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NEW EQUATIONS FOR THE TIME-DEPENDENT REGULATOR PROBLEM

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1. Introduction

In this note we consider the linear time-dependent control problem of minimizing

$$\int_t^T \left[x, Q(t)x + (u,u) \right] dt \quad ,$$

over all piecewise continuous control functions u(t), where u and x are related by the linear equation

$$\frac{dx}{dt} = F(t)x + Gu(t) \quad .$$

Here it is assumed that x is an n-dimensional vector, u an m-dimensional vector, F,Q are nxn piecewise-continuous time-varying matrix functions with $Q(t) \geq 0$ for all $t \leq T$, and G is an nxm constant matrix. Well known results in control theory show that the minimizing control u*(t) is given (in feedback form) by

$$\begin{aligned} u^*(t) &= -G'P(t)x(t) \quad , \\ &= -K(t)x(t) \quad , \end{aligned} \tag{1}$$

where P(t) is the solution of the matrix Riccati equation

$$\begin{aligned} \frac{-dP}{dt} &= Q(t) + PF(t) + F'(t)P - PGG'P \quad , \\ P(T) &= 0 \quad . \end{aligned} \tag{2}$$

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Note that the solution of (2) involves $n(n+1)/2$ equations in the independent components of P. In recent work [1,2,4] it has been shown that when Q,F,G are constant and certain other conditions are satisfied, it is possible to calculate K, the feedback gain, directly with a system of equations whose size is linearly proportional to n, the dimension of the state. However, the approach taken in these works does not appear to be easily extendable to time-dependent systems.

The objective of this note is to pursue a slightly different course in order to arrive at a system of equations suitable for directly computing the gain K, without the need of the intermediate Riccati equation (2). Thus, we shall arrive at a system involving nm equations in the components of K which, if $m \ll N$, significantly reduces the computational burden imposed by the usual Riccati approach. Unfortunately, the current approach is not completely general in that we require the matrix G to be constant. However, at the expense of slight additional complicatations, even this requirement may be partially relaxed. Throughout this note, however, G will be constant and we shall only indicate in the closing remarks how to extend the results to more general G.

2. The Equation for K

Before developing the appropriate equations for K, we state a useful result from [3]:

Theorem 1. (i) Let R be a real, symmetric, positive-definite matrix such that RKG is symmetric. If rank KG = rank K, then all real symmetric P satisfying $G'P = -RK$ are represented in terms of R by

$$P = -K'R(RKG)^{\#}RK + Y \quad ,$$

where $^{\#}$ denotes the Moore-Penrose generalized inverse and where Y is any symmetric matrix satisfying $G'Y = 0$;

(ii) The matrix P above will be positive semi-definite if, and only if, rank KG = rank K, the characteristic values of KG are nonpositive, and $Y \geq 0$.

We now state the main result:

Theorem 2. Let the optimal feedback gain K(t) be given by Eq.(1) Then the components of K may be calculated from the system of nm differential equations

$$\frac{dK}{dt} = - KF(t) + G'\left[Q(t) - F'(t)K'(KG)^{\#}K - K'K\right] \quad , \qquad (3)$$

$$K(T) = 0 \quad .$$

Proof. Since $K = -G'P(t)$, we have $\dot{K}(t) = - G'\dot{P}(t)$ which, by Eq.(2) gives

$$K(t) = G'\,Q(t) + PF(t) + F'(t)P - PGG'P \qquad (4)$$

$$= G'Q(t) - K(t)F(t) + G'F'(t)P - G'K'(t)K(t) \quad .$$

The only offending term in the above expression is $G'F'P$. The proof will be complete as soon as this term is related to K. We assert that

$$G'F'P = -G'F'K'(KG)^{\#}K \quad .$$

This follows immediately from Theorem 1 since we have $R = I$ and since G is of full rank, $Q \geq 0$, the conditions of part (ii) of the theorem are satisfied. Since $P(T) = 0$, we may take $Y = 0$ in the representation formula. Thus $P = -$ '(G)#K, which completes the proof of the theorem.

3. Remarks

(1) Obviously, the equation for K represents a set of nm nonlinear differential equations with known initial conditions and, as such, may be readily integrated using any of the usual numerical methods.

(2) The constancy of G may be weakened to the extent that G satisfies a differential equation of the form $\frac{dG'}{dt} = A(t)G'(t)$. It is a straightforward exercise to generalize our theorem to handle this case.

References

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