Tohoku Math. J.
61 (2009), 441–453

NEW ESTIMATES FOR EIGENVALUES OF THE BASIC DIRAC OPERATOR

SEOUNG DAL JUNG, SOON CHAN KIM AND JEONG HYEONG PARK

(Received December 12, 2008, revised April 16, 2009)

Abstract. On a transverse spin foliation, we give a new lower bound for the square of the eigenvalues of the basic Dirac operator by the smallest eigenvalue of the basic Yamabe operator. Moreover, the limiting foliation is transversally Einsteinian.

1. Introduction. In 2001, S. D. Jung [4] proved that, on a foliated Riemannian manifold with a transverse spin structure, any eigenvalue λ of the basic Dirac operator D_B satisfies the inequality

$$\lambda^2 \geq \frac{q}{4(q-1)} \inf_M (\sigma^\nabla + |\kappa|^2)\,, \tag{1}$$

where $q = \operatorname{codim} \mathcal{F}$, σ^∇ is the transversal scalar curvature and κ is the mean curvature form of $\mathcal{F}$. In the limiting case, the foliation $\mathcal{F}$ is minimal, transversally Einsteinian with constant transversal scalar curvature σ^∇. In 2004, S. D. Jung et al. [6] improved the above inequality (1) by using the basic Yamabe operator Y_B. In fact, any eigenvalue λ of the basic Dirac operator D_B satisfies the inequality

$$\lambda^2 \geq \frac{q}{4(q-1)} (\mu_1 + \inf_M |\kappa|^2)\,, \tag{2}$$

where μ_1 is the first eigenvalue of the basic Yamabe operator. In the inequalities (1) and (2), κ is assumed to be basic.

In this paper, we give an estimate sharper than (1) by using a modified connection $\nabla^{f,g}$ defined by

$$\nabla_X^{f,g} \Psi = \nabla_X \Psi + f\pi(X)\cdot\Psi + g\kappa\cdot\pi(X)\cdot\Psi \tag{3}$$

for any basic functions f and g. Namely, any eigenvalue λ of the basic Dirac operator D_B satisfies

$$\lambda^2 \geq \frac{q}{4(q-1)} \inf_M \left(\sigma^\nabla + \frac{q+1}{q} |\kappa_B|^2\right), \tag{4}$$

2000 *Mathematics Subject Classification.* Primary 53C43; Secondary 58E20.

Key words and phrases. The basic Dirac operator, transverse spin foliation, transversally Einsteinian.

This work was supported by the Korea Science and Engineering Foundation (KOSEF) grant funded by Korea government (MEST) (R01-2008-000-20370-0).

where κ_B is the basic part of κ. Moreover, by using a transversally conformal change of the Riemannian metric, we give an estimate sharper than (2). Namely,

$$\lambda^2 \geq \frac{q}{4(q-1)}\left(\mu_1 + \frac{q+1}{q}\inf_M |\kappa_B|^2\right). \tag{5}$$

Obviously, the inequality (5) is sharper than (4). The limiting foliations of (4) and (5) are transversally Einsteinian with $\kappa_B = 0$, where κ_B is the basic part of κ.

2. Transversal Dirac operator. Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$. Then we have an exact sequence of vector bundles

$$0 \longrightarrow L \longrightarrow TM \stackrel{\pi}{\longrightarrow} Q \longrightarrow 0, \tag{6}$$

where L is the tangent bundle and $Q = TM/L$ is the normal bundle of $\mathcal{F}$. The metric g_M determines an orthogonal decomposition $TM = L \oplus L^\perp$. Identify Q with $L^\perp$ and let g_Q denote the induced metric on Q. The bundle-like condition on g_M means that $\theta(X)g_Q = 0$ for $X \in \Gamma L$, where $\theta(X)$ is the transverse Lie derivative. Let ∇ be the transversal Levi-Civita connection on Q, which is torsion-free and metrical with respect to g_Q. Let R^∇, ρ^∇ and σ^∇ be the transversal curvature tensor, transversal Ricci operator and transversal scalar curvature with respect to ∇, respectively [10]. The foliation $\mathcal{F}$ is *transversally Einsteinian* if $\rho^\nabla = \sigma^\nabla/q$ id. Let $\Omega_B^*(\mathcal{F})$ be the space of all *basic forms* on M, i.e., forms ϕ satisfying $i(X)\phi = i(X)d\phi = 0$ for all $X \in \Gamma L$. Then $\Omega^*(M)$ is decomposed as $\Omega(M) = \Omega_B(\mathcal{F}) \oplus \Omega_B(\mathcal{F})^\perp$ [1, Theorem 2.1]. Let $P : \Omega(M) \to \Omega_B(\mathcal{F})$ be the orthogonal projection onto basic forms [9, Lemma 1.8]. For any r-form ϕ, we denote the basic part of ϕ by $\phi_B := P\phi$. The exterior differential on the de Rham complex $\Omega^*(M)$ is restricted to a differential $d_B : \Omega_B^r(\mathcal{F}) \to \Omega_B^{r+1}(\mathcal{F})$. Let $\kappa \in Q^*$ be the mean curvature form of $\mathcal{F}$. It is well-known [1, Corollary 3.5] that $\kappa_B := P\kappa$ is closed, i.e., $d\kappa_B = 0$. The basic Laplacian Δ_B is given by $\Delta_B = d_B\delta_B + \delta_B d_B$, where δ_B is the formal adjoint operator of d_B.

Let $S(\mathcal{F})$ be a foliated spinor bundle [3, 4] and $\langle\ ,\ \rangle_{g_Q}$ a Hermitian metric on $S(\mathcal{F})$ induced by g_Q. By the Clifford multiplication in the fibers of $S(\mathcal{F})$ for any vector field X in Q and any foliated spinor field Φ, the Clifford product $X \cdot \Phi$, which is also a foliated spinor field, is defined. This product has the following properties: for all $X, Y \in \Gamma Q$ and $\Phi, \Psi \in \Gamma S(\mathcal{F})$,

$$(X \cdot Y + Y \cdot X)\Phi = -2g_Q(X, Y)\Phi, \tag{7}$$

$$\langle X \cdot \Psi, \Phi\rangle_{g_Q} + \langle \Psi, X \cdot \Phi\rangle_{g_Q} = 0, \tag{8}$$

$$\nabla_Y(X \cdot \Psi) = (\nabla_Y X) \cdot \Psi + X \cdot \nabla_Y \Psi, \tag{9}$$

where ∇ is a metric covariant derivation on $S(\mathcal{F})$. Let $\{E_a\}$ be a local orthonormal basic frame of Q. We now define a canonical section $\mathcal{R}^\nabla$ of $\mathrm{Hom}(S(\mathcal{F}), S(\mathcal{F}))$ by the formula

$$\mathcal{R}^\nabla(\Psi) = \sum_{a<b} E_a \cdot E_b \cdot R^S(E_a, E_b)\Psi, \tag{10}$$

where R^S is the curvature tensor of $S(\mathcal{F})$. Then, on the foliated spinor bundle $S(\mathcal{F})$, we have [4, (4.3) and (4.4)]

$$\sum_a E_a \cdot R^S(X, E_a)\Psi = -\frac{1}{2}\rho^\nabla(X)\cdot\Psi\,, \tag{11}$$

$$\mathcal{R}^\nabla = \frac{1}{4}\sigma^\nabla \mathrm{id} \tag{12}$$

for all $X \in \Gamma Q$. The *transversal Dirac operator* D_{tr} acting on sections of $S(\mathcal{F})$ is locally defined [3, 4] by

$$D_{\mathrm{tr}}\Psi = \sum_a E_a \cdot \nabla_{E_a}\Psi - \frac{1}{2}\kappa_B \cdot \Psi\,. \tag{13}$$

Here the Clifford product $\omega\cdot\Psi$ of a 1-form $\omega \in Q^*$ and a foliated spinor field Ψ is defined by $\omega\cdot\Psi \equiv \omega^\sharp\cdot\Psi$, where $\omega^\sharp$ is the g_Q-dual vector field of ω. Then it is well known that D_{tr} is formally self-adjoint. Now we define the subspace $\Gamma_B(S(\mathcal{F}))$ of *basic* or *holonomy invariant* sections of $S(\mathcal{F})$ by

$$\Gamma_B(S(\mathcal{F})) = \{\Psi \in \Gamma S(\mathcal{F})\,;\ \nabla_X\Psi = 0 \quad \text{for } X \in \Gamma L\}\,.$$

Trivially, we see that D_{tr} leaves $\Gamma_B(S(\mathcal{F}))$ invariant. Let $D_B = D_{\mathrm{tr}}|_{\Gamma_B(S(\mathcal{F}))} : \Gamma_B(S(\mathcal{F})) \to \Gamma_B(S(\mathcal{F}))$. This operator D_B is called the *basic Dirac operator* on (smooth) basic sections.

THEOREM 2.1 ([3, 4]). *On a transverse spin foliation $\mathcal{F}$ with $\delta_B\kappa_B = 0$, the Lichnerowicz type formula is given by*

$$D_{\mathrm{tr}}^2\Psi = \nabla_{\mathrm{tr}}^*\nabla_{\mathrm{tr}}\Psi + \frac{1}{4}K_\sigma^\nabla\Psi\,, \tag{14}$$

where $K_\sigma^\nabla = \sigma^\nabla + |\kappa_B|^2$ and

$$\nabla_{\mathrm{tr}}^*\nabla_{\mathrm{tr}}\Psi = -\sum_a \nabla^2_{E_a,E_a}\Psi + \nabla_{\kappa_B^\sharp}\Psi \tag{15}$$

with $\nabla^2_{X,Y} = \nabla_X\nabla_Y - \nabla_{\nabla^M_X Y}$ for all $X, Y \in \Gamma TM$.

Now, we consider, for any real basic function u on M, the transversally conformal metric $\bar g_Q = e^{2u}g_Q$. Let $\bar S(\mathcal{F})$ be the foliated spinor bundles associated with $\bar g_Q$. For any section Ψ of $S(\mathcal{F})$, we write $\bar\Psi \equiv I_u\Psi$, where $I_u : S(\mathcal{F}) \to \bar S(\mathcal{F})$ is an isometry. Then, for any Φ, $\Psi \in \Gamma S(\mathcal{F})$, we have

$$\langle\Phi, \Psi\rangle_{g_Q} = \langle\bar\Phi, \bar\Psi\rangle_{\bar g_Q}\,, \tag{16}$$

and the Clifford multiplication in $\bar S(\mathcal{F})$ is given by

$$\bar X \mathbin{\bar\cdot} \bar\Psi = \overline{X\cdot\Psi} \quad \text{for } X \in \Gamma Q\,. \tag{17}$$

The connections ∇ and $\bar\nabla$ acting respectively on the sections of $S(\mathcal{F})$ and $\bar S(\mathcal{F})$ are related, for any vector field X and any spinor field Ψ, in [6] by

$$\bar\nabla_X\bar\Psi = \overline{\nabla_X\Psi} - \frac{1}{2}\overline{\pi(X)\cdot d_Bu\cdot\Psi} - \frac{1}{2}g_Q(d_Bu, \pi(X))\bar\Psi\,. \tag{18}$$

Let $\bar{D}_{\mathrm{tr}}$ be the transversal Dirac operator associated with the metric $\bar{g}_Q$ and acting on the sections of the foliated spinor bundles $\bar{S}(\mathcal{F})$. Let $\{\bar{E}_a\}$ be a local frame of $\bar{P}_{SO}(\mathcal{F})$. Then $\bar{D}_{\mathrm{tr}}$ is locally expressed by

$$\bar{D}_{\mathrm{tr}}\bar{\Psi} = \sum_a \bar{E}_a \bar{\cdot} \bar{\nabla}_{\bar{E}_a}\bar{\Psi} - \frac{1}{2} P\kappa_{\bar{g}} \bar{\cdot} \bar{\Psi}\,, \tag{19}$$

where $\kappa_{\bar{g}} = e^{2u}\kappa$ is the mean curvature form associated with $\bar{g}_Q$. It is easy to prove that $\bar{D}_{\mathrm{tr}}$ is formally self-adjoint with respect to $\langle\ ,\ \rangle_{\bar{g}_Q}$. Using (17), we have, for any Ψ,

$$\bar{D}_{\mathrm{tr}}\bar{\Psi} = e^{-u}\left(\overline{D_{\mathrm{tr}}\Psi} + \frac{q-1}{2}\,\overline{d_B u \cdot \Psi}\right). \tag{20}$$

For any basic function f, the equality $D_{\mathrm{tr}}(f\Psi) = d_B f \cdot \Psi + f D_{tr}\Psi$ holds. Hence we have

$$\bar{D}_{\mathrm{tr}}(f\bar{\Psi}) = e^{-u}\overline{d_B f \cdot \Psi} + f\bar{D}_{\mathrm{tr}}\bar{\Psi}\,. \tag{21}$$

From (20) and (21), we have the following proposition.

PROPOSITION 2.2 ([6]). *Let $\mathcal{F}$ be the transverse spin foliation of codimension q. Then the transverse Dirac operators D_{tr} and $\bar{D}_{\mathrm{tr}}$ satisfy*

$$\bar{D}_{\mathrm{tr}}(e^{-(q-1)u/2}\bar{\Psi}) = e^{-(q+1)u/2}\overline{D_{\mathrm{tr}}\Psi} \tag{22}$$

for any spinor field $\Psi \in S(\mathcal{F})$.

From Proposition 2.2, if $D_{\mathrm{tr}}\Psi = 0$, then $\bar{D}_{\mathrm{tr}}\bar{\Phi} = 0$, where $\Phi = e^{-(q-1)u/2}\Psi$, and conversely. Therefore the dimension of the space of the foliated harmonic spinors is transversally conformal invariant.

THEOREM 2.3 ([6]). *On the transverse spin foliation $\mathcal{F}$ with $\delta_B\kappa_B = 0$, we have the equality*

$$\bar{D}_{\mathrm{tr}}^2\bar{\Psi} = \bar{\nabla}_{\mathrm{tr}}^*\bar{\nabla}_{\mathrm{tr}}\bar{\Psi} + \frac{1}{4} K_\sigma^{\bar{\nabla}}\bar{\Psi} \tag{23}$$

for every $\bar{\Psi} \in \bar{S}(\mathcal{F})$, where

$$\bar{\nabla}_{\mathrm{tr}}^*\bar{\nabla}_{\mathrm{tr}}\bar{\Psi} = -\sum_a \bar{\nabla}_{\bar{E}_a}\bar{\nabla}_{\bar{E}_a}\bar{\Psi} + \bar{\nabla}_{\sum_a \bar{\nabla}_{\bar{E}_a}\bar{E}_a}\bar{\Psi} + \bar{\nabla}_{(P\kappa_{\bar{g}})^\sharp}\bar{\Psi}\,, \tag{24}$$

$$K_\sigma^{\bar{\nabla}} = \sigma^{\bar{\nabla}} + |\bar{\kappa}_B|^2 + 2(q-2)P\kappa_{\bar{g}}(u)\,. \tag{25}$$

3. The proof of (4). Let $(M, g_M, \mathcal{F}, S(\mathcal{F}))$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M satisfying $\delta_B\kappa_B = 0$. Now, for any basic functions f and g, we define a new connection $\nabla^{f,g}$ on $S(\mathcal{F})$ by

$$\nabla_X^{f,g}\Psi = \nabla_X\Psi + f\pi(X)\cdot\Psi + g\kappa_B\cdot\pi(X)\cdot\Psi \tag{26}$$

for any vector field X and any spinor field Ψ. By a direct calculation, from (26), we have

$$
\begin{aligned}
|\nabla_{\mathrm{tr}}^{f,g}\Psi|^2 &= |\nabla_{\mathrm{tr}}\Psi|^2 + qf^2|\Psi|^2 + qg^2|\kappa_B|^2|\Psi|^2 + g|\kappa_B|^2|\Psi|^2 \\
&\quad - 2f\mathrm{Re}\langle D_{\mathrm{tr}}\Psi, \Psi\rangle_{g_Q} + 2g\mathrm{Re}\langle D_{\mathrm{tr}}\Psi, \kappa_B \cdot \Psi\rangle_{g_Q} - 4g\mathrm{Re}\langle \nabla_{\kappa_B^\sharp}\Psi, \Psi\rangle_{g_Q}\,,
\end{aligned} \tag{27}
$$

where $|\Psi|^2 = \langle \Psi, \Psi\rangle_{g_Q}$. Then we have the following theorem.

THEOREM 3.1. *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q > 1$ and a bundle-like metric g_M satisfying $\delta_B\kappa_B = 0$. Assume that σ^∇ is nonnegative. Then any eigenvalue λ of the basic Dirac operator D_B satisfies*

$$
\lambda^2 \geq \frac{q}{4(q-1)} \inf_M \left(\sigma^\nabla + \frac{q+1}{q}|\kappa_B|^2\right). \tag{28}
$$

PROOF. Since ∇ is metrical and $\delta_B\kappa_B = 0$, we have

$$
\int_M \mathrm{Re}\langle \nabla_{\kappa_B^\sharp}\Psi, \Psi\rangle_{g_Q} = 0\,.
$$

Hence if $D_B\Psi = \lambda\Psi$, from (14) and (27), we have

$$
\int_M |\nabla_{\mathrm{tr}}^{f,g}\Psi|^2 = \int_M \left(qf^2 - 2\lambda f + \lambda^2 + q|\kappa_B|^2 g^2 + |\kappa_B|^2 g - \frac{1}{4}\,K_\sigma^\nabla\right)|\Psi|^2\,.
$$

If we put $f = \lambda/q$ and $g = -1/2q$, then we have

$$
\int_M |\nabla_{\mathrm{tr}}^{f,g}\Psi|^2 = \int_M \frac{q-1}{q}\left(\lambda^2 - \frac{q}{4(q-1)}\left\{K_\sigma^\nabla + \frac{1}{q}|\kappa_B|^2\right\}\right)|\Psi|^2\,, \tag{29}
$$

which proves (28). □

COROLLARY 3.2. *Under the assumptions in Theorem* 3.1, *if the transverse scalar curvature is zero*, *then we get the inequality*

$$
\lambda^2 \geq \frac{q+1}{4(q-1)} \inf_M |\kappa_B|^2\,.
$$

Now we study the limiting case. We define $\mathrm{Ric}_\nabla^{f,g} : \Gamma Q \otimes S(\mathcal{F}) \to S(\mathcal{F})$ by

$$
\mathrm{Ric}_\nabla^{f,g}(X \otimes \Psi) = \sum E_a \cdot R^{f,g}(X, E_a)\Psi\,, \tag{30}
$$

where $R^{f,g}$ is the curvature tensor with respect to $\nabla^{f,g}$. Then we have the following lemma.

LEMMA 3.3. *For any vector field $X \in \Gamma Q$ and spinor field $\Psi \in \Gamma S(\mathcal{F})$, the equality*

$$
\begin{aligned}
\mathrm{Ric}_\nabla^{f,g}(X \otimes \Psi) &= -\frac{1}{2}\,\rho^\nabla(X)\Psi - qX(f)\Psi + 2(q-1)f^2 X \cdot \Psi - d_B f \cdot X \cdot \Psi \\
&\quad + (q-2)X(g)\kappa_B \cdot \Psi + (q-2)g\nabla_X\kappa_B \cdot \Psi + 2qfg g_Q(X, \kappa_B)\Psi \\
&\quad + 2(q-2)g^2|\kappa_B|^2 X \cdot \Psi - 2(q-2)g^2 g_Q(X, \kappa_B)\kappa_B \cdot \Psi \\
&\quad - d_B g \cdot \kappa_B \cdot X \cdot \Psi + 2fg\kappa_B \cdot X \cdot \Psi + g|\kappa_B|^2 X \cdot \Psi
\end{aligned} \tag{31}
$$

holds.

PROOF. From (26), a direct calculation gives

$$\begin{aligned}\nabla_X^{f,g}\nabla_{E_a}^{f,g}\Psi &= \nabla_X\nabla_{E_a}\Psi + X(f)E_a\cdot\Psi + f\nabla_X E_a\cdot\Psi + fE_a\cdot\nabla_X\Psi\\ &\quad + X(g)\kappa_B\cdot E_a\cdot\Psi + g\nabla_X\kappa_B\cdot E_a\cdot\Psi + g\kappa_B\cdot\nabla_X E_a\cdot\Psi\\ &\quad + g\kappa_B\cdot E_a\cdot\nabla_X\Psi + fX\cdot\nabla_{E_a}\Psi + f^2X\cdot E_a\cdot\Psi\\ &\quad + fgX\cdot\kappa_B\cdot E_a\cdot\Psi + g\kappa_B\cdot X\cdot\nabla_{E_a}\Psi + fg\kappa_B\cdot X\cdot E_a\cdot\Psi\\ &\quad + g^2\kappa_B\cdot X\cdot\kappa_B\cdot E_a\cdot\Psi\,.\end{aligned}$$

Moreover, we have

$$\begin{aligned}X\cdot\kappa_B\cdot E_a - E_a\cdot\kappa_B\cdot X &= 2\kappa_B\cdot E_a\cdot X + 2g_Q(X,E_a)\kappa_B - 2g_Q(X,\kappa_B)E_a\\ &\quad + 2g_Q(E_a,\kappa_B)X\,.\end{aligned}$$

Therefore, we have

$$\begin{aligned}R^{f,g}(X,E_a)\Psi &= R^S(X,E_a)\Psi + X(f)E_a\cdot\Psi - X(g)E_a\cdot\kappa_B\cdot\Psi\\ &\quad - 2X(g)g_Q(\kappa_B,E_a)\Psi - gE_a\cdot\nabla_X\kappa_B\cdot\Psi - 2gg_Q(\nabla_X\kappa_B,E_a)\Psi\\ &\quad - 2f^2E_a\cdot X\cdot\Psi - 2f^2g_Q(X,E_a)\Psi - 2fgg_Q(X,\kappa_B)E_a\cdot\Psi\\ &\quad + 2fgg_Q(E_a,\kappa_B)X\cdot\Psi - 2g^2|\kappa_B|^2E_aX\cdot\Psi\\ &\quad - 2g^2g_Q(X,E_a)|\kappa_B|^2\Psi + 2g^2g_Q(X,\kappa_B)E_a\cdot\kappa_B\cdot\Psi\\ &\quad + 4g^2g_Q(X,\kappa_B)g_Q(E_a,\kappa_B)\Psi + 2g^2g_Q(E_a,\kappa_B)\kappa_B\cdot X\cdot\Psi\\ &\quad - E_a(f)X\cdot\Psi - E_a(g)\kappa_B\cdot X\cdot\Psi - g\nabla_{E_a}\kappa_B\cdot X\cdot\Psi\,.\end{aligned}$$

From (11) and (30), we get the equality. □

Hence we have the following theorem.

THEOREM 3.4. *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q > 1$ and a bundle-like metric g_M satisfying $\delta_B\kappa_B = 0$. Assume that σ^∇ is nonnegative. If there exists an eigenspinor field Ψ_1 of the basic Dirac operator D_B for the eigenvalue λ_1 satisfying*

$$\lambda_1^2 = \frac{q}{4(q-1)}\inf_M\left(\sigma^\nabla + \frac{q+1}{q}|\kappa_B|^2\right), \tag{32}$$

then $\mathcal{F}$ is transversally Einsteinian with a positive constant transversal scalar curvature σ^∇ and $\kappa_B = 0$.

PROOF. Let $D_B\Psi_1 = \lambda_1\Psi_1$ with

$$\lambda_1^2 = \frac{q}{4(q-1)}\inf_M\left(\sigma^\nabla + \frac{q+1}{q}|\kappa_B|^2\right).$$

From (29), we see $\nabla_{\rm tr}^{f_1,g_1}\Psi_1 = 0$, where $f_1 = \lambda_1/q$ and $g_1 = -1/2q$. Hence, from (26), we have

$$\nabla_X\Psi_1 = -\frac{\lambda_1}{q}\, X\cdot\Psi_1 + \frac{1}{2q}\,\kappa_B\cdot X\cdot\Psi_1\,. \tag{33}$$

Hence, from (33), we have

$$\begin{aligned}\sum_a E_a\cdot\nabla_{E_a}\Psi_1 &= -\frac{\lambda_1}{q}\sum_a E_a\cdot E_a\cdot\Psi_1 + \frac{1}{2q}\sum_a E_a\cdot\kappa_B\cdot E_a\cdot\Psi_1\\ &= \lambda_1\Psi_1 + \frac{q-2}{2q}\,\kappa_B\cdot\Psi_1\,.\end{aligned}$$

Therefore $D_B\Psi_1 = \lambda_1\Psi_1$ implies $\kappa_B\cdot\Psi_1 = 0$, which means $\kappa_B = 0$. If $\nabla_X^{f,g}\Psi = 0$ for any $X\in\Gamma Q$, then $\mathrm{Ric}_\nabla^{f,g} = 0$. Since $\kappa_B = 0$, from (31), we have

$$\rho^\nabla(X)\cdot\Psi = \frac{4(q-1)}{q^2}\,\lambda_1^2 X\cdot\Psi\,. \tag{34}$$

This means that $\mathcal{F}$ is transversally Einsteinian with a constant transversal scalar curvature $\sigma^\nabla = (4(q-1)/q)\lambda_1^2$. □

4. The proof of (5). Let $(M, g_M, \mathcal{F}, S(\mathcal{F}))$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M satisfying $\delta_B\kappa_B = 0$. In this section, we estimate the eigenvalues of the basic Dirac operator by a transversally conformal change of the metric. Now, we consider, for any real basic function u on M, the transversally conformal metric $\bar g_Q = e^{2u}g_Q$. Let $\bar S(\mathcal{F})$ be its corresponding spinor bundle. For any basic functions f and g, we define the modified connection $\bar\nabla^{f,g}$ on $\bar S(\mathcal{F})$ by

$$\bar\nabla_X^{f,g}\bar\Psi = \bar\nabla_X\bar\Psi + f\pi(X)\,\bar\cdot\,\bar\Psi + g(P\kappa_{\bar g})\,\bar\cdot\,\pi(X)\,\bar\cdot\,\bar\Psi \tag{35}$$

for any vector field X and any spinor field Ψ on M.

LEMMA 4.1. *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a transverse spin foliation $\mathcal{F}$ and a bundle-like metric g_M. Then, for any basic-harmonic 1-form $\omega\in\Omega_B^1(\mathcal{F})$, the equality*

$$\begin{aligned}\bar D_{\rm tr}(f\omega\,\bar\cdot\,\bar\Psi) =& -f\omega\,\bar\cdot\,\bar D_{\rm tr}\bar\Psi - 2f\bar\nabla_\omega\bar\Psi - (q+2)f\omega(u)\bar\Psi\\ &- 2f\overline{\omega\cdot d_Bu\cdot\Psi} + \overline{d_Bf\cdot\omega\cdot\Psi}\end{aligned} \tag{36}$$

holds, where f is a basic function.

PROOF. Note that, for any basic function f, we have

$$D_{\rm tr}(f\omega\cdot\Psi) = -f\omega\cdot D_{\rm tr}\Psi - 2f\nabla_\omega\Psi + d_Bf\cdot\omega\cdot\Psi\,. \tag{37}$$

From (20), we have

$$\begin{aligned}\bar D_{\rm tr}(f\omega\,\bar\cdot\,\bar\Psi) &= e^{-u}\overline{d_Be^u\cdot f\omega\cdot\Psi} + e^u\bar D_{\rm tr}(\overline{f\omega\cdot\Psi})\\ &= f\overline{d_Bu\cdot\omega\cdot\Psi} + \overline{D_{\rm tr}(f\omega\cdot\Psi)} + \frac{q-1}{2}\,\overline{d_Bu\cdot f\omega\cdot\Psi}\,.\end{aligned}$$

From (18), (20) and (37), we have

$$\begin{aligned}\bar{D}_{\rm tr}(f\omega\bar{\cdot}\bar{\Psi}) &= -f\bar{\omega}\bar{\cdot}\overline{D_{\rm tr}\Psi} - 2f\overline{\nabla_\omega\Psi} + \overline{d_Bf\cdot\omega\cdot\Psi} + \frac{q+1}{2}\,f\overline{d_Bu\cdot\omega\cdot\Psi}\\ &= -f\omega\bar{\cdot}\bar{D}_{\rm tr}\bar{\Psi} - 2f\bar{\nabla}_\omega\bar{\Psi} + \frac{q-3}{2}\,f\overline{\omega\cdot d_Bu\cdot\Psi}\\ &\quad + \frac{q+1}{2}\,f\overline{d_Bu\cdot\omega\cdot\Psi} + \overline{d_Bf\cdot\omega\cdot\Psi} - f\omega(u)\bar{\Psi}\,,\end{aligned}$$

which implies (36). □

Let $\mathcal{K} = \{u \in \Omega_B^0(\mathcal{F})\,;\,\kappa(u) = 0\}$. Then we have the following corollary.

COROLLARY 4.2. *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a transverse spin foliation $\mathcal{F}$ and a bundle-like metric g_M. For some transversally conformal metric $\bar{g}_Q = e^{2u}g_Q$ for $u \in \mathcal{K}$, we have*

$$\bar{D}_{\rm tr}(e^{-2u}\kappa_B\bar{\cdot}\bar{\Psi}) = -e^{-2u}\big(\kappa_B\bar{\cdot}\bar{D}_{\rm tr}\bar{\Psi} + 2\bar{\nabla}_{\kappa_B^\sharp}\bar{\Psi}\big)\,. \tag{38}$$

By a long calculation, we have, for any basic functions f and g on M, and for any spinor field Ψ,

$$\begin{aligned}|\bar{\nabla}_{\rm tr}^{f,g}\bar{\Psi}|_{\bar{g}_Q}^2 &= |\bar{\nabla}_{\rm tr}\bar{\Psi}|_{\bar{g}_Q}^2 + qf^2|\bar{\Psi}|_{\bar{g}_Q}^2 + qg^2|P\kappa_{\bar{g}}|_{\bar{g}_Q}^2|\bar{\Psi}|_{\bar{g}_Q}^2 + g|P\kappa_{\bar{g}}|_{\bar{g}_Q}^2|\bar{\Psi}|_{\bar{g}_Q}^2\\ &\quad - 2f\langle\bar{D}_{\rm tr}\bar{\Psi},\bar{\Psi}\rangle_{\bar{g}_Q} - f{\rm Re}\langle P\kappa_{\bar{g}}\bar{\cdot}\bar{\Psi},\bar{\Psi}\rangle_{\bar{g}_Q} + 2g{\rm Re}\langle\bar{D}_{\rm tr}\bar{\Psi}, P\kappa_{\bar{g}}\bar{\cdot}\bar{\Psi}\rangle_{\bar{g}_Q}\\ &\quad - 4g{\rm Re}\langle\bar{\nabla}_{(P\kappa_{\bar{g}})^\sharp}\bar{\Psi},\bar{\Psi}\rangle_{\bar{g}_Q}\,.\end{aligned} \tag{39}$$

Let $D_B\Phi = \lambda\Phi$ for some nonzero Φ. From (22), we have $\bar{D}_{\rm tr}\bar{\Psi} = \lambda e^{-u}\bar{\Psi}$, where $\Psi = e^{-(q-1)u/2}\Phi$. Since $\langle X\cdot\Psi,\Psi\rangle_{g_Q}$ is pure imaginary, we have

$${\rm Re}\langle P\kappa_{\bar{g}}\bar{\cdot}\bar{\Psi},\bar{\Psi}\rangle_{\bar{g}_Q} = 0 \quad\text{and}\quad {\rm Re}\langle\bar{D}_{\rm tr}\bar{\Psi}, P\kappa_{\bar{g}}\bar{\cdot}\bar{\Psi}\rangle_{\bar{g}_Q} = 0\,. \tag{40}$$

By integration, the equation (39) together with (23) gives

$$\begin{aligned}\int_M |\bar{\nabla}_{\rm tr}^{f,g}\bar{\Psi}|_{\bar{g}_Q}^2 &= \int_M e^{-2u}\Big(\lambda^2 - 2fe^u\lambda - \frac{1}{4}\,e^{2u}K_\sigma^{\bar{\nabla}}\Big)|\bar{\Psi}|_{\bar{g}_Q}^2\\ &\quad + \int_M \big(qf^2 + qg^2|P\kappa_{\bar{g}}|_{\bar{g}_Q}^2 + g|P\kappa_{\bar{g}}|_{\bar{g}_Q}^2\big)|\bar{\Psi}|_{\bar{g}_Q}^2\\ &\quad - 4g\int_M {\rm Re}\langle\bar{\nabla}_{(P\kappa_{\bar{g}})^\sharp}\bar{\Psi},\bar{\Psi}\rangle_{\bar{g}_Q}\,.\end{aligned} \tag{41}$$

Let u be in $\mathcal{K}$. Since $\kappa_{\bar{g}} = e^{2u}\kappa$, from (38) and (40), we have

$$\begin{aligned}-2\int_M {\rm Re}\langle\bar{\nabla}_{(P\kappa_{\bar{g}})^\sharp}\bar{\Psi},\bar{\Psi}\rangle_{\bar{g}_Q} &= \int_M {\rm Re}\langle\bar{D}_{\rm tr}(e^{-2u}\kappa_B\bar{\cdot}\bar{\Psi}), e^{4u}\bar{\Psi}\rangle_{\bar{g}_Q}\\ &\quad + \int_M e^{2u}{\rm Re}\langle\kappa_B\bar{\cdot}\bar{D}_{\rm tr}\bar{\Psi},\bar{\Psi}\rangle_{\bar{g}_Q}\\ &= \int_M {\rm Re}\langle e^{-2u}\kappa_B\bar{\cdot}\bar{\Psi}, \bar{D}_{\rm tr}(e^{4u}\bar{\Psi})\rangle_{\bar{g}_Q}\,.\end{aligned}$$

From (20), we have

$$\begin{aligned}\langle e^{-2u}\kappa_B \bar{\cdot} \bar\Psi, \bar D_{\rm tr}(e^{4u}\bar\Psi)\rangle_{\bar g_Q} &= \langle e^{2u}\kappa_B \bar{\cdot} \bar\Psi, \bar D_{\rm tr}\bar\Psi\rangle_{\bar g_Q} + 4\langle e^{u}\kappa_B \bar{\cdot} \bar\Psi, \overline{d_B u\cdot\Psi}\rangle_{\bar g_Q}\\ &= \langle e^{2u}\kappa_B \bar{\cdot} \bar\Psi, \bar D_{\rm tr}\bar\Psi\rangle_{\bar g_Q} + 4\langle e^{2u}\kappa_B\cdot\Psi, d_B u\cdot\Psi\rangle_{g_Q}\,.\end{aligned}$$

On the other hand, for any $u\in\mathcal{K}$, we have

$$\begin{aligned}2\mathrm{Re}\langle\kappa_B\cdot\Psi, d_Bu\cdot\Psi\rangle_{g_Q} &= \langle\kappa_B\cdot\Psi, d_Bu\cdot\Psi\rangle_{g_Q} + \overline{\langle\kappa_B\cdot\Psi, d_Bu\cdot\Psi\rangle}_{g_Q}\\ &= 2\kappa_B(u)|\Psi|^2 = 0\,.\end{aligned}$$

Hence from (40), we have

$$\mathrm{Re}\langle e^{-2u}\kappa_B \bar{\cdot} \bar\Psi, \bar D_{\rm tr}(e^{4u}\bar\Psi)\rangle_{\bar g_Q} = 0\,,$$

which means

$$\int_M \mathrm{Re}\langle\bar\nabla_{(P\kappa_{\bar g})^\sharp}\bar\Psi, \bar\Psi\rangle_{\bar g_Q} = 0\,. \tag{42}$$

Therefore, (41) yields

$$\begin{aligned}\int_M |\bar\nabla^{f,g}_{\rm tr}\bar\Psi|^2_{\bar g_Q} &= \int_M e^{-2u}\big(qf^2 - 2e^u\lambda f + \lambda^2\big)|\bar\Psi|^2_{\bar g_Q}\\ &\quad + \int_M\Big(q|P\kappa_{\bar g}|^2_{\bar g_Q}g^2 + |P\kappa_{\bar g}|^2_{\bar g_Q}g - \frac14 K^{\bar\nabla}_\sigma\Big)|\bar\Psi|^2_{\bar g_Q}\,.\end{aligned} \tag{43}$$

If we put $f = (\lambda/q)e^{-u}$ and $g = -1/2q$, then we have

$$\int_M |\bar\nabla^{f,g}_{\rm tr}\bar\Psi|^2_{\bar g_Q} = \frac{q-1}{q}\int_M e^{-2u}\Big(\lambda^2 - \frac{q}{4(q-1)}\Big\{e^{2u}K^{\bar\nabla}_\sigma + \frac1q|\bar\kappa_B|^2\Big\}\Big)|\bar\Psi|^2_{\bar g_Q}\,. \tag{44}$$

Hence we have the following theorem.

THEOREM 4.3. *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q\ge 2$ and a bundle-like metric g_M satisfying $\delta_B\kappa_B = 0$. Assume that $K^{\bar\nabla}_\sigma$ is nonnegative for some transversally conformal metric $\bar g_Q = e^{2u}g_Q$. Then we have*

$$\lambda^2 \ge \frac{q}{4(q-1)}\sup_{u\in\mathcal{K}}\inf_M\Big(e^{2u}K^{\bar\nabla}_\sigma + \frac1q|\bar\kappa_B|^2\Big)\,, \tag{45}$$

where $K^{\bar\nabla}_\sigma = \sigma^{\bar\nabla} + |\bar\kappa_B|^2$.

The transversal Ricci curvature $\rho^{\bar\nabla}$ of $\bar g_Q = e^{2u}g_Q$ and the transversal scalar curvature $\sigma^{\bar\nabla}$ of $\bar g_Q$ are related to the transversal Ricci curvature ρ^∇ of g_Q and the transversal scalar curvature σ^∇ of g_Q by the following lemma (cf. [6, Lemma 4.3]).

LEMMA 4.4. *On a Riemannian foliation $\mathcal{F}$, we have, for any $X\in Q$,*

$$\begin{aligned}e^{2u}\rho^{\bar\nabla}(X) &= \rho^\nabla(X) + (2-q)\nabla_X d_Bu + (2-q)|d_Bu|^2X + (q-2)X(u)d_Bu\\ &\quad + \{\Delta_Bu - \kappa_B(u)\}X\,,\end{aligned} \tag{46}$$

$$e^{2u}\sigma^{\bar\nabla} = \sigma^{\nabla} + (q-1)(2-q)|d_B u|^2 + 2(q-1)\{\Delta_B u - \kappa_B(u)\}\,. \tag{47}$$

From (47), we have

$$e^{2u}K_\sigma^{\bar\nabla} = \sigma^{\nabla} + |\kappa_B|^2 + 2(q-1)\Delta_B u + (q-1)(2-q)|d_B u|^2 - 2\kappa_B(u)\,.$$

On the other hand, for $q \geq 3$, if we choose the positive function h by $u = 2\ln h/(q-2)$, then we have

$$\Delta_B u = \frac{2}{q-2}\{h^{-2}|d_B h|^2 + h^{-1}\Delta_B h\}\,, \tag{48}$$

$$|d_B u|^2 = \left(\frac{2}{q-2}\right)^2 h^{-2}|d_B h|^2\,. \tag{49}$$

Hence we have

$$e^{2u}K_\sigma^{\bar\nabla} = h^{4/(q-2)}K_\sigma^{\bar\nabla} = h^{-1}Y_B h + |\kappa_B|^2 - \frac{4}{q-2}h^{-1}\kappa_B(h)\,, \tag{50}$$

where Y_B is the basic Yamabe operator of $\mathcal{F}$ defined in [6]. If we choose u in $\mathcal{K}$, then $\kappa_B(h) = 0 = \kappa_B(u)$. From (50), we have

$$e^{2u}K_\sigma^{\bar\nabla} = K_\sigma^{\nabla} + 2(q-1)\Delta_B u = h^{-1}Y_B h + |\kappa_B|^2\,, \tag{51}$$

where $K_\sigma^{\nabla} = \sigma^{\nabla} + |\kappa_B|^2$. From (45), we have the following corollary.

COROLLARY 4.5. *Under the same condition as in Theorem* 4.3, *we have*

$$\lambda^2 \geq \begin{cases} \dfrac{q}{4(q-1)} \sup\limits_{u\in\mathcal{K}} \inf\limits_{M} \Big\{\sigma^{\nabla} + 2(q-1)\Delta_B u \\ \qquad\qquad + (q-1)(2-q)|d_B u|^2 + \dfrac{q+1}{q}\,|\kappa_B|^2\Big\} & \text{if } q \geq 2\,, \\ \dfrac{q}{4(q-1)} \sup\limits_{h\in\mathcal{K}} \inf\limits_{M} \Big\{h^{-1}Y_B h + \dfrac{q+1}{q}\,|\kappa_B|^2\Big\} & \text{if } q \geq 3\,. \end{cases}$$

COROLLARY 4.6. *Let* $(M, g_M, \mathcal{F})$ *be a compact Riemannian manifold with a transverse spin foliation* $\mathcal{F}$ *of codimension* $q \geq 3$ *and a bundle-like metric* g_M *satisfying* $\delta_B\kappa_B = 0$. *Assume that* σ^{∇} *is nonnegative. Then any eigenvalue* λ *of the basic Dirac operator satisfies*

$$\lambda^2 \geq \frac{q}{4(q-1)}\left(\mu_1 + \frac{q+1}{q}\inf|\kappa_B|^2\right), \tag{52}$$

where μ_1 *is the first eigenvalue of the basic Yamabe operator* Y_B *of* $\mathcal{F}$.

Now, we study the limiting case. We define $\mathrm{Ric}^{f,g}_{\bar\nabla} : \Gamma Q \otimes \bar S(\mathcal{F}) \to \bar S(\mathcal{F})$ by

$$\mathrm{Ric}^{f,g}_{\bar\nabla}(X \otimes \bar\Psi) = \sum \bar E_a \,\bar\cdot\, \bar R^{f,g}(X, \bar E_a)\bar\Psi\,, \tag{53}$$

where $\bar{R}^{f,g}$ is the curvature tensor with respect to $\bar{\nabla}^{f,g}$. For $X \in \Gamma Q$ and $\Psi \in \Gamma S(\mathcal{F})$, we have

$$\begin{aligned}
\bar{\nabla}^{f,g}_X \bar{\nabla}^{f,g}_{\bar{E}_a} \bar{\Psi} = {} & \bar{\nabla}_X \bar{\nabla}_{\bar{E}_a} \bar{\Psi} + f X \bar{\cdot} \bar{\nabla}_{\bar{E}_a} \bar{\Psi} + g P\kappa_{\bar{g}} \bar{\cdot} X \bar{\cdot} \bar{\nabla}_{\bar{E}_a} \bar{\Psi} + X(f) \bar{E}_a \bar{\cdot} \bar{\Psi} \\
& + f \bar{\nabla}_X \bar{E}_a \bar{\cdot} \bar{\Psi} + f \bar{E}_a \bar{\cdot} \bar{\nabla}_X \bar{\Psi} + f^2 X \bar{\cdot} \bar{E}_a \bar{\cdot} \bar{\Psi} \\
& + f g P\kappa_{\bar{g}} \bar{\cdot} X \bar{\cdot} \bar{E}_a \bar{\cdot} \bar{\Psi} + X(g) P\kappa_{\bar{g}} \bar{\cdot} \bar{E}_a \bar{\cdot} \bar{\Psi} \\
& + g \bar{\nabla}_X P\kappa_{\bar{g}} \bar{\cdot} \bar{E}_a \bar{\cdot} \bar{\Psi} + g P\kappa_{\bar{g}} \bar{\cdot} \bar{\nabla}_X \bar{E}_a \bar{\cdot} \bar{\Psi} \\
& + g P\kappa_{\bar{g}} \bar{\cdot} \bar{E}_a \bar{\cdot} \bar{\nabla}_X \bar{\Psi} + f g X \bar{\cdot} P\kappa_{\bar{g}} \bar{\cdot} \bar{E}_a \bar{\cdot} \bar{\Psi} \\
& + g^2 P\kappa_{\bar{g}} \bar{\cdot} X \bar{\cdot} P\kappa_{\bar{g}} \bar{\cdot} \bar{E}_a \bar{\cdot} \bar{\Psi}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\bar{R}^{f,g}(X, \bar{E}_a)\bar{\Psi} = {} & \bar{R}^S(X, \bar{E}_a)\bar{\Psi} + X(f)\bar{E}_a \bar{\cdot} \bar{\Psi} - 2f^2 \bar{E}_a \bar{\cdot} X \bar{\cdot} \bar{\Psi} - 2f^2 \bar{g}_Q(X, \bar{E}_a)\bar{\Psi} \\
& - 2fg\bar{g}_Q(P\kappa_{\bar{g}}, X)\bar{E}_a \bar{\cdot} \bar{\Psi} - X(g)\bar{E}_a \bar{\cdot} P\kappa_{\bar{g}} \bar{\cdot} \bar{\Psi} - \bar{E}_a(f) X \bar{\cdot} \bar{\Psi} \\
& - 2X(g)\bar{g}_Q(P\kappa_{\bar{g}}, \bar{E}_a)\bar{\Psi} - g\bar{E}_a \bar{\cdot} \bar{\nabla}_X P\kappa_{\bar{g}} \bar{\cdot} \bar{\Psi} - 2g\bar{g}_Q(\bar{\nabla}_X P\kappa_{\bar{g}}, \bar{E}_a)\bar{\Psi} \\
& - 2g^2 |P\kappa_{\bar{g}}|^2 \bar{E}_a \bar{\cdot} X \bar{\cdot} \bar{\Psi} - 2g^2 |P\kappa_{\bar{g}}|^2 \bar{g}_Q(X, \bar{E}_a)\bar{\Psi} \\
& + 2g^2 \bar{g}_Q(X, P\kappa_{\bar{g}})\bar{E}_a \bar{\cdot} P\kappa_{\bar{g}} \bar{\cdot} \bar{\Psi} + 4g^2 \bar{g}_Q(X, P\kappa_{\bar{g}})\bar{g}_Q(P\kappa_{\bar{g}}, \bar{E}_a)\bar{\Psi} \\
& + 2g^2 \bar{g}_Q(P\kappa_{\bar{g}}, \bar{E}_a) P\kappa_{\bar{g}} \bar{\cdot} X \bar{\cdot} \bar{\Psi} + 2fg\bar{g}_Q(P\kappa_{\bar{g}}, \bar{E}_a) X \bar{\cdot} \bar{\Psi} \\
& - \bar{E}_a(g) P\kappa_{\bar{g}} \bar{\cdot} X \bar{\cdot} \bar{\Psi} - g\bar{\nabla}_{\bar{E}_a} P\kappa_{\bar{g}} \bar{\cdot} X \bar{\cdot} \bar{\Psi}.
\end{aligned}$$

By a simple calculation, we have, from (11) and (53),

$$\begin{aligned}
\mathrm{Ric}^{f,g}_{\bar{\nabla}}(X \otimes \bar{\Psi}) = {} & -\frac{1}{2} \rho^{\bar{\nabla}}(X) \bar{\cdot} \bar{\Psi} - qX(f)\bar{\Psi} + 2(q-1) f^2 X \bar{\cdot} \bar{\Psi} \\
& + 2qfg\bar{g}_Q(P\kappa_{\bar{g}}, X)\bar{\Psi} + (q-2)X(g) P\kappa_{\bar{g}} \bar{\cdot} \bar{\Psi} \\
& + (q-2) g \bar{\nabla}_X P\kappa_{\bar{g}} \bar{\cdot} \bar{\Psi} - \overline{d_B f} \bar{\cdot} X \bar{\cdot} \bar{\Psi} \\
& + 2(q-2) g^2 |P\kappa_{\bar{g}}|^2 X \bar{\cdot} \bar{\Psi} \\
& - 2(q-2) g^2 \bar{g}_Q(X, P\kappa_{\bar{g}}) P\kappa_{\bar{g}} \bar{\cdot} \bar{\Psi} - 2fg P\kappa_{\bar{g}} \bar{\cdot} X \bar{\cdot} \bar{\Psi} \\
& - \overline{d_B g} \bar{\cdot} P\kappa_{\bar{g}} \bar{\cdot} X \bar{\cdot} \bar{\Psi} + g|P\kappa_{\bar{g}}|^2 X \bar{\cdot} \bar{\Psi} \,.
\end{aligned} \tag{54}$$

On the other hand, we have the following proposition.

PROPOSITION 4.7. *If a non-zero spinor field Ψ satisfies $\bar{\nabla}^{f,g}_{\mathrm{tr}} \bar{\Psi} = 0$, then*

$$\begin{aligned}
\nabla_X \Psi = {} & -f e^u \pi(X) \cdot \Psi - g\kappa_B \cdot \pi(X) \cdot \Psi + \frac{1}{2} g_Q(d_B u, \pi(X))\Psi \\
& + \frac{1}{2} \pi(X) \cdot d_B u \cdot \Psi \,.
\end{aligned} \tag{55}$$

PROOF. Let $\bar{\nabla}^{f,g}_{\mathrm{tr}} \bar{\Psi} = 0$. From (35), we have

$$\bar{\nabla}_X \bar{\Psi} + f\pi(X) \bar{\cdot} \bar{\Psi} + g P\kappa_{\bar{g}} \bar{\cdot} \pi(X) \bar{\cdot} \bar{\Psi} = 0 \,.$$

Hence, from (18), we have

$$\overline{\nabla_X \Psi} - \frac{1}{2}\,\overline{\pi(X)\cdot d_B u \cdot \Psi} - \frac{1}{2} X(u)\bar{\Psi} + f e^u \overline{\pi(X)\cdot\Psi} + g\overline{\kappa_B \cdot \pi(X)\cdot \Psi} = 0\,,$$

which proves (55). □

THEOREM 4.8. *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 3$ and a bundle-like metric g_M satisfying $\delta_B \kappa_B = 0$. Assume that σ^∇ is nonnegative. If there exists an eigenspinor field Φ_1 of the basic Dirac operator D_B for the eigenvalue λ_1 satisfying*

$$\lambda_1^2 = \frac{q}{4(q-1)}\left(\mu_1 + \frac{q+1}{q}\inf|\kappa_B|^2\right),$$

then $\mathcal{F}$ is transversally Einsteinian with a positive constant transversal scalar curvature σ^∇ and $\kappa_B = 0$.

PROOF. Let $D_B\Phi_1 = \lambda_1\Phi_1$ with

$$\lambda_1^2 = \frac{q}{4(q-1)}\left(\mu_1 + \frac{q+1}{q}\inf|\kappa_B|^2\right).$$

Let $\Psi = e^{-(q-1)u/2}\Phi_1$. From (44), we see that $\bar{\nabla}^{f_1,g_1}_{\mathrm{tr}}\bar{\Psi} = 0$, where $f_1 = (\lambda_1/q)e^{-u}$ and $g_1 = -1/2q$. Hence we have, from (35),

$$\bar{\nabla}_{\bar{E}_a}\bar{\Psi} + f\bar{E}_a \,\bar{\cdot}\, \bar{\Psi} + gP\kappa_{\bar{g}} \,\bar{\cdot}\, \bar{E}_a \,\bar{\cdot}\, \bar{\Psi} = 0\,.$$

Therefore, we have

$$\sum_a \bar{E}_a \,\bar{\cdot}\, \bar{\nabla}_{\bar{E}_a}\bar{\Psi} = qf\bar{\Psi} - (q-2)gP\kappa_{\bar{g}} \,\bar{\cdot}\, \bar{\Psi}\,,$$

and then

$$\bar{D}_{\mathrm{tr}}\bar{\Psi} + \frac{1}{2}\,P\kappa_{\bar{g}} \,\bar{\cdot}\, \bar{\Psi} = qf\bar{\Psi} - (q-2)gP\kappa_{\bar{g}} \,\bar{\cdot}\, \bar{\Psi}\,.$$

Since $\bar{D}_B\bar{\Psi} = \lambda_1 e^{-u}\bar{\Psi}$, we have

$$\lambda_1 e^{-u}\bar{\Psi} + \frac{1}{2}\,P\kappa_{\bar{g}} \,\bar{\cdot}\, \bar{\Psi} = \lambda_1 e^{-u}\bar{\Psi} + \frac{q-2}{2q}\,P\kappa_{\bar{g}} \,\bar{\cdot}\, \bar{\Psi}\,.$$

Hence we have $\kappa_B \cdot \Psi = 0$, which implies $\kappa_B = 0$. If $\bar{\nabla}^{f,g}_X\bar{\Psi} = 0$ for any $X \in \Gamma Q$, then $\mathrm{Ric}^{f,g}_{\bar{\nabla}} = 0$. Let $X = (d_B f)^\sharp$. Then, from (54), we get

$$\left\langle\left(-\frac{1}{2}\,\rho^{\bar{\nabla}}(X) + 2(q-1)f^2 X\right) \bar{\cdot}\, \bar{\Psi}, \bar{\Psi}\right\rangle_{\bar{g}_Q} = (q-1)|d_B f|^2_{\bar{g}_Q}|\bar{\Psi}|^2_{\bar{g}_Q}\,. \tag{56}$$

Hence the left-hand side in the equation (56) is pure imaginary while the right-hand side in the equation (56) is real, and so both sides are all zero. That is, $d_B f = 0$. So u is constant. Also, we have, from (54),

$$\rho^{\bar{\nabla}}(X) = 4(q-1)f^2 X \tag{57}$$

for any $X \in \Gamma Q$. Since u is constant, from (46), we have

$$\rho^{\nabla}(X) = \frac{4(q-1)}{q^2}\lambda_1^2 X\,. \tag{58}$$

Hence $\mathcal{F}$ is transversally Einsteinian with a constant transversal scalar curvature $\sigma^{\nabla} = (4(q-1)/q)\lambda_1^2$. □

REMARK 4.9. The existence of the bundle-like metric such that κ is basic-harmonic is assured from [2, Theorem 4], [7, Theorem 2.10] and [8, Theorem 6.2]. So Theorems 3.4 and 4.8 imply that $\mathcal{F}$ is minimal, transversal Einsteinian.

Acknowledgments. The authors are grateful to the referee for his/her kind suggestions and comments.

REFERENCES

[1] J. A. ALVAREZ LÓPEZ, The basic component of the mean curvature of Riemannian foliations, Ann. Global Anal. Geom. 10 (1992), 179–194.

[2] D. DOMÍNGUEZ, A tenseness theorem for Riemannian foliations, C. R. Acad. Sci. Paris Sér. I Math. 320 (1995), 1331–1335.

[3] J. F. GLAZEBROOK AND F. W. KAMBER, Transversal Dirac families in Riemannian foliations, Comm. Math. Phys. 140 (1991), 217–240.

[4] S. D. JUNG, The first eigenvalue of the transversal Dirac operator, J. Geom. Phys. 39 (2001), 253–264.

[5] S. D. JUNG, Eigenvalue estimates for the basic Dirac operator on a Riemannian foliation admitting a basic harmonic 1-form, J. Geom. Phys. 57 (2007), 1239–1246.

[6] S. D. JUNG, B. H. KIM AND J. S. PAK, Lower bounds for the eigenvalues of the basic Dirac operator on a Riemannian foliation, J. Geom. Phys. 51 (2004), 116–182.

[7] P. MARCH, M. MIN-OO AND E. A. RUH, Mean curvature of Riemannian foliations, Canad. Math. Bull. 39 (1996), 95–105.

[8] A. MASON, An application of stochastic flows to Riemannian foliations, Houston J. Math. 26 (2000), 481–515.

[9] E. PARK AND K. RICHARDSON, The basic Laplacian of a Riemannian foliation, Amer. J. Math. 118 (1966), 1249–1275.

[10] Ph. TONDEUR, Foliations on Riemannian manifolds, Springer-Verlag, New-York, 1988.

DEPARTMENT OF MATHEMATICS
JEJU NATIONAL UNIVERSITY
JEJU 690–756
KOREA

E-mail addresses: sdjung@cheju.ac.kr
kscyh@dreamwiz.com

DEPARTMENT OF MATHEMATICS
SUNGKYUNGKWAN UNIVERSITY
SUWON 440–746
KOREA

E-mail address: parkj@skku.edu