomorphism. In fact, it turns out, as shown in [1], that this last form of Vaught's Conjecture is equivalent to TVC(S_∞). This is still open, but Hjorth [11] has recently made major progress by characterizing group theoretically the Polish groups G for which TVC(G) holds in a stronger form, namely for *analytic sets*, i.e., for any Borel action of G on X and any analytic invariant set $A \subseteq X$, A contains either countably many or perfectly many orbits. This fails for S_∞ and thus for any G which has a closed subgroup with quotient S_∞. Hjorth's result is that the TVC(G) for analytic sets holds exactly for all Polish groups G that do not contain closed subgroups with quotient S_∞. This has as a corollary that if the Vaught Conjecture (even for $L_{\omega_1\omega}$) fails, which some people believe to be the case, then TVC(G) is completely solved, because it holds for exactly the same G for which Hjorth's above mentioned characterization works.

It is clear that the classification problems corresponding to equivalence relations (X, E) with $E \leq_B \mathbb{R}$ are exactly those for which there is a Borel map $f : X \to Y$, Y some Polish space, with

$$x \, E \, y \iff f(x) = f(y),$$

i.e., for which complete invariants can be found which are real numbers, complex numbers, or more generally members of a Polish space, and are therefore fairly concrete. We thus call such E *concretely classifiable*. (Other terminologies that are used for this concept are: E is *smooth* or E is *tame*.) In that case X/E can simply be viewed as an analytic subset of a Polish space, so it is well understood. It is therefore those E for which $\mathbb{R} <_B E$ that represent genuinely different, i.e., singular, quotient spaces X/E. The first main fact about these singular spaces is that there is a smallest possible one. Recall the Vitali equivalence relation E_0. As mentioned earlier, $\mathbb{R} <_B E_0$, so E_0 is not concretely classifiable. We now have the *General Glimm-Effros Dichotomy*: (Harrington-Kechris-Louveau [10]) If E is a Borel equivalence relation, then exactly one of the following holds:

(i) $E \leq_B \mathbb{R}$

or

(ii) $E_0 \leq_B E$.

Thus the following is an initial segment of $\leq_B$:

$$1 <_B 2 <_B 3 <_B \cdots <_B \mathbb{N} <_B \mathbb{R} <_B E_0,$$

and any Borel equivalence relation E not in this list satisfies $E_0 <_B E$.

The attribute "Glimm-Effros" in this theorem signifies the fact that this result generalizes some special cases originally proved by Glimm [9] and Effros [5], motivated by the theory of operator algebras. One interesting point concerning the general version of this theorem is that the only known proof makes crucial use of *effective* descriptive set theory, although the statement of the result is clearly understood in the classical Borel theoretic context.

Beyond E_0 the linearity of $\leq_B$ breaks down and this order becomes quite complex. This non-linearity seems to be a basic feature of Borel cardinality as compared with classical Cantor cardinality. There are in fact uncountably many incomparable under $\leq_B$ Borel equivalence relations (Woodin) and, even more, the partial order $(p(\mathbb{N}), \subseteq^*)$, where $\subseteq^*$ is inclusion modulo finite sets, embeds into $\leq_B$ (Louveau-Velickovic [24]). It also turns out that $\leq_B$ is unbounded. In fact, there is an analog of Cantor's Theorem: For each Borel equivalence relation E on a Polish space X, if F on $X^{\mathbb{N}}$ is defined by:

$$(x_n)\ F\ (y_n) \iff \{\, x_n : n \in \mathbb{N} \,\}, \{\, y_n : n \in \mathbb{N} \,\} \text{ meet the same } E\text{-classes}$$

(so that X/F is essentially the same as $p_{\aleph_0}(X/E) = \{\, A \subseteq X/E : |A| \leq \aleph_0 \,\}$), then $E <_B F$, provided E has at least two classes (Friedman-Stanley [7]).

§7. Most of the natural examples of classification problems that can be represented by Borel equivalence relations have special properties and this, as usual, motivates restricting attention to important subclasses.

The first one that I will consider here is the class of *countable Borel equivalence relations*, where E is *countable* if every equivalence class is countable. Examples include E_0, the Turing equivalence relation, and any orbit equivalence relation induced by a Borel action of a countable group. In fact, Feldman-Moore [6] showed that any countable Borel equivalence relation E is induced by such an action of a countable group. The results below, unless otherwise stated, come from Dougherty-Jackson-Kechris [4] and Jackson-Kechris-Louveau [18]. Often methods and results of ergodic theory play a crucial role in this study.

First, among the countable Borel equivalence relations it turns out that there is a largest one, in the sense of $\leq_B$, denoted by E_∞, which is naturally called *universal*. Thus, among non-concretely classifiable countable E, there is a smallest one, E_0, and a largest one, E_∞, so they all fall in the interval

$$E_0 \leq_B E \leq_B E_\infty.$$

There are also intermediate ones

$$E_0 <_B E <_B E_\infty,$$

but remarkably at this time only two distinct ones are known, say E, F, and they satisfy $E_0 <_B E <_B F <_B E_\infty$. In particular, it is unknown if there exist infinitely many or two incomparable ones. [*Addendum.* It has been now shown in S. Adams and A. S. Kechris, *Linear algebraic groups and countable Borel equivalence relations*, preprint, 1999, that there are indeed uncountably many incomparable countable Borel equivalence relations.]

It turns out that a lot of classification problems are represented by Borel equivalence relations E which, although not necessarily themselves countable, are $\sim_B$ to some countable F, and so their complexity can be measured in the hierarchy of countable Borel equivalence relations. They include, for example, any orbit equivalence relation induced by a Borel action of a Polish *locally compact* group (e.g., a Lie group) or the isomorphism relation on various classes of countable models that in some sense have "finite type", for instance finitely generated groups or locally finite (i.e., having finite degree at each vertex) connected graphs.

Let me next discuss some results concerning classification of equivalence relations within the interval $[E_0, E_\infty]$.

(A) E_0.

The countable E which are $\leq_B E_0$ turn out to be exactly those induced by a Borel action of the simplest (infinite) countable group, $\mathbb{Z}$, i.e., by the orbits of a single Borel automorphism. They can be also characterized as those of the form $E = \bigcup_n E_n$, with $E_n \subseteq E_{n+1}$ *finite* Borel equivalence relations, i.e., having finite classes (Weiss [32], Slaman-Steel [28]). For that reason they are called *hyperfinite*. So, up to $\sim_B$, E_0 is the unique non-concretely classifiable hyperfinite equivalence relation.

It is natural to ask what countable groups G always produce hyperfinite Borel equivalence relations. It can be shown that any such G must be amenable, i.e., carry a left-invariant finitely additive probability measure (see, e.g., Kechris [19]). Weiss [32] raised the question of whether, conversely, any Borel action of a countable amenable group gives rise to a hyperfinite orbit equivalence relation. (This turns out to be true in the measure theoretic context, i.e., neglecting null sets, for any given Borel probability measure on the underlying space, see Ornstein-Weiss [26], Connes-Feldman-Weiss [3].) Weiss proved it is true for $G = \mathbb{Z}^n$, and this was extended later in [18] to the following theorem which is essentially the best result known to date: Any orbit equivalence relation induced by a Borel action of a finitely generated group of polynomial growth is hyperfinite. (These groups can be also characterized as the finitely generated nilpotent-by-finite groups, by a theorem of Gromov.)

Here are some other examples of equivalence relations E which, up to $\sim_B$, are hyperfinite:

(i) Any orbit equivalence relation induced by a Borel action of $\mathbb{R}$ (i.e., a Borel flow) or even $\mathbb{R}^n$.
(ii) The isomorphism relation on torsion-free abelian groups of rank 1 (i.e., subgroups of $(\mathbb{Q}, +)$). Up to $\sim_B$ this turns out to be exactly E_0.

A lot more is known about hyperfinite Borel equivalence relations, including a complete classification up to Borel isomorphism. But many important open questions still remain, for which I refer to the above papers.

(B) E_∞.

Here are some examples of equivalence relations and classifications of complexity E_∞ (i.e., $\sim_B E_\infty$).

(i) The equivalence relation induced by the translation action of the free group on two generators, F_2, on its subsets. (Note here that replacing F_2 by $\mathbb{Z}$ we get E_0.)
(ii) (Thomas-Velickovic [31]) Conjugacy of subgroups of F_2.
(iii) Isomorphism of locally finite connected graphs or trees.
(iv) (Thomas-Velickovic [31] and [30]) Isomorphism of finitely generated groups, and also fields of finite transcendence degree over $\mathbb{Q}$.
(v) (Slaman-Steel) Arithmetic equivalence for subsets of $\mathbb{N}$.
(vi) (Hjorth-Kechris [14]) Conformal equivalence of planar domains and also Riemann surfaces. Thus the "moduli space" of general Riemann surfaces is very complicated. In contrast, conformal equivalence of *compact* Riemann surfaces is concretely classifiable and the "moduli space" in this particular case has a well-known rich geometric structure.

There are also some important conjectures here:

(a) Let $\equiv_T$ be the Turing equivalence relation on $p(\mathbb{N})$. *Conjecture*: $(\equiv_T) \sim_B E_\infty$. By results of Slaman-Steel [28], $E_0 <_B (\equiv_T)$. A positive answer would compute the exact Borel cardinality of the set of Turing degrees but would also have some other interesting implications. It would disprove, for example, the well-known Martin Conjecture concerning the structure of Turing-invariant definable functions (the Fifth Victoria Delfino Problem; see [22]), which is open since the 1970s, since it would imply the existence of strange functions, like for instance a Borel pairing function on the Turing degrees.

(b) Let $\cong_n$ be the isomorphism relation of rank $\leq n$ torsion free abelian groups (i.e., subgroups of $(\mathbb{Q}^n, +)$). *Conjecture*: For $n \geq 2$, we have that $(\cong_n) \sim_B E_\infty$ (see Hjorth-Kechris [15].) The problem of finding a reasonable classification of such groups is a classical question in abelian group theory (see Fuchs [8]). A well known result of Baer provides a satisfactory classification in the rank 1 case and implies that it has complexity E_0. Hjorth [12] has already shown that $E_0 <_B (\cong_2)$ (and in fact a combination of Hjorth's work and recent work of Kechris shows that $E_0 \ll_B (\cong_2)$). So the rank 2 case necessarily involves a more complex classification problem. A proof of the above conjecture would be strong evidence that there cannot be a reasonable classification in this case.

Finally, there are as yet very few natural examples of classification problems that correspond to intermediate $E_0 <_B E <_B E_\infty$. One such is the isomorphism on *rigid* locally finite trees (as opposed to general locally finite trees which have complexity E_∞). S. Thomas has also suggested the possibility that another example might be the conjugacy equivalence of subgroups of certain Burnside groups. [*Addendum.* Recently, S. Adams and A. S. Kechris, *Linear algebraic groups and countable Borel equivalence relations*, preprint, 1999, have found many new such examples.]

§8. Beyond the countable Borel equivalence relations, another natural subclass consists of the orbit equivalence relations induced by Borel actions of the infinite symmetric group, S_∞, and its closed subgroups. These include all the countable ones. It turns out (see Becker-Kechris [1]) that up to $\sim_B$ these are the same as the isomorphism relations $\cong_\sigma$ on the countable models of some $L_{\omega_1\omega}$ theory σ, hence their obvious interest to logicians.

It should be pointed out that there are theories σ for which $\cong_\sigma$ might not be Borel, e.g., $\cong_\gamma$, where γ is the theory of graphs. In fact, $\cong_\gamma$ is the largest possible $\cong_\sigma$ in the sense of the order $\leq_B$.

Restricting attention to isomorphism relations $\cong_\sigma$ which are Borel (model theoretically this means that the Scott ranks of countable models of σ are bounded below ω_1), we have the following general picture: There is a transfinite sequence of theories σ_α, with Borel $F_\alpha = (\cong_{\sigma_\alpha})$, $\alpha < \omega_1$, so that

$$F_1 = \mathbb{R} <_B E_\infty <_B F_2 <_B \cdots <_B F_\alpha < \cdots$$

and the transfinite sequence (F_α) is cofinal among the equivalence relations $\cong_\sigma$ which are Borel. Roughly speaking, F_α is such that its quotient space is the αth iterated countable powerset of $\mathbb{R}$, so that the quotient space of F_2 is $p_{\aleph_0}(\mathbb{R})$, of F_3 is $p_{\aleph_0}(p_{\aleph_0}(\mathbb{R}))$, etc. (Friedman-Stanley [7]). Since every Borel $\cong_\sigma$ is $\leq_B$ some F_α, this can be used to measure the set theoretic complexity of complete invariants for the isomorphism of countable models of σ. Thus $E \leq_B F_2$ means that invariants are countable (unordered) sets of reals (or some other Polish space), $E \leq_B F_3$ means that invariants are countable sets of countable sets of reals, etc.

In Hjorth-Kechris-Louveau [17], it was shown that there is a precise relationship between the descriptive complexity of $\cong_\sigma$, appropriately measured, and the type of complete invariants for $\cong_\sigma$. Saying that $\cong_\sigma$ is *potentially of class* $\mathbf{\Pi}^0_\alpha$ ($\mathbf{\Sigma}^0_\alpha$) if $(\cong_\sigma) \sim_B E$ for some equivalence relation E which is $\mathbf{\Pi}^0_\alpha$ ($\mathbf{\Sigma}^0_\alpha$), we have, as special cases of the results in that paper, the following:

(i) $\cong_\sigma$ is potentially $\mathbf{\Pi}^0_2$ if and only if $\cong_\sigma$ is concretely classifiable (i.e., the invariants are reals or members of some Polish space).
(ii) $\cong_\sigma$ is potentially $\mathbf{\Sigma}^0_3$ if and only if $\cong_\sigma$ is potentially $\mathbf{\Sigma}^0_2$ if and only if $(\cong_\sigma) \sim_B E$ for some countable Borel E (so we are now in the domain of §7).
(iii) $\cong_\sigma$ is potentially $\mathbf{\Pi}^0_3$ if and only if $(\cong_\sigma) \leq_B F_2$ (so the invariants are countable sets of reals or some other Polish space).
(iv) $\cong_\sigma$ is potentially $\mathbf{\Pi}^0_4$ if and only if $(\cong_\sigma) \leq_B F_3$ (so the invariants are countable sets of countable sets of reals), etc.

For instance, it turns out that isomorphism of locally finite (not necessarily connected) graphs is $\sim_B F_2$. (Recall that for connected graphs we have complexity $E_\infty <_B F_2$.) Also isomorphism of countable archimedean totally ordered abelian groups with a distinguished positive element is $\sim_B F_2$. In a different context, it follows from the classical Halmos-von Neumann Theorem that conjugacy of ergodic measure preserving transformations with discrete spectrum has also complexity F_2.

§9. It is now interesting to isolate the class of classification problems which can be represented by (X, E) which satisfy $E \leq_B (\cong_\sigma)$ for some theory σ (with $\cong_\sigma$ not necessarily Borel). This means that we can assign in a Borel way to each $x \in X$ a countable model $f(x)$, with universe $\mathbb{N}$, so that

$$x\, E\, y \iff f(x) \cong f(y).$$

If this happens, it is natural to say that E *admits classification by countable structures*, since complete invariants of E are isomorphism types of countable structures. I emphasize that I am not necessarily assuming here that E is Borel. For example, all the specific classification problems that I have discussed until now admit classification by countable structures. However, H. Friedman has found some time ago an example of a Borel equivalence relation which does not admit classification by countable structures. Recently Hjorth [13] has developed a powerful machinery, called the *theory of turbulence*, which in the case that E is induced by a continuous action of a Polish group, allows one to analyze in an appropriate sense when E cannot be classified by countable structures, in terms of the topological dynamics (more specifically the *local* structure of the orbits) of the action.

As an application of this theory, various interesting classification problems have been shown to be complex enough so that they do not admit classification by countable structures. Here is a sample:

(i) (Hjorth [13]) Conjugacy on the group of homeomorphisms of the unit square, $H(I^2)$. (On the other hand, replacing I^2 by the unit interval I one has classification by countable structures.)

(ii) (Hjorth [13]) Conjugacy of ergodic measure preserving transformations. (This should be contrasted with the special case of discrete spectrum transformations, where a classification by countable structures also exists, as mentioned in §8.)
(iii) (Kechris-Sofronidis [23]) Unitary equivalence of unitary or self-adjoint operators.
(iv) (Hjorth-Kechris [14]) Biholomorphic equivalence of 2-dimensional complex manifolds. (Again in contrast with the 1-dimensional case, i.e., Riemann surfaces, which admit classification by countable structures.)

§10. Finally, there are Borel equivalence relations which are not below, in the sense of $\leq_B$, an orbit equivalence relation induced by a Borel action of *any* Polish group (and not just S_∞). Thus complete invariants for them cannot be represented by orbits of such actions. The canonical example is the equivalence relation E_1 on $\mathbb{R}^{\mathbb{N}}$ defined as follows:

$$x\ E_1\ y \iff \exists n\, \forall m \geq n\ (x_m = y_m).$$

(It should be noted here that if we replace $\mathbb{R}$ by any *countable* set with more than one element, then the corresponding equivalence relation is, up to $\sim_B$, the same as E_0.) It was shown in Kechris-Louveau [21] that for any orbit equivalence relation E induced by a Borel action of a Polish group we have

$$E_1 \not\leq_B E.$$

Moreover, E_1 has the following minimality property

$$E \leq_B E_1 \Longrightarrow E \leq_B E_0 \text{ or } E \sim_B E_1.$$

It turns out that E_1 has many manifestations. To mention a particularly interesting one, Solecki showed that if for any *indecomposable continuum* C (i.e., a continuum which cannot be written as the union of two proper subcontinua), we denote by E_C the equivalence relation induced by its composants (where two points are in the same composant if they belong to a proper subcontinuum), then $E_C \sim_B E_0$ or $E_C \sim_B E_1$.

In particular, E_C is never concretely classifiable, which answered an old problem in the theory of continua. Moreover, Solecki has shown that there are indecomposable continua of both types. In fact, it seems that the topologically simpler ones correspond to E_0, while the topologically more complex ones correspond to E_1.

It has been conjectured (see Kechris-Louveau [21] and Hjorth-Kechris [16]) that E_1 is the precise obstruction for non-reducibility into the orbit equivalence relation of a Polish group action, i.e., that if E is a Borel equivalence relation, then exactly one of the following holds: (i) $E_1 \leq_B E$ or (ii) $E \leq_B F$, for some F induced by a Borel action of a Polish group.

This is open, and perhaps too optimistic in general, but an important special case has been verified by Solecki [29], namely when E is of the form E_I, for some Borel ideal I on $\mathbb{N}$, where for $x, y \subseteq \mathbb{N}$

$$x \, E_I \, y \iff x \bigtriangleup y \in I.$$

This relates this area with the theory of Borel p-ideals on the integers where Farah, Solecki, Todorcevic and Velickovic have recently obtained many interesting results and have found surprising connections with the theory of Banach spaces.

§11. To summarize, it is clear at this stage that the study of singular spaces and the hierarchy of classification problems uncovers intriguing new phenomena and presents many challenging problems. It also leads to novel interactions between descriptive set theory and other areas of logic and mathematics. So it seems to be a very promising area for further investigations.

REFERENCES

[1] H. Becker and A. S. Kechris, ***The descriptive set theory of Polish group actions***, London Mathematical Society Lecture Note Series, no. 232, Cambridge University Press, 1966.

[2] A. Connes, ***Noncommutative geometry***, Academic Press, 1994.

[3] A. Connes, J. Feldman, and B. Weiss, *An amenable equivalence relation is generated by a single transformation*, ***Ergodic Theory and Dynamical Systems***, vol. 1 (1981), pp. 430–450.

[4] R. Dougherty, S. Jackson, and A. S. Kechris, *The structure of hyperfinite Borel equivalence relations*, ***Transactions of the American Mathematical Society***, vol. 341 (1994), no. 1, pp. 193–225.

[5] E. G. Effros, *Transformation groups and C^*-algebras*, ***Annals of Mathematics***, vol. 81 (1965), pp. 38–55.

[6] J. Feldman and C. C. Moore, *Ergodic equivalence relations and von Neumann algebras, I*, ***Transactions of the American Mathematical Society***, vol. 234 (1977), pp. 289–324.

[7] H. Friedman and L. Stanley, *A Borel reducibility theory for classes of countable models*, this Journal, vol. 54 (1989), pp. 894–914.

[8] L. Fuchs, ***Abelian groups***, Academic Press, 1967.

[9] J. Glimm, *Locally compact transformation groups*, ***Transactions of the American Mathematical Society***, vol. 101 (1961), pp. 124–138.

[10] L. Harrington, A. S. Kechris, and A. Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations*, ***Journal of the American Mathematical Society***, vol. 3 (1990), no. 4, pp. 903–928.

[11] G. Hjorth, *Vaught's conjecture on analytic sets*, preprint, 1997.

[12] ———, *Around nonclassifiability for countable torsion-free abelian groups*, preprint, 1998.

[13] ———, *Classification and orbit equivalence relations*, preprint, 1998.

[14] G. Hjorth and A. S. Kechris, *The complexity of the classification of Riemann surfaces and complex manifolds*, ***Illinois Journal of Mathematics***, to appear.

[15] ———, *Borel equivalence relations and classifications of countable models*, ***Annals of Pure and Applied Logic***, vol. 82 (1996), pp. 221–272.

[16] ———, *New dichotomies for Borel equivalence relations*, ***Bulletin of Symbolic Logic***, vol. 3 (1997), no. 3, pp. 329–346.

[17] G. Hjorth, A. S. Kechris, and A. Louveau, *Borel equivalence relations induced by actions of the symmetric group*, ***Annals of Pure and Applied Logic***, vol. 92 (1998), pp. 63–112.

[18] S. Jackson, A. S. Kechris, and A. Louveau, *Countable Borel equivalence relations*, in preparation.

[19] A. S. Kechris, *Amenable equivalence relations and Turing degrees*, this Journal, vol. 56 (1991), pp. 182–194.

[20] ———, ***Classical descriptive set theory***, Graduate Texts in Mathematics, no. 156, Springer-Verlag, 1995.

[21] A. S. Kechris and A. Louveau, *The classification of hypersmooth Borel equivalence relations*, ***Journal of the American Mathematical Society***, vol. 10 (1997), no. 1, pp. 215–242.

[22] A. S. Kechris and Y. N. Moschovakis (editors), ***Cabal seminar 76–77***, Lecture Notes in Mathematics, no. 689, Springer-Verlag, 1978.

[23] A. S. Kechris and N. Sofronidis, *A strong generic ergodicity property of unitary conjugacy*, preprint, 1997.

[24] A. Louveau and B. Velickovic, *A note on Borel equivalence relations*, ***Proceedings of the American Mathematical Society***, vol. 120 (1994), pp. 255–259.

[25] Y. N. Moschovakis, ***Descriptive set theory***, North-Holland, 1980.

[26] D. Ornstein and B. Weiss, *Ergodic theory of amenable group actions, I. The Rohlin lemma*, ***Bulletin of the American Mathematical Society***, vol. 2 (1980), pp. 161–164.

[27] J. Silver, *Counting the number of equivalence classes of Borel and co-analytic equivalence relations*, ***Annals of Mathematical Logic***, vol. 18 (1980), pp. 1–28.

[28] T. Slaman and J. Steel, *Definable functions on degrees*, ***Cabal seminar 81–85***, Lecture Notes in Mathematics, no. 1333, Springer-Verlag, 1988, pp. 37–55.

[29] S. Solecki, *Analytic ideals*, ***Bulletin of Symbolic Logic***, vol. 2 (1996), no. 3, pp. 339–348.

[30] S. Thomas and B. Velickovic, *On the complexity of the isomorphism relation for fields of finite transcendence degree*, preprint, 1998.

[31] ———, *On the complexity of the isomorphism relation for finitely generated groups*, preprint, 1998.

[32] B. Weiss, *Measurable dynamics*, ***Contemporary Mathematics***, vol. 26 (1984), pp. 395–421.

DEPARTMENT OF MATHEMATICS
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CA 91125, USA
E-mail: kechris@caltech.edu