NEW ELEMENTARY COMPONENTS OF THE GORENSTEIN LOCUS OF THE HILBERT SCHEME OF POINTS

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Abstract. We construct new explicit examples of nonsmoothable Gorenstein algebras with Hilbert function $(1, n, n, 1)$. This gives a new infinite family of elementary components in the Gorenstein locus of the Hilbert scheme of points and solves the cubic case of Iarrobino's conjecture.

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1. Introduction

Hilbert schemes of points were first constructed by Grothendieck in 1960-61 [Gro95]. Since then they have found many applications, notably in combinatorics [Hai03] and in constructing hyperkähler manifolds [Bea83]. Hilbert schemes of points also appear in complexity theory while studying tensor and border ranks [Lan17]. One of the more important results of the theory is that by Fogarty stating that the Hilbert scheme of points of a smooth, irreducible surface is itself smooth and irreducible [Fog68]. The Hilbert scheme of points for three- and higher-dimensional varieties is singular and not well understood.

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The topology of the Hilbert scheme of points is still poorly understood and finding its irreducible components remains a challenge. The building blocks for them are the elementary components, those parametrizing subschemes with one-point support. Points of the Hilbert scheme of points corresponding to Gorenstein zero-dimensional subschemes form an open set, called the *Gorenstein locus*. Few components of this locus are known. Additionally, the smooth points of these components are often not explicitly given. Explicit points outside of the smoothable component are of interest in applications to tensors [JLP22] and in computations.

Let $S = k[\alpha_1, ..., \alpha_n]$ and $P = k[x_1, ..., x_n]$ be polynomial rings of n variables over a field k of characteristic 0. There is an action of S on P defined as follows

$$\alpha_1^{b_1}\alpha_2^{b_2}...\alpha_n^{b_n} \cdot x_1^{a_1}x_2^{a_2}...x_n^{a_n} = \begin{cases} x_1^{a_1-b_1}x_2^{a_2-b_2}...x_n^{a_n-b_n} & \text{if } \forall_i\, a_i \geq b_i \\ 0 & \text{otherwise.} \end{cases}$$

This action is called *contraction* and is somewhat similar to differentiation. For any polynomial $f \in P$ the set $\mathrm{Ann}(f) = \{s \in S : s \cdot f = 0\}$ forms a homogeneous ideal of S, called the *apolar ideal*. In this paper, we present the following result.

Theorem 1. *If $n \geq 6$ (except for $n = 7$), then for a general $f \in P$ homogeneous of degree 3 the ideal $\mathrm{Ann}(f)$ is a smooth point of an elementary component of the Hilbert scheme.*

This is a corollary of the following theorem. The *apolar algebra* $S/\mathrm{Ann}(f)$, denoted $\mathrm{Apolar}(f)$, with Hilbert function $(1, n, n, 1)$ is said to satisfy the small tangent space condition if the k-algebra $S/\mathrm{Ann}(f)^2$ has the smallest Hilbert function possible.

Theorem 2. *If $n \geq 6$ (except for $n = 7$), then for a general $f \in P$ homogeneous of degree 3 the apolar algebra $\mathrm{Apolar}(f)$ satisfies the small tangent space condition.*

Loosely speaking, Theorem 2 asserts that $\mathrm{Apolar}(f)$ has only trivial deformations of second order. For why it is false when $n \leq 5$ or $n = 7$ see [CN11] and [BCR22].

The characteristic 0 assumption can be removed for $n \geq 18$, see Theorem 3.1. We believe this to be true for all n, but small n would probably require a direct verification. We make this verification on computer for characteristics 0,2, and 3. For n less than 13 this was also done by Iarrobino and Kanev [IK99, Lemma 6.21].

Summing up, we show Theorem 2 to hold for all characteristics when $n \geq 18$ and for characteristics 0,2, and 3 in general. This resolves the following conjecture, posed by Iarrobino and Kanev, in the case $d = 3$.

Conjecture 3 ([IK99], Conjecture 6.30)**.** *Let d be an odd integer. If one of the following conditions holds*

(1) *$n = 4$ and $d \geq 15$,*
(2) *$n = 5$ and $d \geq 5$,*
(3) *$n \geq 6$ and $d \geq 3$ (except for $(n, d) = (7, 3)$),*

then for a general $f \in P$ homogeneous of degree d the apolar algebra $\mathrm{Apolar}(f)$ satisfies the small tangent space condition.

For $d > 3$ essentially nothing is known.

In order to prove Theorem 2 it suffices to give, for every n, a single example of a polynomial whose apolar algebra satisfies the small tangent space condition. In our proof, we give three rather simple ones covering all n greater than 8. For n divisible by 3 we take the following polynomial

$$F = \sum_{i=1}^{m} a_i b_i c_i + a_i a_{i+1}^2 + b_i b_{i+1}^2 + c_i c_{i+1}^2.$$

As a consequence, we provide an explicit description of a smooth point of an elementary component of the Hilbert scheme. Since the apolar ideals associated to our examples admit a set of generators consisting only of monomials and binomials they are also convenient from a computational point of view. Moreover, for $n \geq 18$, our proof does not use any computer computations. This is important in complexity theory, where structure tensors of such algebras correspond to 1-generic tensors [Lan17, Section 5.6.1].

We begin, in chapter 2, by giving all necessary background such as contraction, apolar algebras, and Gorenstein rings. It is also there where we compute the tangent space to the Hilbert scheme and present equivalent descriptions of the small tangent space condition. Then, in chapter 3, we prove Theorem 2 for sufficiently large n. Small n are taken care of in chapter 4 where we verify them on a computer.

2. Preliminaries

This chapter introduces all notions related to our study. In section 2.1, we define the Hilbert scheme and describe its tangent space. In section 2.2, we introduce apolar algebras and divided power rings associated to polynomial rings. In section 2.3, we introduce the dualizing functor $(-)^\vee$ and define zero-dimensional Gorenstein local rings. In section 2.4, we define the small tangent space condition and relate it to the tangent space of the Hilbert scheme. Finally, in section 2.5, we give a link between the small tangent space condition and smooth points on elementary components of the Hilbert scheme.

2.1. **Hilbert scheme.**

In this section, we introduce the notion of deformation. The deformation functor turns out to be representable by a scheme, called the Hilbert scheme of points.

Let k be a field. Given two k-algebras S and A, we write S_A for the ring $S \otimes_k A$ treated as an A-algebra.

Definition 2.1. Let S be a fixed, finitely generated k-algebra. The *embedded deformation functor* $\mathrm{Def}_{\mathrm{emb}}\colon k\text{-Alg} \to \mathrm{Set}$ assigns to a k-algebra A the set of isomorphism classes of ideals $I \triangleleft S_A$ such that S_A/I is a locally free A-module of finite rank. To a morphism $A \to B$ of k-algebras the functor $\mathrm{Def}_{\mathrm{emb}}$ assigns the function taking $I \in \mathrm{Def}_{\mathrm{emb}}(A)$ to $IS_B \in \mathrm{Def}_{\mathrm{emb}}(B)$.

We consider the following theorem as the definition of the Hilbert scheme.

Theorem 2.2 ([HS04], Theorem 1.1)**.** *Let S be a fixed, finitely generated k-algebra. Then, there exists a finite type k-scheme $\mathcal{H}$, called the Hilbert scheme of points, representing the deformation functor $\mathrm{Def}_{\mathrm{emb}}$ in the sense that there is an isomorphism of sets $\mathrm{Def}_{\mathrm{emb}}(A) \cong \mathrm{Mor}_{\mathrm{Sch}}(\mathrm{Spec}\, A, \mathcal{H})$ natural in A.*

Note that $\mathrm{Def}_{\mathrm{emb}}(k)$, and hence $\mathcal{H}(k)$, is the set of ideals $I \triangleleft S$ such that $\dim_k S/I$ is finite. Since S/I is Noetherian $\dim_k S/I$ being finite is equivalent to S/I being zero-dimensional.

Theorem 2.3 ([Str96], Theorem 10.1)**.** *Let S be a finitely generated k-algebra, and let $\mathcal{H}$ be its associated Hilbert scheme. For an ideal $I \lhd S$ such that $\dim_k S/I$ is finite, hence for a rational point of $\mathcal{H}$, the tangent space of $\mathcal{H}$ at I is isomorphic to $\mathrm{Hom}_S(I, S/I)$.*

2.2. Apolar algebras.

In this section, following [Jel17], we introduce the notion of apolar algebra. This is the easiest to construct and, in the case of zero-dimensional, graded local rings, the only example of a Gorenstein ring (see Theorem 2.11).

Consider a polynomial ring $S = k[\alpha_1, .., \alpha_n]$ over a field k. Recall that S is a graded k-algebra with the ideal S_+ equal to $(\alpha_1, ..., \alpha_n)$. We denote by $S^\vee$ the S-module $\mathrm{Hom}_k(S, k)$ of k-linear functionals on S. Let $\langle -, - \rangle \colon S \times S^\vee \to k$ be the natural map given by evaluation.

Definition 2.4. Let P be the submodule $\{f \in S^\vee \colon \forall_{N \gg 0} \langle (S_+)^N, f \rangle = 0\}$ of $S^\vee$. The induced action of S on P is called *contraction*.

We now give a more concrete description of contraction. If $\mathbf{a} = (a_1, a_2, ..., a_n)$ is a multi-index, we write $\alpha^{\mathbf{a}}$ for the monomial $\alpha_1^{a_1}\alpha_2^{a_2}...\alpha_n^{a_n} \in S$. For every multi-index $\mathbf{a}$ there is a unique functional $\mathbf{x}^{[\mathbf{a}]} \in P$ dual to $\alpha^{\mathbf{a}}$ in the sense that for all multi-indices $\mathbf{b}$ we have

$$\langle \alpha^{\mathbf{b}}, \mathbf{x}^{[\mathbf{a}]} \rangle = \begin{cases} 1 & \text{if } \mathbf{a} = \mathbf{b} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathbf{x}^{[\cdot]}$ form a k-basis of P. The quantity $\sum \mathbf{a} :\equiv \sum a_i$ is called the *degree* of $\mathbf{x}^{[\mathbf{a}]}$. An element $f \in P$ is called *homogeneous* of degree d if f is contained in $\mathrm{span}_k(\mathbf{x}^{[\mathbf{a}]} \colon \sum \mathbf{a} = d)$. Contraction behaves on the basis as follows

$$\alpha^{\mathbf{b}} \cdot \mathbf{x}^{[\mathbf{a}]} = \begin{cases} \mathbf{x}^{[\mathbf{a}-\mathbf{b}]} & \text{if } \mathbf{a} \geq \mathbf{b} \ (\forall_i\, a_i \geq b_i) \\ 0 & \text{otherwise.} \end{cases}$$

Though we do not need this, we can equip P with a ring structure. Multiplication on P is given by the formula

$$\mathbf{x}^{[\mathbf{a}]}\mathbf{x}^{[\mathbf{b}]} = \binom{\mathbf{a}+\mathbf{b}}{\mathbf{a}} \mathbf{x}^{[\mathbf{a}+\mathbf{b}]}$$

where $\binom{\mathbf{a}+\mathbf{b}}{\mathbf{a}} = \prod \binom{a_i+b_i}{a_i}$. In this way, P is a *divided power ring.*

Definition 2.5. Let $f \in P$, and let $\mathrm{Ann}(f)$ denote the ideal $\{s \in S \colon s \cdot f = 0\}$ of S. The k-algebra $S/\mathrm{Ann}(f)$ is called the *apolar algebra* of f, and is denoted $\mathrm{Apolar}(f)$.

2.3. Zero-dimensional Gorenstein local rings.

Throughout this section let $(A, \mathfrak{m}, k)$ be a zero-dimensional, finitely generated local k-algebra. We denote by A-mod the category of finitely generated modules over A.

We recall basic definitions and properties concerning zero-dimensional Gorenstein rings following [Eis95, Chapter 21].

Definition 2.6. A functor $E : A\text{-mod}^{\mathrm{op}} \to A\text{-mod}$ is called *dualizing* if $E^2 \cong 1$.

Proposition 2.7 ([Eis95], Proposition 21.1)**.** *If $E \colon A\text{-mod}^{\mathrm{op}} \to A\text{-mod}$ is dualizing, then there is an isomorphism of functors $E \cong \mathrm{Hom}_A(-, E(A))$. Moreover, up to isomorphism there exists at most one dualizing functor on A-mod.*

Consider the functor $(-)^\vee :\equiv \mathrm{Hom}_k(-,k)$. For an A-module M, the vector space $\mathrm{Hom}_k(M,k)$ naturally forms an A-module with the A-action given by

$$(a\cdot\varphi)(m) = \varphi(am)$$

where $\varphi \in M^\vee$, $m \in M$, and $a \in A$. Therefore, we can view $(-)^\vee$ as a functor $A\text{-mod}^{\mathrm{op}} \to A\text{-mod}$.

Proposition 2.8 ([Eis95], Section 21.1)**.** *The functor $(-)^\vee$ is dualizing.*

Combining Propositions 2.7 and 2.8 shows that up to isomorphism there exists a unique dualizing functor on A-mod.

Definition 2.9. We say that A is *Gorenstein* if $A^\vee \cong A$.

If A is Gorenstein, then in view of Proposition 2.7 we have an isomorphism of functors $\mathrm{Hom}_A(-,A) \cong \mathrm{Hom}_A(-,A^\vee) \cong (-)^\vee$. In particular $\mathrm{Hom}_A(-,A)$ is dualizing. Conversely, if $\mathrm{Hom}_A(-,A)$ is dualizing, then, by Proposition 2.7 and Yoneda lemma, A is isomorphic to $A^\vee$, so A is Gorenstein.

We have the following characterization of zero-dimensional Gorenstein rings.

Proposition 2.10 ([Eis95], Proposition 21.5)**.** *Let $(A,\mathfrak{m},k)$ be a zero-dimensional, finitely generated local k-algebra. Then, the following conditions are equivalent.*

1. *A is Gorenstein.*
2. *A is injective as an A-module.*
3. *The annihilator of the maximal ideal $\mathrm{Ann}(\mathfrak{m}) \subset A$ is one dimensional.*
4. *$\mathrm{Hom}_A(-,A)$ is dualizing.*

2.4. Small tangent space condition.

In this section, we introduce the small tangent space condition and better describe the tangent space of the Hilbert scheme associated to a polynomial ring. We also prove Proposition 2.18, which is needed in the proof of Theorem 3.1.

As in section 2.2, let S be a polynomial ring of n variables over a field k, and let P be its associated divided power ring. We denote by $\mathcal{H}$ the Hilbert scheme associated to S. Recall that $\mathrm{Apolar}(f) = S/\mathrm{Ann}(f)$. If $I = \mathrm{Ann}(f)$ we write S/I and $\mathrm{Apolar}(f)$ interchangeably.

When saying that a graded module M has Hilbert function $(h_0, h_1, ..., h_j)$, $h_i \in \mathbb{N}$ we mean that $H(M)_i$ is equal to h_i for $i \in \{0,1,...,j\}$ and that $H(M)_i$ is equal to 0 for $i \notin \{0,1,...,j\}$. For example, S has Hilbert function $(1, n, \binom{n+1}{2}, \binom{n+2}{3}, ...)$. We denote a shift of gradation in square brackets, so $M[d]_i = M_{d+i}$.

Theorem 2.11 ([Iar94], Lemma 1.2 and Theorem 1.5)**.** *For every nonzero $f \in P$ homogeneous of degree d the apolar algebra $\mathrm{Apolar}(f)$ is a graded zero-dimensional Gorenstein local ring. Moreover, there is a graded isomorphism $\mathrm{Apolar}(f) \cong \mathrm{Apolar}(f)^\vee[-d]$.*

For every homogeneous ideal $I \lhd S$, if S/I is a zero-dimensional Gorenstein local ring, then there exists homogeneous $f \in P$ such that $I = \mathrm{Ann}(f)$.

We can now describe the tangent space of the Hilbert scheme more concretely.

Proposition 2.12. *Let $f \in P$ be homogeneous of degree d, and let $I = \mathrm{Ann}(f)$. Then, the tangent space $T_I\mathcal{H}$ is isomorphic as a graded module to $(I/I^2)^\vee[-d]$.*

Proof. By Theorem 2.3 the tangent space to $\mathcal{H}$ at I is isomorphic to $\mathrm{Hom}_S(I, S/I)$. By the tensor-hom adjunction and the isomorphism $I \otimes_S S/I \cong I/I^2$ we get

$$\mathrm{Hom}_S(I, S/I) \cong \mathrm{Hom}_{S/I}(I/I^2, S/I).$$

Then, Theorem 2.11 yields

$$\mathrm{Hom}_{S/I}(I/I^2, S/I) \cong \mathrm{Hom}_{S/I}(I/I^2, (S/I)^\vee)[-d].$$

Now, again by the tensor-hom adjunction, we obtain

$$\mathrm{Hom}_{S/I}(I/I^2, (S/I)^\vee)[-d] \cong (I/I^2)^\vee[-d].$$

Hence $T_I\mathcal{H} \cong (I/I^2)^\vee[-d]$ as required. □

From now on, since we are mainly interested in degree 3 homogeneous elements of P, we reduce ourselves to this special case.

Let $f \in P$ be homogeneous of degree 3, and let $I = \mathrm{Ann}(f)$. Since $S_{\geq 4}$ is contained in I the Hilbert function of S/I can be nonzero only in degrees $0, 1, 2$, and 3. Moreover, since $(S/I)^\vee \cong (S/I)[3]$ the Hilbert function is symmetric in the sense that $H(S/I)_0 = H(S/I)_3$ and $H(S/I)_1 = H(S/I)_2$. Clearly, $H(S/I)_0 = 1$ and $H(S/I)_1 \leq n$. In view of the following proposition, the case $H(S/I)_1 < n$ might be considered degenerate.

Proposition 2.13 ([IK99], Proposition 3.12)**.** *There is an open, dense subset U of the space of cubics* $\mathrm{Spec}\,\mathrm{Sym}(P_3)^\vee$ *such that for all rational points $f \in U(k)$ the Hilbert function of* $\mathrm{Apolar}(f)$ *is* $(1, n, n, 1)$*.*

Proposition 2.14. *Let $f \in P$ be homogeneous of degree* 3*, and let $I = \mathrm{Ann}(f)$. If* $\mathrm{Apolar}(f)$ *has Hilbert function* $(1, n, n, 1)$*, then $H(S/I^2)_4 \geq n$.*

Proof. This follows from [Jel18, Lemma 3.4]. □

Definition 2.15. Let $f \in P$ be homogeneous of degree 3, and let $I = \mathrm{Ann}(f)$. Then, we say that $\mathrm{Apolar}(f)$ satisfies the *small tangent space condition* if $\mathrm{Apolar}(f)$ has Hilbert function $(1, n, n, 1)$ and $H(S/I^2)_4 = n$, $H(S/I^2)_5 = 0$.

Proposition 2.16. *Let $f \in P$ be homogeneous of degree* 3*. Then,* $\mathrm{Apolar}(f)$ *satisfies the small tangent space condition if and only if the tangent space of $\mathcal{H}$ at $I = \mathrm{Ann}(f)$ has Hilbert function n, $\binom{n+2}{3} - 1$, $\binom{n+1}{2} - n$ in degrees $-1, 0, 1$ respectively, and* 0 *elsewhere.*

Proof. First suppose that $\mathrm{Apolar}(f)$ satisfies the small tangent space condition. In view of Proposition 2.12 we need to compute the Hilbert function of I/I^2. The ring S/I has Hilbert function $(1, n, n, 1)$, so $I_{\leq 1} = 0$. It follows that S/I^2 is all of S in degrees $0, 1, 2$, and 3. Furthermore, since S/I satisfies the small tangent space condition S_5 is contained in I^2, so $S_{\geq 6}$ is contained in I^2 as well, which means that $H(S/I^2)_{\geq 6}$ is 0. Thus, the Hilbert function of S/I^2 is $(1, n, \binom{n+1}{2}, \binom{n+2}{3}, n)$. Furthermore, since S/I has Hilbert function $(1, n, n, 1)$, we get that the Hilbert function of I/I^2 is equal to $(0, 0, \binom{n+1}{2} - n, \binom{n+2}{3} - 1, n)$, hence the tangent space $T_I\mathcal{H} \cong (I/I^2)^\vee[-3]$ has the desired Hilbert function.

Now suppose that $T_I\mathcal{H}$ has the given Hilbert function. Then, I/I^2 has Hilbert function $(0, 0, \binom{n+1}{2} - n, \binom{n+2}{3} - 1, n)$. Since $S_{\geq 4} \subset I$ this means that $H(S/I^2)_4 = n$ and $H(S/I^2)_5 = 0$. Moreover, since I^2 is not all of S we have $H(I)_0 = 0$, so $H(I^2)_1 = 0$. Thus, $H(S/I)_0 = 1$ and $H(S/I)_1 = n$, which since S/I is Gorenstein implies that S/I has Hilbert function $(1, n, n, 1)$ as required. □

Corollary 2.17. *Let $f \in P$ be homogeneous of degree 3 such that* $\operatorname{Apolar}(f)$ *has Hilbert function* $(1, n, n, 1)$. *Then,* $\operatorname{Apolar}(f)$ *satisfies the small tangent space condition if and only if the tangent space of $\mathcal{H}$ at* $\operatorname{Ann}(f)$ *has the smallest Hilbert function possible.*

Proposition 2.18. *Let $f \in P$ be homogeneous of degree* 3 *such that* $\operatorname{Apolar}(f)$ *satisfies the small tangent space condition. Then, for a general $g \in P$ homogeneous of degree* 3 *the apolar algebra* $\operatorname{Apolar}(g)$ *satisfies the small tangent space condition.*

Proof. Let U be the open subscheme of $\operatorname{Spec}\operatorname{Sym}(P_3)^\vee$ from Proposition 2.13. By [Jel18, Section 2.2] there is a family $Z \subset U \times \mathbb{A}^n_k \to U$ such that the fibre over $f \in U(k)$ is $\operatorname{Spec}\operatorname{Apolar}(f)$. Hence, the claim follows from Corollary 2.17 and upper semi-continuity of rank for the quasicoherent sheaf $I(Z)/I(Z)^2$. □

2.5. **Elementary components.**

We now describe the connection between the small tangent space condition and smooth points on elementary components of the Hilbert scheme.

As in section 2.4, we only consider degree 3 homogeneous elements of P.

Definition 2.19. An irreducible component Z of the Hilbert scheme is called *elementary* if for all rational points $I \in Z(k)$ the S-module S/I is supported at a single point.

Proposition 2.20. *Let $f \in P$ be homogeneous of degree 3. If* $\operatorname{Apolar}(f)$ *satisfies the small tangent space condition, then $I = \operatorname{Ann}(f)$ is a smooth point of an elementary component of $\mathcal{H}$.*

Proof. Since $\operatorname{Apolar}(f)$ satisfies the small tangent space condition, by Proposition 2.16, we have $\dim_k \operatorname{Hom}_S(I, S/I)_{<0} = n$. Hence, by [Jel19, Theorem 4.5 and Corollary 4.7], all irreducible components containing I are elementary. Smoothness follows from the discussion in [IK99, Proof of Lemma 6.21]. □

3. Small tangent space condition in degree 3

In this chapter, we prove Theorem 3.1, which confirms Conjecture 3 in the case where $d = 3$ and $n \geq 18$.

Theorem 3.1. *Let S be a polynomial ring of n variables over a field k, and let P be its associated divided power ring. If $n \geq 18$, then for a general $f \in P$ homogeneous of degree* 3 *the apolar algebra* $\operatorname{Apolar}(f)$ *satisfies the small tangent space condition.*

Proof. In view of Proposition 2.18 it suffices to find, for each $n \geq 18$, a single $f \in P$ such that $\operatorname{Apolar}(f)$ satisfies the small tangent space condition. We divide the proof into three cases: $n = 3m$, $n = 3m + 1$, and $n = 3m + 2$, where $m \geq 6$. They are resolved by Propositions 3.11, 3.18, and 3.24 respectively. □

Corollary 3.2. *Let S be a polynomial ring of n variables over a field k, and let P be its associated divided power ring. If $n \geq 18$, then for a general $f \in P$ homogeneous of degree* 3 *the ideal* $\operatorname{Ann}(f)$ *is a smooth point of an elementary component of the Hilbert scheme.*

Proof. This follows by combining Proposition 2.20 and Theorem 3.1. □

3.1. **Proof of Theorem 3.1; case $n = 3m$.**

Let $S = k[a_i, b_i, c_i]_{i=1}^m$ be a polynomial ring of $n = 3m$ variables. Recall that we assume $m \geq 6$. When writing indices we treat them modulo m.

Consider the following polynomial

$$F = \sum_{i=1}^{m} a_i b_i c_i + a_i a_{i+1}^2 + b_i b_{i+1}^2 + c_i c_{i+1}^2.$$

Also by F we denote its dual element in the divided power ring associated to S.

Let I be the smallest ideal such that the following remark holds.

Remark 3.3. For all $x, y \in \{a, b, c\}$, and for all indices i, j, if $j \notin \{i-1, i, i+1\}$, then $x_i y_j \in I$.

For all $x, y \in \{a, b, c\}$, and for all indices i, j, if $x \neq y$, then $x_i y_{i+1} - x_j y_{j+1} \in I$.

For all $x, y \in \{a, b, c\}$, and for all indices i, j, one of the following holds:

1. $x_i y_j$ is contained in I.
2. $x \neq y$ and $j \in \{i-1, i+1\}$.
3. There exists an index k and $p, q \in \{a, b, c\}$, $p \neq q$, such that $x_i y_j - p_k q_k \in I$.

The polynomial F is chosen such that $I \subset \operatorname{Ann} F$.

Let J denote the ideal $I^2 + \langle a_i a_{i+1} a_{i+2}^2, b_i b_{i+1} b_{i+2}^2, c_i c_{i+1} c_{i+2}^2 \mid i = 1, ..., m \rangle$. Note that F, I, and J are invariant under index translation and permutations of the set $\{a, b, c\}$.

We want to show that Apolar(F) satisfies the small tangent space condition. The main part of the proof is checking that all polynomials of degree 4 are contained in J, hence that $H(S/I^2)_4 = n$.

In this section, we use the following notation. For polynomials $Q, R \in S$ we write $Q \equiv R$ if Q is equal to R in S/I^2.

Lemma 3.4. *For all $x, y, z, w, p, q \in \{a, b, c\}$, and for all indices i, j, k,*

1. *there exists an index t such that $x_i y_j z_t w_{t+1}$ is in I^2.*
2. *there exist indices s, t such that each of $x_i y_{i+1} z_t w_{t+1}$, $x_s y_{s+1} z_j w_{j+1}$, and $x_s y_{s+1} z_t w_{t+1}$ is in I^2.*
3. *if $j, k \in \{i-1, i, i+1\}$, then there exists an index s such that each of $x_i y_j z_s w_{s+1}$ and $z_s w_{s+1} p_k q_k$ is in I^2.*

Proof. Throughout the proof we use the assumption $m \geq 6$.

We first prove (1). If $j \notin \{i+1, i+2, i+3\}$, then we take $t = i+2$, so that $x_i y_j z_{i+2} w_{i+3} = (x_i w_{i+3})(y_j z_{i+2}) \equiv 0$. If $j \in \{i+1, i+2, i+3\}$, then we take $t = i-2$, so that $x_i y_j z_{i-2} w_{i-1} = (x_i z_{i-2})(y_j w_{i-1}) \equiv 0$.

Now we show (2). If $j \notin \{i+1, i+2, i+3\}$, then we take $s = j+2$, $t = j-2$, so that

$$x_i y_{i+1} z_{j-2} w_{j-1} = (x_i z_{j-2})(y_{i+1} w_{j-1}) \equiv 0$$
$$x_{j+2} y_{j+3} z_j w_{j+1} = (x_{j+2} z_j)(y_{j+3} w_{j+1}) \equiv 0$$
$$x_{j+2} y_{j+3} z_{j-2} w_{j-1} = (x_{j+2} z_{j-2})(y_{j+3} w_{j-1}) \equiv 0.$$

If $j \in \{i+1, i+2, i+3\}$, then we take $s = j-2$, $t = j+1$, so that

$$x_i y_{i+1} z_{j+1} w_{j+2} = (x_i z_{j+1})(y_{i+1} w_{j+2}) \equiv 0$$
$$x_{j-2} y_{j-1} z_j w_{j+1} = (x_{j-2} z_j)(y_{j-1} w_{j+1}) \equiv 0$$
$$x_{j-2} y_{j-1} z_{j+1} w_{j+2} = (x_{j-2} z_{j+1})(y_{j-1} w_{j+2}) \equiv 0.$$

Finally, we prove (3). If $k = i+1$, then we take $s = i-3$, so that

$$x_i y_j z_{i-3} w_{i-2} = (x_i w_{i-2})(y_j z_{i-3}) \equiv 0$$
$$z_{i-3} w_{i-2} p_{i+1} q_{i+1} = (z_{i-3} p_{i+1})(w_{i-2} q_{i+1}) \equiv 0.$$

If $k \in \{i-1, i\}$, then we take $s = i+2$, so that

$$x_i y_j z_{i+2} w_{i+3} = (x_i z_{i+2})(y_j w_{i+3}) \equiv 0$$
$$z_{i+2} w_{i+3} p_k q_k = (z_{i+2} p_k)(w_{i+3} q_k) \equiv 0.$$

This finishes the proof. □

Lemma 3.5. *For all $x, y, z, w \in \{a, b, c\}$, $z \neq w$, and for all indices i, j, k, if either $x_j y_k \in I$ or $x \neq y$ and $k = j+1$, then the monomial $z_i w_{i+1} x_j y_k$ is contained in I^2.*

Proof. By symmetry we can assume $z = a$, $w = b$. Hence, we only need to examine monomials of the form $a_i b_{i+1} x_j y_k$, where either $x_j y_k \in I$ or $x \neq y$ and $k = j+1$.

First consider the case $x_j y_k \in I$. By Lemma 3.4 there exists an index t such that $a_t b_{t+1} x_j y_k \in I^2$. Therefore, $a_i b_{i+1} x_j y_k = (a_i b_{i+1} - a_t b_{t+1})(x_j y_k) + a_t b_{t+1} x_j y_k \equiv 0$.

Now consider the case where $x \neq y$ and $k = j+1$. Then, by Lemma 3.4, there exist indices t and s such that $a_t b_{t+1} x_j y_{j+1}$, $a_i b_{i+1} x_s y_{s+1}$, and $a_t b_{t+1} x_s y_{s+1}$ are in I^2. Therefore, $a_i b_{i+1} x_j y_{j+1} = (a_i b_{i+1} - a_t b_{t+1})(x_j y_{j+1} - x_s y_{s+1}) + a_i b_{i+1} x_s y_{s+1} + a_t b_{t+1} x_j y_{j+1} - a_t b_{t+1} x_s y_{s+1} \equiv 0$. □

Lemma 3.6. *For all indices i and j, the monomial $a_i a_j b_i c_i$ is contained in J.*

Proof. First assume that $j \notin \{i-2, i-1, i, i+1\}$. Then, we can rewrite $a_i a_j b_i c_i$ as follows.

$$\begin{aligned} a_i a_j b_i c_i &= (a_i b_i - c_{i-1} c_i)(a_j c_i) + (a_j c_{i-1})(c_i^2 - a_{i-1} b_{i-1}) + \\ &\quad + a_{i-1} a_j b_{i-1} c_{i-1} \equiv a_{i-1} a_j b_{i-1} c_{i-1} \end{aligned}$$

Hence, iterating this procedure, we get $a_i a_j b_i c_i \equiv a_j a_{j+2} b_{j+2} c_{j+2}$. Therefore, we just need to examine monomials $a_{i-2} a_i b_i c_i$, $a_{i-1} a_i b_i c_i$, $a_i^2 b_i c_i$, and $a_i a_{i+1} b_i c_i$.

Monomial $a_{i-2} a_i b_i c_i$ can be rewritten in the following way.

$$\begin{aligned} a_{i-2} a_i b_i c_i &= (a_i b_i - c_{i-1} c_i)(a_{i-2} c_i) + (a_{i-2} c_{i-1} - a_{i+2} c_{i+3})(c_i^2 - a_{i-1} b_{i-1}) + \\ &\quad + a_{i-2} a_{i-1} b_{i-1} c_{i-1} + (a_{i+2} c_i)(c_i c_{i+3}) - (a_{i-1} a_{i+2})(b_{i-1} c_{i+3}) \equiv \\ &\equiv a_{i-2} a_{i-1} b_{i-1} c_{i-1} \end{aligned}$$

Hence, we are reduced to considering $a_{i-1} a_i b_i c_i$, $a_i^2 b_i c_i$, and $a_i a_{i+1} b_i c_i$. Before we do so, we make some auxiliary computations.

$$\begin{aligned} a_i^4 &= (a_i^2 - b_{i-1} c_{i-1})^2 - (b_{i-1}^2 - b_{i-3} b_{i-2})(c_{i-1}^2 - c_{i-3} c_{i-2}) + \\ &\quad - b_{i-1}^2 c_{i-3} c_{i-2} - b_{i-3} b_{i-2} c_{i-1}^2 + b_{i-3} b_{i-2} c_{i-3} c_{i-2} + \\ &\quad + 2 a_i^2 b_{i-1} c_{i-1} \\ a_i^2 a_{i+1}^2 &= (a_i^2 - b_{i-1} c_{i-1})(a_{i+1}^2 - b_i c_i) + a_i^2 b_i c_i + (a_{i+1} b_{i-1})(a_{i+1} c_{i-1}) + \\ &\quad - b_{i-1} b_i c_{i-1} c_i \end{aligned}$$

Hence, Lemma 3.5 shows $a_i^4 \equiv 0$ and $a_i^2 a_{i+1}^2 \equiv a_i^2 b_i c_i$. We make further computations, where we assume $j \notin \{i-1, i, i+1\}$.

$$a_i^2 b_i c_i \equiv a_i^2 a_{i+1}^2 = (a_{i+2}^2 - a_i a_{i+1})^2 - a_{i+2}^4 + 2 a_i a_{i+1} a_{i+2}^2 \equiv 2 a_i a_{i+1} a_{i+2}^2 \in J$$
$$a_i a_j a_{j+1}^2 = (a_i a_j)(a_{j+1}^2 - a_{j-1} a_j) + a_i a_{j-1} a_j^2 \equiv a_i a_{j-1} a_j^2$$

Iterating the last computation we get $a_ia_ja_{j+1}^2 \equiv a_ia_{i+1}a_{i+2}^2$. We are now ready to rewrite $a_{i-1}a_ib_ic_i$ and $a_ia_{i+1}b_ic_i$.

$$\begin{aligned}
a_{i-1}a_ib_ic_i &= (a_{i-1}a_i - a_{i+1}^2)(b_ic_i - a_{i+1}^2) + a_{i-1}a_ia_{i+1}^2 + \\
&+ (a_{i+1}b_i - a_{i+3}b_{i+2})(a_{i+1}c_i - a_{i-1}c_{i-2}) + (a_{i-1}a_{i+1})(b_ic_{i-2}) + \\
&+ (a_{i+1}a_{i+3})(b_{i+2}c_i) - (a_{i-1}a_{i+3})(b_{i+2}c_{i-2}) - a_{i+1}^4 \equiv \\
&\equiv a_{i-1}a_ia_{i+1}^2 \in J \\
a_ia_{i+1}b_ic_i &= (a_ia_{i+1} - a_{i+2}^2)(b_ic_i - a_{i+1}^2) + (a_ia_{i+1} - a_{i+2}^2)(a_{i+1}^2 - a_{i-1}a_i) + \\
&+ a_{i-1}a_i^2a_{i+1} + a_{i+1}^2a_{i+2}^2 - (a_{i-1}a_{i+2})(a_ia_{i+2}) + (a_{i+2}b_i)(a_{i+2}c_i) + \\
&- a_{i+1}^2a_{i+2}^2 \equiv a_{i+1}a_{i+2}a_{i+3}^2 \in J
\end{aligned}$$

This finishes the proof. □

Lemma 3.7. *For all $x, y, z, w \in \{a, b, c\}$, $z \neq w$, and for all indices i, j, k, the monomial $z_iw_{i+1}x_jy_k$ is contained in J.*

Proof. By symmetry we can assume $z = a$, $w = b$. Hence, we only need to examine monomials of the form $a_ib_{i+1}x_jy_k$.

Lemma 3.5 covers some of the cases. In any other there exist $p, q \in \{a, b, c\}$, $p \neq q$, and an index t such that $x_jy_k - p_tq_t \in I$. Furthermore, since $x_jy_k \notin I$ we have $k, t \in \{j-1, j, j+1\}$, hence, by Lemma 3.4, we can choose an index s such that both $a_sb_{s+1}x_jy_k$ and $a_sb_{s+1}p_tq_t$ are in I^2. Then, we have $a_ib_{i+1}x_jy_k = (a_ib_{i+1} - a_sb_{s+1})(x_jy_k - p_tq_t) + a_ib_{i+1}p_tq_t + a_sb_{s+1}x_jy_k - a_sb_{s+1}p_tq_t \equiv a_ib_{i+1}p_tq_t$. Hence, to finish the proof it suffices to consider monomials of the form $a_ib_{i+1}x_jy_j$ with $x \neq y$.

If $j \notin \{i-1, i, i+1, i+2\}$, then $a_ib_{i+1}x_jy_j = (a_ix_j)(b_{i+1}y_j) \equiv 0$.

Since $x \neq y$ one of them is not a, say $x \neq a$. Then, if $j = i-1$, we can write $a_ib_{i+1}x_{i-1}y_{i-1}$ as $x_{i-1}a_iy_{i-1}b_{i+1}$, and since $y_{i-1}b_{i+1} \in I$, Lemma 3.5 applies showing $x_{i-1}a_iy_{i-1}b_{i+1} \equiv 0$. Similarly, if $j = i+2$, since one of x, y in not b, say $x \neq b$, we know, by Lemma 3.5, that $b_{i+1}x_{i+2}a_iy_{i+2} \equiv 0$.

In the case $j = i+1$ we need to consider monomials $a_ia_{i+1}b_{i+1}^2$, $a_ia_{i+1}b_{i+1}c_{i+1}$, and $a_ib_{i+1}^2c_{i+1}$. Monomial $a_ia_{i+1}b_{i+1}c_{i+1}$ is a special case of Lemma 3.6. Others can be rewritten as follows.

$$\begin{aligned}
a_ia_{i+1}b_{i+1}^2 &= (a_ib_{i+1} - a_{i-2}b_{i-1})(a_{i+1}b_{i+1} - c_{i+2}^2) + a_ib_{i+1}c_{i+2}^2 + \\
&+ (a_{i-2}a_{i+1})(b_{i-1}b_{i+1}) - (a_{i-2}c_{i+2})(b_{i-1}c_{i+2}) \equiv 0 \\
a_ib_{i+1}^2c_{i+1} &= (a_ib_{i+1} - a_{i-2}b_{i-1})(b_{i+1}c_{i+1} - a_{i+2}^2) + a_ia_{i+2}^2b_{i+1} + \\
&+ (a_{i-2}b_{i+1})(b_{i-1}c_{i+1}) - (a_{i-2}a_{i+2})(b_{i-1}a_{i+2}) \equiv 0
\end{aligned}$$

where $a_ib_{i+1}c_{i+2}^2$ and $a_ia_{i+2}^2b_{i+1}$ are in I^2 by Lemma 3.5.

It remains to consider $j = i$. We need to examine three monomials, $a_i^2b_ib_{i+1}$, $a_i^2b_{i+1}c_i$, and $a_ib_ib_{i+1}c_i$. We can rewrite them as follows.

$$\begin{aligned}
a_i^2b_ib_{i+1} &= (a_i^2 - a_{i-2}a_{i-1})(b_ib_{i+1} - b_{i+2}^2) + (a_ib_{i+2})^2 + \\
&+ (a_{i-2}b_i)(a_{i-1}b_{i+1}) - (a_{i-2}b_{i+2})(a_{i-1}b_{i+2}) \equiv 0 \\
a_i^2b_{i+1}c_i &= (a_i^2 - a_{i-2}a_{i-1})(b_{i+1}c_i - b_{i+3}c_{i+2}) + (a_ib_{i+3})(a_ic_{i+2}) + \\
&+ (a_{i-2}c_i)(a_{i-1}b_{i+1}) - (a_{i-2}c_{i+2})(a_{i-1}b_{i+3}) \equiv 0
\end{aligned}$$

Since, by symmetry, $a_ib_ib_{i+1}c_i$ is a special case of Lemma 3.6 the proof is finished. □

Lemma 3.8. *For all $x, y, z, w \in \{a, b, c\}$, and for all indices i, j, k, t such that $x_i y_j \in I$, the monomial $x_i y_j z_k w_t$ is in J.*

Proof. If $z_k w_t$ is contained in I as well, then $x_i y_j z_k w_t \in I^2$. If $z_k w_t$ is of the form $p_s q_{s+1}$ for some index s and $p, q \in \{a, b, c\}$, $p \neq q$, then Lemma 3.7 applies.

In any other case there exist $p, q \in \{a, b, c\}$, $p \neq q$, and an index s such that $z_k w_t - p_s q_s \in I$. Therefore, we have $x_i y_j z_k w_t = (x_i y_j)(z_k w_t - p_s q_s) + x_i y_j p_s q_s \equiv x_i y_j p_s q_s$. Hence, by symmetry, it suffices to examine monomials of the form $a_i b_i x_j y_k$ with $x_j y_k \in I$.

If any of j, k are in $\{i-1, i+1\}$, then Lemma 3.7 applies. If both j and k are not in $\{i-1, i, i+1\}$, then $a_i b_i x_j y_k = (a_i x_j)(b_i y_k) \equiv 0$. Thus, we can assume one of j, k equal to i, say $k = i$.

Now, if $y \neq c$, since $x_j y_i \in I$, we get $a_i b_i x_j y_i = (a_i b_i - c_{i-1} c_i)(x_j y_i) + c_{i-1} y_i c_i x_j$, hence Lemma 3.7 applies. We are therefore reduced to monomials of the form $a_i b_i c_i x_j$. By symmetry we can assume $x = a$, and use Lemma 3.6 to finish the proof. □

Proposition 3.9. *Every degree 4 homogenous polynomial of S is contained in J.*

Proof. In view of Lemma 3.8 it suffices to verify monomials where no two indices differ by more than 1. Moreover, if two indices differ exactly by 1, and not all letters are equal, then Lemma 3.7 applies. Hence, by symmetry, it suffices to examine $a_i^2 b_i c_i, a_i^2 b_i^2, a_i^3 b_i, a_i^4, a_i^3 a_{i+1}, a_i a_{i+1}^3$, and $a_i^2 a_{i+1}^2$. Clearly, $a_i^2 b_i c_i$ is in J. Monomial a_i^4 was shown to be in I^2 in the proof of Lemma 3.6, hence, by symmetry, c_i^4 is also in I^2. We can rewrite the remaining monomials as follows.

$$\begin{aligned}
a_i^2 b_i^2 &= (a_i b_i - c_{i+1}^2)^2 + 2(a_i c_{i+1})(b_i c_{i+1}) - c_{i+1}^4 \equiv 0 \\
a_i^3 b_i &= (a_i b_i - c_{i+1}^2)(a_i^2 - b_{i-1} c_{i-1}) + (a_i b_{i-1})(b_i c_{i-1}) + (a_i c_{i+1})^2 + \\
&\quad - (c_{i-1} c_{i+1})(b_{i-1} c_{i+1}) \equiv 0 \\
a_i^3 a_{i+1} &= (a_i^2 - b_{i-1} c_{i-1})(a_i a_{i+1} - b_{i+1} c_{i+1}) + a_i^2 b_{i+1} c_{i+1} + \\
&\quad + a_i a_{i+1} b_{i-1} c_{i-1} - (b_{i-1} b_{i+1})(c_{i-1} c_{i+1}) \\
a_i a_{i+1}^3 &= (a_i a_{i+1} - b_{i+1} c_{i+1})(a_{i+1}^2 - b_i c_i) + a_i a_{i+1} b_i c_i + \\
&\quad + a_{i+1}^2 b_{i+1} c_{i+1} - b_i b_{i+1} c_i c_{i+1} \\
a_i^2 a_{i+1}^2 &= (a_i^2 - a_{i-2} a_{i-1})(a_{i+1}^2 - b_i c_i) + a_i^2 b_i c_i + (a_{i-2} a_{i+1})(a_{i-1} a_{i+1}) + \\
&\quad - (a_{i-2} b_i)(a_{i-1} c_i - a_{i+1} c_{i+2}) - (a_{i-2} a_{i+1})(b_i c_{i+2})
\end{aligned}$$

Then, Lemmas 3.6 and 3.7 finish the proof. □

Lemma 3.10. *For all $x, y \in \{a, b, c\}$, and for all indices i, j, the monomial $x_i y_j a_j b_j c_j$ is in I^2.*

Proof. By symmetry we can assume $y = a$. Note that $a_j^2 b_j$ annihilates F, so is in I. If $i \notin \{j-1, j, j+1\}$, then $x_i c_j \in I$, so $x_i a_j^2 b_j c_j = (x_i c_j)(a_j^2 b_j) \in I^2$. Now suppose $i \in \{j-1, j, j+1\}$. Either $x \neq b$ or $x \neq c$, by symmetry we can assume $x \neq c$. Then, in the case $i \in \{j-1, j+1\}$, we obtain $x_i a_j^2 b_j c_j = (x_i c_j - x_{i+3} c_{j+3})(a_j^2 b_j) + (x_{i+3} a_j)(a_j c_{j+3}) b_j \in I^2$. In the case $i = j$ we need to consider monomials $a_i^3b_ic_i$ and $a_i^2b_i^2c_i$. We rewrite them as follows.

$$a_i^3b_ic_i = (a_i^2b_i)(a_ic_i - b_{i-1}b_i) + (a_ib_i^2)(a_ib_{i-1} - a_{i-2}b_{i-3}) + (a_{i-2}a_i)(b_{i-3}b_i)b_i \equiv 0$$
$$a_i^2b_i^2c_i = (a_i^2c_i)(b_i^2 - b_{i-2}b_{i-1}) + (a_i^2b_{i-1})(b_{i-2}c_i) \equiv 0$$

This finishes the proof. □

Proposition 3.11. *The apolar algebra* $\mathrm{Apolar}(F)$ *satisfies the small tangent space condition.*

Proof. It is easy to check that no linear form annihilates F, hence $\mathrm{Apolar}(F)$ has Hilbert function $(1, n, n, 1)$. Proposition 3.9 implies that $H(S/I^2)_4 \leq n$, so $H(S/\mathrm{Ann}(F)^2) \leq n$. Thus, by Proposition 2.14, $H(S/\mathrm{Ann}(F)^2)_4 = n$. Finally, since monomials of the form $x_ia_ib_ic_i$ generate $(S/I^2)_4$ Lemma 3.10 implies that $H(S/I^2)_5 = 0$, so also $H(S/\mathrm{Ann}(F)^2) = 0$. □

3.2. **Proof of Theorem 3.1; case $n = 3m + 1$.**

Let $S' = k[a_i, b_i, c_i, d]_{i=1}^m$ be a polynomial ring of $n = 3m + 1$ variables. Recall that we assume $m \geq 6$. When writing indices we treat them modulo m.

Consider the following polynomial

$$F' = \sum_{i=1}^{m} a_ib_ic_i + a_ia_{i+1}^2 + b_ib_{i+1}^2 + c_ic_{i+1}^2 + a_ib_{i+1}d.$$

Also by F' we denote its dual element in the divided power ring associated to S'.

Let I' be the smallest ideal such that the following remark holds.

Remark 3.12. For all $x, y \in \{a, b, c\}$, and for all indices i, j, if $j \notin \{i - 1, i, i + 1\}$, then $x_iy_j \in I'$.

For all $x, y \in \{a, b, c\}$, and for all indices i, j, if $x \neq y$, then $x_iy_{i+1} - x_jy_{j+1} \in I'$.

For all $x, y \in \{a, b, c\}$, and for all indices i, j, one of the following holds.

(1) x_iy_j is contained in I'.
(2) $x \neq y$ and $j \in \{i - 1, i + 1\}$.
(3) There exists an index k and $p, q \in \{a, b, c\}$, $p \neq q$, such that $x_iy_j - p_kq_k \in I'$.

For any $x \in \{a, b, c\}$, and any index i, one of the following holds.

(1) x_id is contained in I'.
(2) There exists an index j and $p, q \in \{a, b, c\}$, $p \neq q$, such that $x_id - p_jq_j \in I'$.

The polynomial F' is chosen such that $I' \subset \mathrm{Ann}\, F'$.

Let J' denote the ideal $(I')^2 + \langle a_ia_{i+1}a_{i+2}^2, b_ib_{i+1}b_{i+2}^2, c_ic_{i+1}c_{i+2}^2 \mid i = 1, ..., m\rangle + \langle a_1b_1c_1d\rangle$. Note that F', I', and J' are invariant under index translation.

We want to show that $\mathrm{Apolar}(F')$ satisfies the small tangent space condition. The main part of the proof is checking that all polynomials of degree 4 are contained in J', hence that $H(S'/(I')^2)_4 = n$.

In this section, we use the following notation. For polynomials $Q, R \in S'$ we write $Q \equiv R$ if Q is equal to R in $S'/(I')^2$.

Lemma 3.13. *All monomials of degree* 4*, not divisible by* d *are contained in* J'.

Proof. We have a natural inclusion of rings $S \subset S'$. Note that $I \subset I' \cap S$, so also $J \subset J' \cap S$. Since every monomial not divisible by d is contained in S, the claim follows from Proposition 3.9. □

Lemma 3.14. *For all $x, y, z \in \{a, b, c\}$, and for all indices i, j, k, the monomial $x_i y_j z_k d$ is contained in J'.*

Proof. If any of $x_i y_j$, $x_i z_k$, $y_j z_k$ is in I', say $x_i y_j \in I'$, then either $z_k d \in I'$, so that $x_i y_j z_k d \in (I')^2$, or there exist $p, q \in \{a, b, c\}$ and an index t such that $z_k d - p_t q_t \in I'$, so $x_i y_j z_k d = (x_i y_j)(z_k d - p_t q_t) + x_i y_j p_t q_t$. Monomial $x_i y_j p_t q_t$ is contained in J' by Lemma 3.13.

If any of $x_i y_j$, $x_i z_k$, $y_j z_k$ is of the form $w_t v_{t+1}$, $w \neq v$, say $j = i + 1$, $x \neq y$, then either $z_k d \in I'$ and we can rewrite $x_i y_{i+1} z_k d = (x_i y_{i+1} - x_{k+1} y_{k+2})(z_k d) + x_{k+1} y_{k+2} z_k d$, or $z_k d \notin I'$ and there exist $p, q \in \{a, b, c\}$ and an index t such that $z_k d - p_t q_t \in I'$, hence $x_i y_{i+1} z_k d = (x_i y_{i+1} - x_{k+1} y_{k+2})(z_k d - p_t q_t) + x_{k+1} y_{k+2} z_k d + x_i y_{i+1} p_t q_t - x_{k+1} y_{k+2} p_t q_t$. In any case the claim follows by the previous paragraph and Lemma 3.13.

It remains to consider the case where either $x = y = z$ and $j, k \in \{i, i+1\}$, or $i = j = k$. We first consider the case where $i = j = k$ and not all x, y, z are the same. If x, y, z are not mutually different, say $x = y$, then since $y \neq z$ there exists $w \in \{a, b, c\}$, $w \neq x$, such that $y_i z_i - w_{i-1} w_i \in I'$. Hence, if $x_i d \in I'$ we get $x_i y_i z_i d = (x_i d)(y_i z_i - w_{i-1} w_i) + x_i w_{i-1} w_i d$, and if $x_i d \notin I'$, then there are $p, q \in \{a, b, c\}$ and an index s such that $x_i d - p_s q_s \in I'$, so $x_i y_i z_i d = (x_i d - p_s q_s)(y_i z_i - w_{i-1} w_i) + x_i w_{i-1} w_i d + y_i z_i p_s q_s - w_{i-1} w_i p_s q_s$. Thus, the claim follows by the previous parts of the proof and Lemma 3.13. Now we consider the case where x, y, z are mutually different, hence we need to examine the monomial $a_i b_i c_i d$. We rewrite it as follows.

$$a_i b_i c_i d = (a_i b_i - c_{i+1}^2)(c_i d) + (c_i c_{i+1} - a_{i+1} b_{i+1})(c_{i+1} d) + a_{i+1} b_{i+1} c_{i+1} d$$

Therefore, $a_i b_i c_i d \equiv a_{i+1} b_{i+1} c_{i+1} d$, and so $a_i b_i c_i d \equiv a_1 b_1 c_1 d \in J'$.

We now consider the case where $x = y = z$. If $i = j = k$, then there are three monomials to consider, $a_i^3 d$, $b_i^3 d$, and $c_i^3 d$. We rewrite them in the following way.

$$\begin{aligned} a_i^3 d &= (a_i^2 - b_{i-1} c_{i-1})(a_i d - a_{i+1} c_{i+1}) + a_i^2 a_{i+1} c_{i+1} + \\ &\quad + (a_i b_{i-1})(c_{i-1} d) - (a_{i+1} b_{i-1})(c_{i-1} c_{i+1}) \equiv 0 \\ b_i^3 d &= (b_i^2 - a_{i-1} c_{i-1})(b_i d - a_i^2) + a_i^2 b_i^2 + a_{i-1} b_i c_{i-1} d + \\ &\quad - a_{i-1} a_i^2 c_{i-1} \equiv 0 \\ c_i^3 d &= (c_i^2 - c_{i-2} c_{i-1})(c_i d) + (c_{i-2} c_i)(c_{i-1} d) \equiv 0 \end{aligned}$$

Hence, Lemma 3.13 and previous parts of the proof apply. Now suppose $x = y = z$ and at least one of j, k is $i + 1$, say $j = i + 1$, hence there exist $p, q \in \{a, b, c\}$, $p \neq q$, and an index t such that $x_i x_{i+1} - p_t q_t \in I'$. Then, either $x_k d \in I'$ and we get $x_i x_{i+1} x_k d = (x_i x_{i+1} - p_t q_t)(x_k d) + x_k p_t q_t d$, or there exist $w, v \in \{a, b, c\}$ and an index s such that $x_k d - w_s v_s \in I'$, so we get $x_i x_{i+1} x_k d = (x_i x_{i+1} - p_t q_t)(x_k d - w_s v_s) + x_i x_{i+1} w_s v_s + x_k p_t q_t d - p_t q_t w_s v_s$. Either case follows from the previous parts of the proof and Lemma 3.13. □

Lemma 3.15. *For all $x, y \in \{a, b, c\}$, and all indices i, j, the monomial $x_i y_j d^2$ is contained in J'.*

Proof. If both $x_i d$ and $y_j d$ are in I', then $x_i y_j d^2 \in (I')^2$. If only one of them is in I', say $x_i d \in I'$, then there exist $z, w \in \{a, b, c\}$ and an index k such that $y_j d - z_k w_k \in I'$, so $x_i y_j d^2 = (x_i d)(y_j d - z_k w_k) + x_i z_k w_k d$. Finally, if $x_i d \notin I'$, $y_j d \notin I'$, then there exist $w, z, p, q \in \{a, b, c\}$ and indices k, t such that $x_i d - w_k z_k \in I'$ and $y_j d - p_t q_t \in I'$, so $x_iy_jd^2 = (x_id - p_tq_t)(y_jd - z_kw_k) + x_iz_kw_kd + y_jp_tq_td - z_kw_kp_tq_t$. Hence, the claim follows from Lemmas 3.13 and 3.14. □

Proposition 3.16. *Every degree 4 homogenous polynomial of S' is contained in J'.*

Proof. Lemmas 3.13, 3.14, and 3.15 cover most of the cases. It remains to check that for any $x \in \{a, b, c\}$ and any index i both x_id^3 and d^4 are in J'. Thus, we need to consider four monomials, which we rewrite as follows.

$$\begin{aligned} a_id^3 &= (a_id - a_{i+1}c_{i+1})(d^2) + a_{i+1}c_{i+1}d^2 \\ b_id^3 &= (b_id - b_{i-1}c_{i-1})(d^2) + b_{i-1}c_{i-1}d^2 \\ c_id^3 &= (c_id)(d^2) \equiv 0 \\ d^4 &= (d^2)^2 \equiv 0 \end{aligned}$$

Thus, Lemma 3.15 finishes the proof. □

Lemma 3.17. *For all $x, y \in \{a, b, c\}$, and for all indices i, j, the monomials $x_iy_ja_jb_jc_j$, $x_ia_jb_jc_jd$ and $a_jb_jc_jd^2$ are in $(I')^2$.*

Proof. That $x_iy_ja_jb_jc_j$ is in $(I')^2$ follows from the inclusion of rings $S \subset S'$ and Lemma 3.10. Note that a_jb_jd annihilates F', so is in I'. If $x_i \neq c_j$, then $x_ia_jb_j \in I'$, so $x_ia_jb_jc_jd = (x_ia_jb_j)(c_jd) \in (I')^2$. Thus, it remains to consider $a_jb_jc_j^2d$ and $a_jb_jc_jd^2$. We rewrite them as follows.

$$\begin{aligned} a_jb_jc_j^2d &= (a_jb_jd)(c_j^2 - c_{j-2}c_{j-1}) + (a_jc_{j-2})(c_{j-1}d)b_j \equiv 0 \\ a_jb_jc_jd^2 &= (a_jb_jd)(c_jd) \equiv 0 \end{aligned}$$

Hence, the proof is complete. □

Proposition 3.18. *The apolar algebra* $\mathrm{Apolar}(F')$ *satisfies the small tangent space condition.*

Proof. It is easy to check that no linear form annihilates F', hence $\mathrm{Apolar}(F')$ has Hilbert function $(1, n, n, 1)$. Proposition 3.16 implies that $H(S'/(I')^2)_4 \leq n$, so $H(S'/\mathrm{Ann}(F')^2) \leq n$. Thus, by Proposition 2.14, $H(S'/\mathrm{Ann}(F')^2)_4 = n$. Finally, since monomials of the form $x_ia_ib_ic_i$ and $a_1b_1c_1d$ generate $(S'/(I')^2)_4$ Lemma 3.17 implies that $H(S'/(I')^2)_5 = 0$, so also $H(S'/\mathrm{Ann}(F')^2) = 0$. □

3.3. **Proof of Theorem 3.1; case $n = 3m + 2$.**

Let $S'' = k[a_i, b_i, c_i, d, e]_{i=1}^m$ be a polynomial ring of $n = 3m + 2$ variables. Recall that we assume $m \geq 6$. When writing indices we treat them modulo m.

Consider the following polynomial

$$F'' = \sum_{i=1}^{m} a_ib_ic_i + a_ia_{i+1}^2 + b_ib_{i+1}^2 + c_ic_{i+1}^2 + a_ib_{i+1}d + b_ic_{i+1}e.$$

Also by F'' we denote its dual element in the divided power ring associated to S''.

Let I'' be the smallest ideal such that the following remark holds.

Remark 3.19. For all $x, y \in \{a, b, c\}$, and for all indices i, j, if $j \notin \{i - 1, i, i + 1\}$, then $x_iy_j \in I''$.

For all $x, y \in \{a, b, c\}$, and for all indices i, j, if $x \neq y$, then $x_iy_{i+1} - x_jy_{j+1} \in I''$.

For all $x, y \in \{a, b, c\}$, and for all indices i, j, one of the following holds.

(1) x_iy_j is contained in I''.
(2) $x \neq y$ and $j \in \{i-1, i+1\}$.
(3) There exists an index k and $p, q \in \{a, b, c\}$, $p \neq q$, such that $x_iy_j - p_kq_k \in I''$.

For any $x \in \{a, b, c\}$, any $y \in \{d, e\}$, and any index i, one of the following holds.

(1) x_iy is contained in I''.
(2) There exists an index j and $p, q \in \{a, b, c\}$, $p \neq q$, such that $x_iy - p_jq_j \in I''$.

The polynomial F'' is chosen such that $I'' \subset \operatorname{Ann} F''$.

Let J'' denote the ideal $(I'')^2 + \langle a_ia_{i+1}a_{i+2}^2, b_ib_{i+1}b_{i+2}^2, c_ic_{i+1}c_{i+2}^2 \,|\, i = 1, ..., m\rangle + \langle a_1b_1c_1d, a_1b_1c_1e\rangle$. Note that F'', I'', and J'' are invariant under index translation.

We want to show that $\operatorname{Apolar}(F'')$ satisfies the small tangent space condition. The main part of the proof is checking that all polynomials of degree 4 are contained in J'', hence that $H(S''/(I'')^2)_4 = n$.

In this section, we use the following notation. For polynomials $Q, R \in S''$ we write $Q \equiv R$ if Q is equal to R in $S''/(I'')^2$.

Lemma 3.20. *All monomials of degree* 4*, not divisible by* de *are contained in* J''*.*

Proof. We have two inclusions of rings $S' \subset S''$, one takes a, b, c, d to a, b, c, d respectively, the other takes a, b, c, d to b, c, a, e respectively. Note that, in both cases, $I' \subset I'' \cap S'$, so $J' \subset J'' \cap S'$. Since every monomial not divisible by de is contained in at least one of those subrings, the claim follows from Proposition 3.16. □

Lemma 3.21. *For all* $x, y \in \{a, b, c\}$*, and for all indices* i, j*, the monomial* x_iy_jde *is contained in* J''*.*

Proof. If both x_id and y_je are in I'', then $x_iy_jde \in (I'')^2$. If only x_id is in I'', then there exist $z, w \in \{a, b, c\}$ and an index k such that $y_je - z_kw_k \in I''$, so $x_iy_jde = (x_id)(y_je - z_kw_k) + x_iz_kw_kd$. Similarly, if only y_je is in I'', then there exist $z, w \in \{a, b, c\}$ and an index k such that $x_id - z_kw_k \in I''$, so $x_iy_jde = (x_id - z_kw_k)(y_je) + y_jz_kw_ke$. If both x_id and y_je are not in I'', then there exist $w, z, p, q \in \{a, b, c\}$ and indices k, t such that $x_id - w_kz_k \in I''$ and $y_jd - p_tq_t \in I''$, so $x_iy_jde = (x_id - p_tq_t)(y_je - z_kw_k) + x_iz_kw_kd + y_jp_tq_te - z_kw_kp_tq_t$. Hence, the claim follows from Lemma 3.20. □

Proposition 3.22. *Every degree* 4 *homogenous polynomial of* S'' *is contained in* J''*.*

Proof. Lemmas 3.20 and 3.21 cover most of the cases. The rest we rewrite as follows.

$$
\begin{aligned}
a_id^2e &= (a_ie)(d^2) \equiv 0 \\
b_id^2e &= (b_ie - c_{i+2}^2)(d^2) + (c_{i+2}d)^2 \equiv 0 \\
c_id^2e &= (c_id)(de) \equiv 0 \\
a_ide^2 &= (a_ie)(de) \equiv 0 \\
b_ide^2 &= (b_id - a_i^2)(e^2) + (a_ie)^2 \equiv 0 \\
c_ide^2 &= (c_id)(e^2) \equiv 0 \\
d^3e &= (d^2)(de) \equiv 0 \\
d^2e^2 &= (de)^2 \equiv 0 \\
de^3 &= (de)(e^2) \equiv 0
\end{aligned}
$$

This finishes the proof. □

Lemma 3.23. *For all $x, y \in \{a, b, c\}$, $z, w \in \{d, e\}$, and for all indices i, j, the monomials $x_i y_j a_j b_j c_j$, $x_i a_j b_j c_j z$ and $a_j b_j c_j z w$ are in $(I'')^2$.*

Proof. We have two inclusions of rings $S' \subset S''$, one takes a, b, c, d to a, b, c, d respectively, the other takes a, b, c, d to b, c, a, e respectively. Hence, in view of Lemma 3.17 it suffices to consider $a_j b_j c_j de$. We have $a_j b_j c_j de = (a_j e)(c_j d)b_j \in (I'')^2$. □

Proposition 3.24. *The apolar algebra* Apolar(F'') *satisfies the small tangent space condition.*

Proof. It is easy to check that no linear form annihilates F'', hence Apolar(F'') has Hilbert function $(1, n, n, 1)$. Proposition 3.22 implies that $H(S''/(I'')^2)_4 \leq n$, so $H(S''/\mathrm{Ann}(F'')^2) \leq n$. Thus, by Proposition 2.14, $H(S''/\mathrm{Ann}(F'')^2)_4 = n$. Finally, since monomials of the form $x_i a_i b_i c_i$, $a_1 b_1 c_1 d$, and $a_1 b_1 c_1 e$ generate $(S''/(I'')^2)_4$ Lemma 3.23 implies that $H(S''/(I'')^2) = 0$, so also $H(S''/\mathrm{Ann}(F'')^2) = 0$. □

4. Computer computations for $n < 18$

Let S be a polynomial ring of n variables. In this chapter, we give examples of degree 3 polynomials F such that Apolar(F) satisfies the small tangent space condition for $n = 6$ and $7 < n < 18$ (the case $n \geq 18$ is covered by Theorem 3.1).

We have checked on computer, using Macaulay2 [GS], that they are indeed correct for fields of characteristic 0, 2, and 3. We believe that they work in any characteristic, though a proof would probably require a direct verification, so we restrict ourselves to supplying a computer code which one can use to verify these examples in any given characteristic.

Note that in order to verify Conjecture 3 for a field k of characteristic 0 it suffices to check $k = \mathbb{Q}$. Similarly, for a field k of characteristic p is suffices to check $k = \mathbb{F}_p$.

Our examples from chapter 3 work also for $n \geq 9$. For $n = 6$ and $n = 8$ we construct different polynomials. For $n = 6$ we have chosen the polynomial

$$F = a_1 b_1 c_1 + a_2 b_2 c_2 + a_1 a_2^2 + b_1 b_2^2 + c_1 c_2^2 + a_1^3 + b_1^3 + c_1^3.$$

For $n = 8$ we have chosen

$$F = a_1 b_1 c_1 + a_2 b_2 c_2 + a_1 a_2^2 + b_1 b_2^2 + c_1 c_2^2 + a_1 de + b_1^2 d + c_1^2 e.$$

4.1. **Macaulay2 code.**

In this section, we describe the computer code we have used to verify our examples. First one needs to chose a field, hence to type

```
kk = QQ;
```

or (replacing p by a prime number of choice)

```
kk = ZZ/p;
```

into the Macaulay2 console. Then, one needs to specify the number of variables of the polynomial ring typing

```
n = ?
```

with ? replaced by the chosen number. If n was chosen to be 6, then the following code generates the appropriate polynomial.

```
S = kk[a_1,a_2,b_1,b_2,c_1,c_2];
F = a_1*b_1*c_1 + a_2*b_2*c_2 + a_1*a_2^2 + b_1*b_2^2 +
    c_1*c_2^2 + a_1^3 + b_1^3 + c_1^3;
```

If $n = 8$ one needs to enter the following lines.

```
S = kk[a_1,a_2,b_1,b_2,c_1,c_2,d,e];
F = a_1*b_1*c_1 + a_2*b_2*c_2 + a_1*a_2^2 + b_1*b_2^2 +
    c_1*c_2^2 + a_1*d*e + b_1^2*d + c_1^2*e;
```

If n is divisible by 3 and greater than 8, then the following code needs to be entered.

```
m = n//3;
S = kk[a_1..a_m,b_1..b_m,c_1..c_m];
F = 0;
for i in 1..m-1 do F = F + a_i*b_i*c_i + a_i*a_(i+1)^2 +
    b_i*b_(i+1)^2 + c_i*c_(i+1)^2;
F = F + a_m*b_m*c_m + a_m*a_1^2 + b_m*b_1^2 + c_m*c_1^2;
```

If n gives remainder 1 upon division by 3 and is greater than 8, then one uses the following code.

```
m = (n-1)//3;
S = kk[a_1..a_m,b_1..b_m,c_1..c_m,d];
F = 0;
for i in 1..m-1 do F = F + a_i*b_i*c_i + a_i*a_(i+1)^2 +
    b_i*b_(i+1)^2 + c_i*c_(i+1)^2 + a_i*b_(i+1)*d;
F = F + a_m*b_m*c_m + a_m*a_1^2 + b_m*b_1^2 + c_m*c_1^2 +
    a_m*b_1*d;
```

Lastly, if n gives remainder 2 upon division by 3 and is greater than 8, then the following code needs to be used.

```
m = (n-2)//3;
S = kk[a_1..a_m,b_1..b_m,c_1..c_m,d,e];
F = 0;
for i in 1..m-1 do F = F + a_i*b_i*c_i + a_i*a_(i+1)^2 +
    b_i*b_(i+1)^2 + c_i*c_(i+1)^2 + a_i*b_(i+1)*d +
    b_i*c_(i+1)*e;
F = F + a_m*b_m*c_m + a_m*a_1^2 + b_m*b_1^2 + c_m*c_1^2 +
    a_m*b_1*d + b_m*c_1*e;
```

To verify whether the apolar algebra induced by the generated polynomial satisfies the small tangent space condition one can run the following lines.

```
I = ideal fromDual(matrix{{F}}, DividedPowers => true);
if (hilbertFunction(0,S/I) == 1 and
    hilbertFunction(1,S/I) == n and
    hilbertFunction(4,S/I^2) == n and
    hilbertFunction(5,S/I^2) == 0)
    then print True else print False;
```

If the answer given by Macaulay2 reads "True", then Apolar(F) satisfies the small tangent space condition. If on the other hand the answer reads "False", then Apolar(F) does not satisfy the small tangent space condition.

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