10-1993

# New Exactly Solvable Hamiltonians - Shape Invariance and Self-Similarity

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## Recommended Citation

Barclay, D.T., Dutt, R., Gangopadhyaya, A., Khare, A., Pagnamenta, A., and Sukhatme, U., (1993). New exactly solvable Hamiltonians- Shape invariance and self-similarity. Phys. Rev. A. 48(4): 2786-2797.

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PHYSICAL REVIEW A VOLUME 48, NUMBER 4 OCTOBER 1993

# New exactly solvable Hamiltonians: Shape invariance and self-similarity

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(Received 3 May 1993)

We discuss in some detail the self-similar potentials of Shabat [Inverse Prob. **8**, 303 (1992)] and Spiridonov [Phys. Rev. Lett. **69**, 298 (1992)] which are reflectionless and have an infinite number of bound states. We demonstrate that these self-similar potentials are in fact shape-invariant potentials within the formalism of supersymmetric quantum mechanics. In particular, using a scaling *Ansatz* for the change of parameters, we obtain a large class of new, reflectionless, shape-invariant potentials of which the Shabat-Spiridonov ones are a special case. These new potentials can be viewed as $q$ deformations of the single-soliton solution corresponding to the Rosen-Morse potential. Explicit expressions for the energy eigenvalues, eigenfunctions, and transmission coefficients for these potentials are obtained. We show that these potentials can also be obtained numerically. Included as an intriguing case is a shape-invariant double-well potential whose supersymmetric partner potential is only a single well. Our class of exactly solvable Hamiltonians is further enlarged by examining two new directions: (i) changes of parameters which are different from the previously studied cases of translation and scaling and (ii) extending the usual concept of shape invariance in one step to a multistep situation. These extensions can be viewed as $q$ deformations of the harmonic oscillator or multisoliton solutions corresponding to the Rosen-Morse potential.

PACS number(s): 03.65.Ge, 11.30.Pb

## I. INTRODUCTION

Recently Shabat [1] and Spiridonov [2] have discussed potentials which are reflectionless and have an infinite number of bound states. In addition, these potentials have the remarkable property of being self-similar and can be looked upon as $q$ deformations of the single-soliton solution corresponding to the Rosen-Morse potential. Normally, in quantum Lie algebras one takes the underlying space to be noncommutative and the deformation parameter $q$ measures deviation from normal analysis. In contrast, an interesting feature of Shabat [1] and Spiridonov's work [2] is that they have considered the same problem in a commutative space with the $q$ deformation arising from the specific nature of the potential. Spiridonov [3] has also considered the deformation of parasupersymmetric quantum mechanics [4] and obtained potentials which can be regarded as a $q$ deformation of the two-soliton solution corresponding to the Rosen-Morse potential.

Another interesting advance of recent years has been in supersymmetric quantum mechanics [5] where new insight into exactly solvable potentials has been obtained through the concept of shape invariance. It has been shown [6] that for supersymmetric partner potentials $V_\pm(x,a_0)$ satisfying the properties of shape invariance and unbroken supersymmetry, one can write down the energy eigenvalues algebraically. Subsequently it was shown that both the eigenfunctions [7] and the scattering matrix [8] can also be obtained algebraically for these potentials. The shape-invariance condition is given by

$$V_+(x,a_0) = V_-(x,a_1) + R(a_0), \tag{1.1}$$

where $a_0$ is a set of parameters, $a_1 = f(a_0)$ is an arbitrary function describing the change of parameters, and the remainder $R(a_0)$ is independent of $x$. Certain solutions to the shape-invariance condition are known [9] including essentially all the standard problems discussed in quantum mechanics textbooks. In all these cases $a_1$ and $a_0$ have been related by a translation ($a_1 = a_0 + \alpha$). Careful analyses with this *Ansatz* have failed to yield any additional shape-invariant potentials [10]. Indeed it has been suggested [11] that there are no other shape-invariant potentials. Although a rigorous proof has never been given, no counter examples have so far been found either.

 

In this paper we show that the Shabat-Spiridonov (SS) self-similar potentials can be understood within the framework of shape invariance. In particular, we show that by using a scaling *Ansatz* ($a_1 = qa_0$) for the change of parameter $a_0$, a large class of new shape-invariant potentials which are reflectionless and possess an infinite number of bound states can be obtained [12]. [This is slightly misleading in that a reparametrization of the form $a_1 = qa_0$ can be recast as $a'_1 = a'_0 + \alpha$ merely by taking logarithms. However, since the choice of parameter is usually an integral part of constructing a shape-invariant potential, it is in practice part of the *Ansatz*. For instance, in Sec. III, we will construct potentials by expanding in $a_0$, a procedure whose legitimacy and outcome are clearly dependent on our choice of parameter (and hence reparametrization). Note that, although the construction is noninvariant, the resulting potentials will still be invariant under redefinitions of $a_0$.] Our potentials contain the self-similar potentials of Shabat [1] and Spiridonov [2] but are considerably more general.

The plan of the paper is the following. In Sec. II, we discuss in some detail the self-similar potentials of Shabat and Spiridonov (SS). An unfortunate feature of these potentials is that they are not known in analytical form for all $x$ and we therefore give graphs of these potentials for some values of the deformation parameter, which we denote by $p$ $(0 < p < 1)$.

In Sec. III we briefly discuss the shape-invariance condition within the formalism of supersymmetric quantum mechanics. Using a scaling *Ansatz* ($a_1 = qa_0$), we obtain a large class of new reflectionless shape-invariant potentials. Explicit expressions for the eigenvalues, eigenfunctions, and transmission coefficients for these potentials are derived. These potentials can be viewed as $q$ deformations of a one-dimensional harmonic oscillator or of the single-soliton solution corresponding to the Rosen-Morse potential. The self-similar potentials of SS [1, 2] are rederived as a special case.

In Sec. IV we discuss a new technique that essentially solves the inverse scattering problem for this wider class numerically and so enables us to calculate these potentials. Examples of the results obtained are discussed.

In Sec. V we give the Taylor-series expansion of the potentials for large $x$. By using these reflectionless potentials as solutions of the Korteweg–de Vries (KdV) equation, we then estimate the area under them and indicate how one can also estimate higher moments for these cases rigorously even though the potentials are not known in analytical form. Using the ground-state wave function for these potentials, we also give graphs of a continuous parameter family of potentials which are strictly isospectral to one of the self-similar potentials.

Section VI introduces multistep scaling *Ansätze* for the change of parameters and hence obtains new shape-invariant potentials which can be looked upon as $q$ deformations of the multisoliton solutions corresponding to the Rosen-Morse potential.

In Sec. VII we discuss various other *Ansätze* for connection between parameters $a_1$ and $a_0$ and obtain yet more new shape-invariant potentials. Explicit expressions for the eigenvalues and the eigenfunctions of these potentials are also given.

Finally, in Sec. VIII, we summarize the results of this paper and indicate some open problems.

## II. SELF-SIMILAR POTENTIALS

Shabat [1] and Spiridonov [2] discussed an infinite chain of reflectionless Hamiltonians given by ($\hbar = 2m = 1$)

$$H_n = P^2 + V_n(x), \qquad n = 0, 1, 2, \ldots, \tag{2.1}$$

with

$$V_0(x) = W_0^2 - W_0' + C_0 \tag{2.2}$$

and

$$V_{n+1}(x) = V_n(x) + 2W_n'(x). \tag{2.3}$$

The various superpotentials $W_n(x)$ satisfy the following set of differential equations:

$$W_n^2 + W_n' = W_{n+1}^2 - W_{n+1}' + C_{n+1}, \qquad n = 0, 1, 2, \ldots, \tag{2.4}$$

where the $C_n$ are arbitrary positive constants. It is amusing to note that relations (2.4) arise naturally in the framework of parasupersymmetric quantum mechanics [4]. Let us assume at this stage that all $W_n(x)$ are such that the functions

$$\psi_0^{(n)}(x) \propto \exp\Big[-\int^x W_n(y)dy\Big] \tag{2.5}$$

are square integrable, and hence correspond to the ground-state wave function of the Hamiltonian $H_n$. In this case one has the standard situation of unbroken supersymmetry and the potential $V_{i+1}(x)$ has one bound state less than $V_i(x)$. Using Eqs. (2.1)–(2.4), it follows that the eigenvalues of $H_0$ are given by

$$E_m^{(0)} = \sum_{i=0}^{m} C_i, \qquad m = 0, 1, 2, \ldots \tag{2.6}$$

while the corresponding eigenfunctions are given by

$$\psi_m^{(0)}(x) \propto (P + iW_0)(P + iW_1) \cdots (P + iW_{m-1})\psi_0^{(m)}(x). \tag{2.7}$$

It should be noted here that the Hamiltonian $H_j$ has the same spectrum as $H_0$ except that the lowest $j$ levels of $H_0$ are missing.

In general, it is not possible to determine these potentials unless one imposes some extra constraints. SS specify superpotentials by demanding that all superpotentials $W_n(x)$ satisfy the self-similar *Ansatz*

$$W_i(x) = p^i W(p^i x), \tag{2.8}$$

with $0 < p < 1$. Thus there is just one unknown function $W(x)$. On using $i = 0$ and 1 in Eqs. (2.8) and (2.4), one obtains the following finite-difference differential equation defining $W(x)$:

$$W^2(x) + W'(x) = p^2 W^2(px) - p^2 W'(px) + C_1, \quad (2.9)$$

where the prime denotes differentiation with respect to the argument. Equations (2.8) and (2.9) are the key statements underlying the concept of self-similarity. On using $i = 2, 3, \ldots$ and Eqs. (2.4), (2.8), and (2.9) one then concludes that

$$C_n = (p^2)^{n-1} C_1 \quad (2.10)$$

and hence the $m$th eigenvalue of $H_0$ is given by

$$E_m^{(0)} = C_0 + \frac{C_1(1-p^{2m})}{(1-p^2)}, \qquad m = 0, 1, 2, \ldots, \quad (2.11)$$

One can choose the arbitrary constant $C_0$ to be zero, which corresponds to taking $E_0^{(0)} = 0$. An alternate convenient choice is to pick $C_0$ such that

$$\lim_{m \to \infty} E_m^{(0)} = 0, \quad (2.12)$$

which gives

$$E_m^{(0)} = -\frac{C_1 p^{2m}}{(1-p^2)}, \quad m = 0, 1, 2, \ldots . \quad (2.13)$$

One can try to find $W(x)$ by solving the finite-difference differential Eq. (2.9) in a Taylor-series form near $x = 0$; if

$$W(x) = \sum_{j=0}^{\infty} b_j x^j \quad (2.14)$$

then

$$b_1(1+p^2) + b_0^2(1-p^2) = C_1 \quad (2.15)$$

and

$$b_{j+1} = -\frac{(1-p^{j+2})}{(j+1)(1+p^{j+2})} \sum_{m=0}^{j} b_m b_{j-m}, \qquad j = 1, 2, \ldots . \quad (2.16)$$

Normalizability of wave functions is ensured if $W(x)$ is a continuous function, positive at $x \to \infty$ and negative at $x \to -\infty$. This is the case if one chooses $b_0 = 0$. In this case, it follows from (2.16) that all even coefficients $b_j$ $(j = 0, 2, 4, \ldots)$ are zero and hence $W(x) = -W(-x)$. In particular one finds

$$W(x) = \frac{C_1}{(1+p^2)} x - \frac{1}{3}\left(\frac{C_1}{1+p^2}\right)^2 \frac{(1-p^4)}{(1+p^4)} x^3 + O(x^5). \quad (2.17)$$

Some special cases are worth noting. At $p = 1$ the solution of Eq. (2.9) is the standard one-dimensional harmonic oscillator with $W(x) = C_1 x/2$, while in the limit $p \to 0$, the solution of (2.9) is the one-soliton superpotential corresponding to the Rosen-Morse potential given by

$$W(x) = \sqrt{C_1} \tanh(\sqrt{C_1} x). \quad (2.18)$$

Hence the general solution to Eq. (2.9) with $0 < p < 1$ can be looked upon as a deformation of the hyperbolic-tangent function with $p$ acting as the deformation parameter. Notice that the number of bound states as given by (2.13) increases discontinuously from just one at $p = 0$ to infinity for $p > 0$.

It is important to note that the superpotentials $W(x, C_1)$ which solve the self-similarity condition (2.9) have the simple scaling property

$$W(x, C_1) = \sqrt{C_1} F(\sqrt{C_1} x), \quad (2.19)$$

where $F(x) \equiv W(x, 1)$. Thus for a particular $p$ one only needs to find $W(x, C_1)$ for any one nontrivial value of $C_1$. Knowing $W(x)$, one immediately knows $V_0(x)$ and the other potentials $V_n(x)$ can be recursively obtained from $V_0(x)$ using Eqs. (2.1)–(2.4), (2.8), and (2.10), so that the whole chain of potentials is known in principle once $V_0(x)$ is known.

Unfortunately, merely knowing the Taylor series about $x = 0$ is not sufficient if one wants to carry out this program, since simple arguments show this series (2.17) to have a radius of convergence $R$, where

$$\frac{\pi}{2} \le \sqrt{C_1} R \le \frac{\pi}{2} \sqrt{\frac{1+p^2}{1-p^2}}, \qquad 0 < p \le 1. \quad (2.20)$$

The lower bound derives from noticing that the series coefficients are smaller for $0 < p < 1$ than for $p = 0$, the latter being essentially the expansion of $\tanh(\sqrt{C_1} x)$ and known to have radius of convergence $\pi/2$ (see also [13]). The upper bound involves realizing that there still has to be a pole on the imaginary axis when $p > 0$: from (2.17) one sees that along that axis $W(ix) = i\omega(x)$ inside any radius of convergence, where $\omega(x)$ is a real function satisfying

$$-\omega^2(x) + \omega'(x) = -p^2 \omega^2(px) - p^2 \omega'(px) + C_1, \quad (2.21)$$

but also that $\omega(x) > \omega(px)$ and $\omega'(x) > \omega'(px)$, so that

$$\omega'(x)(1+p^2) > \omega^2(x)(1-p^2) + C_1 \quad (2.22)$$

implying that $\omega(x)$ grows faster than $\tan(\sqrt{C_1}\sqrt{1-p^2} x / \sqrt{1+p^2})$ and so has the requisite singularity.

In the absence of a solution to (2.9) in terms of elementary functions for all $p$, one must resort to some sort of numerical determination other than by trying to sum the Taylor series. The most direct approach relies on noticing that since $px < x$, if one already knows $W(x)$ in an interval, then one knows the right hand side of (2.9) in a larger interval. This allows us to treat the right hand side as an already known function of $x$ and thus integrate up the equation using a (fourth order) Runge-Kutta method. To start this process off, the Taylor series was summed to 75 terms on an interval well inside the radius of convergence. The superpotentials thus obtained have been checked both by comparing them to the summed series throughout the region where the latter is still valid and by direct numerical integration of the Schrödinger equation

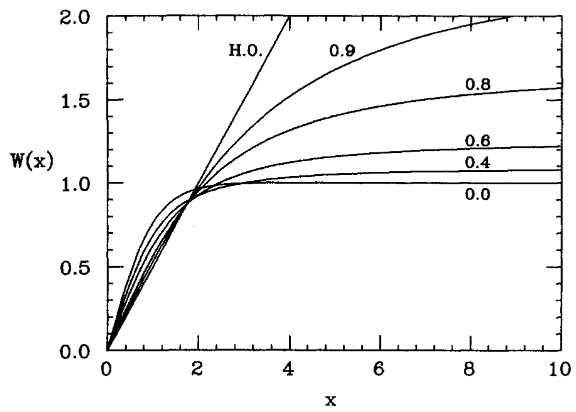


FIG. 1. Self-similar superpotentials $W(x)$ for various values of the deformation parameter $p$. The Taylor series are summed for $x < 0.4$ and the functions extrapolated from there by numerically solving the self-similarity condition, Eq. (2.9). The curve labeled H.O. (harmonic oscillator) corresponds to the $p = 1$ limit.

with the spectra checked against (2.13). In both cases the degree of agreement is extremely high. Examples of the kind of superpotentials and potentials obtained are shown in Figs. 1 and 2. Note that, having independently calculated these functions, we see no evidence of the oscillations in them reported in [13].

Aside from its practical utility, the insight underlying the numerical determination also strengthens confidence that (2.9) actually has a solution. Because any solution in a finite interval can be analytically continued to arbitrarily large $x$, the question of existence clearly reduces to establishing it in a neighborhood around $x = 0$. But this is precisely the place where a convergent Taylor series is known to exist and this is sufficient.

Additional analytic properties of these potentials will be described in Sec. V, but first we wish to introduce our

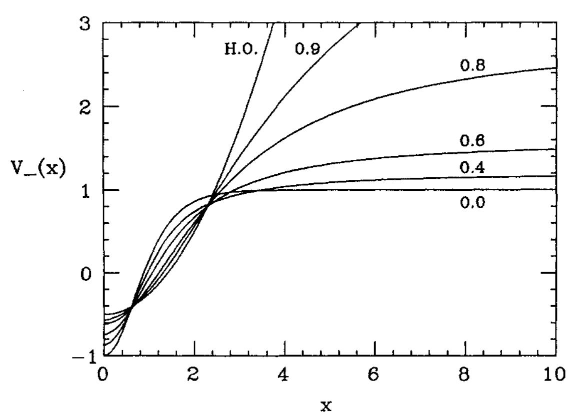


FIG. 2. Self-similar potentials $V_-(x)$ (symmetric about $x = 0$) corresponding to the superpotentials graphed in Fig. 1. The curve labeled H.O. (harmonic oscillator) corresponds to the $p = 1$ limit.

more general class of potentials to which the methods of that section will also apply.

## III. SHAPE INVARIANCE WITH SCALING *ANSATZ*

A fresh impetus to the study of exactly solvable problems in nonrelativistic quantum mechanics was provided by Gendenshteĭn [6] with the introduction of shape-invariant partner potentials within the framework of supersymmetric quantum mechanics. To set our notations, we give a quick review of both supersymmetric quantum mechanics and shape invariance [9]. The partner Hamiltonians $H_\pm$ are given by

$$H_- = A^+ A, \qquad H_+ = AA^+, \tag{3.1}$$

where ($\hbar = 2m = 1$)

$$A = \frac{d}{dx} + W(x), \qquad A^+ = \frac{d}{dx} - W(x), \tag{3.2}$$

so that the two partner potentials $V_\pm(x)$ can be expressed in terms of the superpotential $W(x)$ thus

$$V_\pm(x) = W^2(x) \pm W'(x). \tag{3.3}$$

From here it follows that all the energy eigenvalues of $H_\pm$ are positive semidefinite. Further, it turns out that in case supersymmetry (SUSY) is unbroken the ground-state energy of one of the two Hamiltonians is zero and all other energy eigenvalues of $H_\pm$ are paired

$$E_0^{(-)} = 0, \qquad E_{n+1}^{(-)} = E_n^{(+)}. \tag{3.4}$$

Here, for convention's sake, we always consider the situation of unbroken SUSY and so the ground-state energy of $H_-$ is zero. The corresponding eigenfunction $\psi_0^{(-)}(x)$ [which satisfies $A\psi_0^{(-)}(x) = 0$] turns out to be

$$\psi_0^{(-)}(x) = Ne^{-\int^x W(y)dy}. \tag{3.5}$$

One can also show that because of SUSY the eigenfunctions and scattering amplitudes of the two partner Hamiltonians are also related,

$$\psi_n^{(+)} = \frac{A\psi_{n+1}^{(-)}}{\sqrt{E_n^{(+)}}}, \qquad \psi_{n+1}^{(-)} = \frac{A^+\psi_n^{(+)}}{\sqrt{E_n^{(+)}}}, \tag{3.6}$$

$$R_-(k) = \frac{W_- + ik}{W_- - ik} R_+(k), \tag{3.7}$$

$$T_-(k) = \frac{W_+ - ik'}{W_- - ik} T_+(k), \tag{3.8}$$

where $k = (E - W_-^2)^{1/2}$ and $k' = (E - W_+^2)^{1/2}$ with $W_\pm = W(x = \pm\infty)$.

If the pair of SUSY partner potentials $V_\pm(x)$ defined by Eq. (3.3) differ only via the parameters that appear in them, then they are said to be shape invariant [6]; that is, if the partner potentials $V_\pm(x, a_0)$ satisfy the condition (1.1). In terms of the superpotential $W$, this shape-invariance condition reads

$$W^2(x,a_0)+W'(x,a_0) = W^2(x,a_1)-W'(x,a_1)+R(a_0). \quad (3.9)$$

The common $x$ dependence in $V_-$ and $V_+$ allows a full determination of energy eigenvalues [6], eigenfunctions [7], and scattering matrices [8] algebraically. One finds

$$E_n^{(-)}(a_0)=\sum_{k=0}^{n-1}R(a_k), \qquad E_0^{(-)}(a_0)=0, \quad (3.10)$$

$$\psi_n^{(-)}(x,a_0) = A^+(x,a_0)A^+(x,a_1)\cdots A^+(x,a_{n-1})\psi_0^{(-)}(x,a_n). \quad (3.11)$$

It is still a challenging open problem to identify and classify all the solutions to the shape-invariance condition (3.9). Certain solutions to it are known [9] and they include essentially all exactly solvable problems discussed in standard texts on quantum mechanics. For all these, $a_1$ and $a_0$ are related by a translation. Careful analysis with this *Ansatz* has failed to uncover any additional shape-invariant potentials [10] and in fact it has been suggested that there are no others [11]. We shall now show that this is not the case since a large number of new shape-invariant potentials can result from a new scaling *Ansatz*

$$a_1 = qa_0, \quad (3.12)$$

where $0<q<1$, a choice motivated by recent interest in $q$-deformed Lie algebras. Our approach includes the self-similar potentials discussed in the preceding section as a special case.

Consider an expansion of the superpotential of the form

$$W(x,a_0)=\sum_{j=0}^{\infty}g_j(x)a_0^j. \quad (3.13)$$

Using Eqs. (3.12) and (3.13) in the shape-invariance condition (3.9), writing $R(a_0)$ in the form

$$R(a_0)=\sum_{j=0}^{\infty}R_j a_0^j, \quad (3.14)$$

and equating powers of $a_0$ yields

$$2g_0'(x)=R_0, \qquad g_1'(x)+2d_1g_0(x)g_1(x)=r_1d_1, \quad (3.15)$$

$$g_n'(x)+2d_ng_0(x)g_n(x)=r_nd_n-d_n\sum_{j=1}^{n-1}g_j(x)g_{n-j}(x), \quad (3.16)$$

where

$$r_n\equiv\frac{R_n}{(1-q^n)}, \quad d_n\equiv\frac{(1-q^n)}{(1+q^n)}, \quad n=1,2,3,\ldots\,. \quad (3.17)$$

This set of linear differential equations is easily solvable in succession to give a general solution of Eq. (3.9). Let us first consider the special case

$$g_0(x)=0, \quad (3.18)$$

which implies $R_0=0$. The general solution of (3.16) is then

$$g_1(x)=r_1d_1x,$$

$$g_n(x)=d_n\int dx\Big[r_n-\sum_{j=1}^{n-1}g_j(x)g_{n-j}(x)\Big], \quad n=2,3,\ldots\,, \quad (3.19)$$

where without any loss of generality we have assumed all the constants of integration to be zero. The shape-invariance condition thus essentially fixes the $g_n(x)$ [and hence $W(x,a_0)$ via (3.13)] once $R(a_0)$ is specified, i.e., once the set of $r_n$ are chosen. Implicit constraints on this choice are that the resulting ground-state wave function (3.5) be normalizable and, so that the spectrum (3.10) is sensibly ordered, that $R(q^na_0)>0$.

The simplest case is $r_1>0$ (positivity required to ensure normalizable wave functions) and $r_n=0$, $n\geq 2$. Here (3.19) takes on a particularly simple form $g_n(x)=\beta_nx^{2n-1}$ with

$$\beta_1=d_1r_1, \quad \beta_n=-\frac{d_n}{(2n-1)}\sum_{j=1}^{n-1}\beta_j\beta_{n-j}, \quad n=2,3,\ldots \quad (3.20)$$

and so

$$W(x,a_0)=\sum_{i=1}^{\infty}\beta_ia_0^ix^{2i-1}=\sqrt{a_0}F(\sqrt{a_0}x). \quad (3.21)$$

For $a_1=qa_0$ this now gives

$$W(x,a_1)=\sqrt{q}W(\sqrt{q}x,a_0), \quad (3.22)$$

hence in this special case the shape-invariance condition (3.9) becomes

$$W^2(x,a_0)+W'(x,a_0) = qW^2(\sqrt{q}x,a_0)-q\frac{dW}{d\sqrt{q}x}(\sqrt{q}x,a_0)+a_0r_1(1-q). \quad (3.23)$$

Comparing this to (2.9), one thus sees that the case $r_n=0$, $n\geq 2$ corresponds to the self-similar $W$ of Shabat and Spiridonov provided one writes $\gamma^2\equiv d_1r_1a_0$ and $q\equiv p^2$. In fact, if instead of choosing $r_n=0$, $n\geq 2$, any one $r_n$ (say $r_j$) is taken to be nonzero and $q^j$ is replaced by $p^2$ then one again obtains the self-similar potentials. In these instances the results obtained from shape invariance and self-similarity are entirely equivalent and the Shabat-Spiridonov self-similarity condition turns out to be a special case of the shape-invariance one (3.9).

However, it is necessary to emphasize here that shape invariance is a much more general concept than self-similarity. For example, if we choose more than one $r_n$ to be nonzero, then shape-invariant potentials are obtained which are not self-similar. As an illustration, consider $r_n = 0$, $n \geq 3$. Using Eq. (3.16) one can readily calculate all the $g_n(x)$, of which the first three are

$$g_1(x) = d_1 r_1 x, \qquad g_2(x) = d_2 r_2 x - \tfrac{1}{3} d_1^2 r_1^2 d_2 x^3, \tag{3.24}$$
$$g_3(x) = -\tfrac{2}{3} d_1 r_1 d_2 r_2 d_3 x^3 + \tfrac{2}{15} d_1^3 r_1^3 d_2 d_3 x^5.$$

Notice that in this case $W(x)$ contains only odd powers of $x$. This makes the potentials $V_\pm(x)$ symmetric in $x$ and also guarantees unbroken SUSY. It is convenient to define the combinations $\Gamma_1 \equiv d_1 r_1 a_0 = \gamma^2$ and $\Gamma_2 \equiv d_2 r_2 a_0^2$. Then, the energy eigenvalues follow immediately from Eqs. (3.10) and (3.14) $(0 < q < 1)$:

$$E_n^{(-)}(a_0) = \Gamma_1 \frac{(1+q)(1-q^n)}{(1-q)} + \Gamma_2 \frac{(1+q^2)(1-q^{2n})}{(1-q^2)}, \quad n = 0, 1, 2, \ldots \tag{3.25}$$

while the (unnormalized) ground-state wave function is

$$\psi_0^{(-)}(x, a_0) = \exp\left[ -\frac{x^2}{2}(\Gamma_1 + \Gamma_2) + \frac{x^4}{4}(d_2\Gamma_1^2 + 2d_3\Gamma_1\Gamma_2 + d_4\Gamma_2^2) + O(x^6) \right]. \tag{3.26}$$

The excited wave functions can be recursively calculated using Eq. (3.11), though usually it is more convenient to use the relation

$$\psi_n^{(-)}(x, a_0) = A^+(x, a_0)\psi_{n-1}^{(-)}(x, a_1). \tag{3.27}$$

We can also calculate the transmission coefficient of this symmetric potential $(k = k')$ using relation (3.8) and the fact that for this shape-invariant potential [8]

$$T_+(k, a_0) = T_-(k, a_1 = q a_0). \tag{3.28}$$

Repeated application of Eqs. (3.8) and (3.28) gives

$$T_-(k, a_0) = \frac{[ik - W(\infty, a_0)][ik - W(\infty, a_1)] \cdots [ik - W(\infty, a_{n-1})]}{[ik + W(\infty, a_0)][ik + W(\infty, a_1)] \cdots [ik + W(\infty, a_{n-1})]} T_-(k, a_n), \tag{3.29}$$

where

$$W(\infty, a_j) = \sqrt{E_\infty^{(-)} - E_j^{(-)}}. \tag{3.30}$$

Now, as $n \to \infty$, $a_n = q^n a_0 \to 0$ $(0 < q < 1)$ and, since we have taken $g_0(x) = 0$, one gets $W(x, a_n) \to 0$. This corresponds to a free particle, for which the transmission coefficient is unity; as a result the reflection coefficient $R_-(x, a_0)$ vanishes and the transmission coefficient is given by

$$T_-(k, a_0) = \prod_{j=0}^{\infty} \frac{\left[ik - W(\infty, a_j)\right]}{\left[ik + W(\infty, a_j)\right]}. \tag{3.31}$$

The above discussion keeping only $r_1, r_2 \neq 0$ can readily be generalized to an arbitrary number of nonzero $r_j$. The energy eigenvalues for this case are given by $(\Gamma_j \equiv d_j r_j a_0^j)$

$$E_n^{(-)}(a_0) = \sum_j \Gamma_j \frac{(1+q^j)(1-q^{nj})}{(1-q^j)}, \quad n = 0, 1, 2, \ldots . \tag{3.32}$$

Not only are these potentials also symmetric and reflectionless, $T_-$ is again given by Eq. (3.31) since it was derived using only $a_n = q^n a_0$ and the fact that $V_-(x)$ is symmetric, without any assumption being made regarding the coefficients $r_n$.

The limit $q \to 0$ is particularly simple and again yields the one-soliton Rosen-Morse potential with $W = \alpha \tanh \alpha x$. Thus results corresponding to different choices of $R_n$ can be regarded as multiparameter deformations of this potential.

Finally, let us consider the solution to the shape-invariance condition (3.9) in the case where $R_0$ is nonzero, so that $g_0(x) = \frac{1}{2}R_0 x$ from (3.15) rather than being zero. One can again solve the set of linear differential Eqs. (3.16) in succession using this $g_0(x)$ and hence obtain $g_1(x), g_2(x), \ldots$ . Further, the spectrum can be immediately found using Eqs. (3.10) and (3.14); for example, in the case of an arbitrary number of nonzero $R_j$ (in addition to $R_0$), the spectrum is given by

$$E_n = nR_0 + \sum_j \Gamma_j \frac{(1+q^j)(1-q^{nj})}{(1-q^j)}, \tag{3.33}$$

which is the spectrum of a $q$-deformed harmonic oscillator [14]. It should be noted here that, unlike the usual $q$ oscillator where the space is noncommutative, but the potential is normal $(w^2x^2)$, in our approach the space is commutative, but the potential is deformed, giving rise to such a multiparameter deformed oscillator spectrum.

## IV. NUMERICAL RESULTS

Explicit determination of the SS potentials (described in Sec. II) crucially depended on the scaling property

$$W(x, a_0) = \sqrt{a_0} F(\sqrt{a_0} x), \tag{4.1}$$

displayed by the solutions, which allowed $W'(x,a_0)$ to be related to $W(\sqrt{q}x,a_0)$ instead of merely $W(x,a_1) = W(x,qa_0)$. However, such scaling is not a property of the solutions in Sec. III when more than one $r_n$ is nonzero, as can be seen from the series expansion (3.24). When only $r_1$ and $r_2$ are nonzero, there is a generalization of the form $W(x,a_0) = \sqrt{r_1 a_0} F(\sqrt{r_1 a_0}x, r_2 a_0/r_1)$ which relates the behavior at $x$ to that of *another problem* [that corresponding to calculating $W(x,a_0)$ with a different $r_2$] at $\sqrt{q}x$. By forming a ladder of these potentials (in which $r_2$ is tending to zero and hence the problem towards the special SS case), it should be possible to determine $W(x,a_0)$ using this fact. However, we chose to devote this section to a method that emphasizes $W(x,a)$ as a function of both $x$ and $a$ and which more readily generalizes to arbitrary $r_n$.

Intuitively one would still expect the series (3.13) to be convergent for either $x$ or $a$ sufficiently small. Thus in the $(x,a)$ plane we can assume that $W(x,a)$ is calculable to arbitrary accuracy close to either axis and the problem reduces to continuing this knowledge out into the plane. Defining sum and difference functions

$$S(x,a) \equiv W(x,a) + W(x,qa), \tag{4.2a}$$

$$D(x,a) \equiv W(x,a) - W(x,qa), \tag{4.2b}$$

for $a_1 = qa_0$, the shape-invariance condition (3.9) becomes

$$\frac{dS}{dx} = -S(x,a)D(x,a) + R(a), \tag{4.3}$$

where $R(a)$ is a known function. Now if one knows $W(x,a)$ for $x < X$, $a < A$, an Euler step (since $D$ is only known in $x < X$, there is not enough information available for that to be a Runge-Kutta one) will give $S(X+h,a)$, again for $a < A$. One must now invert (4.2a) to convert this knowledge of $S(X+h,a)$ into information about $W(X+h,a)$. Iterated use of (4.2a) relates $W(X+h,a)$ to $W(X+h,q^n a)$ and $n$ known values of $S$. For some sufficiently large $n$, $W(X+h,q^n a)$ can be calculated using the Taylor series (3.13) and so one can indeed determine $W(X+h,a)$ for all $a < A$. Breaking the $(x,a)$ plane up into a grid and using some suitable interpolation method for the points in between, one can iterate this Euler step up through $x$ and so obtain $W(x,a)$ numerically for values of $x$ and $a$ limited by computing constraints only. The only inputs are $R(a)$ and the series approximations for small $x$ and $a$.

A program to implement this scheme has been developed and its results for the special SS case were shown to agree very well with the earlier (more accurate, but overly specialized) program. As an example of the new potentials this permits us to consider the potential corresponding to $r_1 = 1$, $r_2 = -1$ with all other $r_n = 0$. Provided $a < 1/(1+q)$ the spectrum given by (3.25) is well ordered. In Fig. 3 we display the potential calculated for $q = 0.3$ and $a = 0.75$, along with its partner potential and the exact spectrum found from (3.25). (These eigenvalues have also been checked numerically.) Note that for this choice of parameters, $V_-(x)$ is a double-well potential, whereas its shape-invariant partner $V_+(x)$ is a single well. This situation is unlike the (non-shape-invariant) examples discussed in Ref. [15] where the SUSY partner of the initial double-well potential has a sharp $\delta$-like spike at its center. Apart from being the first shape-invariant double well, this example stretches naïve intuition concerning shape invariance. Furthermore, this example barely indicates the variety of behavior available by altering $q$, $a$, and the $r_n$ in this new class of potentials.

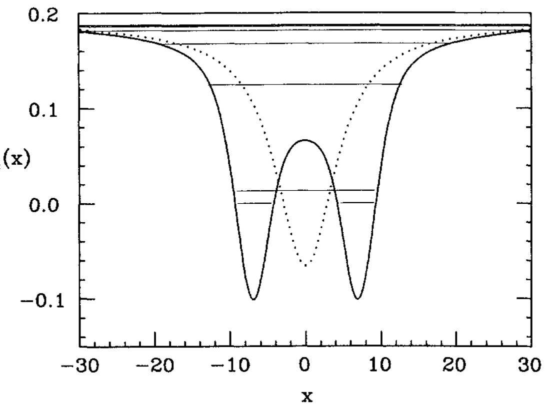


FIG. 3. A double-well potential $V_-(x)$ (solid line) and its shape-invariant, single-well supersymmetric partner $V_+(x)$ (dotted line). The exact spectra are also displayed. Parameter values are $r_1 = 1$, $r_2 = -1$, $q = 0.3$, and $a = 0.75$.

Finally, we note that the basic idea of this section can be divorced from the details of the Taylor series. The restriction to symmetric potentials gives $W(0,a) = 0$, to be used as a boundary condition for the intial Euler step. To invert (4.2a) one can use the infinite series

$$W(x,a) = S(x,a) - S(x,qa) + S(x,q^2 a) - S(x,q^3 a) + \cdots, \tag{4.4}$$

which is convergent provided $W(x,a) \to 0$ as $a \to 0$. More useful numerically is the fact that, because it is alternating, this series can be truncated with a rigorous bound on the error and without needing to calculate some $W(X+h,q^n a)$ by use of a Taylor series. Neither is the reliance on $a_1 = qa_0$ terribly restrictive: one can always redefine the parameters to obtain this. The only crucial constraint is that the method applies exclusively to symmetric potentials holding infinitely many bound states and corresponding to a chain of Hamiltonians ($H_-$, $H_+$, etc.) which tends asymptotically towards a free-particle one [i.e., $W(x,a_n) \to 0$ as $n \to \infty$]. Otherwise the sole input is $R(a)$ (expressed in terms of the appropriately defined $a$), which in general should be deducible from any desired (possible) spectrum.

## V. ANALYTIC RESULTS

Although undeniably useful, simply being able to calculate $W$ is an unsatisfactory state of affairs unless there is also a body of complementary analytic results. This section therefore brings together several approaches by which such results can be gathered.

In our method of constructing the potentials, the Taylor series about $x = 0$ played an essential role (as it did in [1] and [2]). However, to get a better insight into the potentials, it may be worthwhile to also know the Taylor series of $W(x)$ around $x \to \infty$. For simplicity, we now restrict our attention to the SS family and return to using the original parameter $p = \sqrt{q}$. Substituting $t = 1/x$ in Eq. (2.9) yields

$$W^2(t) - t^2W'(t) = p^2W^2\left(\frac{t}{p}\right) + t^2W'\left(\frac{t}{p}\right) + C_1. \tag{5.1}$$

We assume, in the absence of evidence to the contrary, that the simple Taylor series is the appropriate expansion. On substituting

$$W(t) = \sum_{j=0}^{\infty} a_j t^j \tag{5.2}$$

in this equation one obtains

$$a_0^2 = C_1/(1-p^2), \qquad a_1 = 0 \tag{5.3}$$

and

$$a_j = -(j-1)\left(\frac{1+p^{j-2}}{1-p^{j-2}}\right)\frac{a_{j-1}}{2a_0} - \frac{1}{2a_0}\sum_{m=2}^{j-2} a_m a_{j-m}, \quad j = 3, 4, \ldots . \tag{5.4}$$

Thus we find that as $x \to +\infty$

$$W(x) = \sqrt{\frac{C_1}{(1-p^2)}} + \frac{a_2}{x^2} - \frac{a_2}{a_0}\left(\frac{1+p}{1-p}\right)\frac{1}{x^3} + \cdots , \tag{5.5}$$

where $a_2$ is an arbitrary constant. This arbitrariness is due to the fact that $W(0) = 0$ has not been imposed while deriving (5.5). In fact it is not easy to do so since the series (5.5) is valid for large $x$.

It has already been established that we are dealing with reflectionless, symmetric potentials for which the infinite spectra of eigenvalues are known exactly in closed form. This type of problem has already been well studied, but the standard inverse scattering method [16] has proved too cumbersome to be of much practical use in deriving these potentials in this nontrivial context. However, certain well-known, related results can be used to quite strongly constrain the potentials: the point is, being reflectionless, these can be regarded as a solution of the KdV equation at time $t = 0$ [17]. Now it is known that such a solution as $t \to \pm\infty$ will break up into an infinite number of solitons of the form $2k_i^2 \text{sech}^2 k_i x$. On using the fact that KdV solitons obey an infinite number of conservation laws corresponding to mass, momentum, energy, etc., one can immediately obtain constraints on the reflectionless potentials by using the known solutions at $t \to \pm\infty$. For example, the first three conservation laws are

$$\int_{-\infty}^{\infty} V_0(x)dx = \sum_{i=0}^{\infty}\int_{-\infty}^{\infty} V^{(i)}(x)dx, \tag{5.6}$$

$$\int_{-\infty}^{\infty} V_0^2(x)dx = \sum_{i=0}^{\infty}\int_{-\infty}^{\infty} [V^{(i)}(x)]^2dx, \tag{5.7}$$

$$\int_{-\infty}^{\infty}\left[V_0^3(x) + \frac{1}{2}\left(\frac{dV_0}{dx}\right)^2\right]dx = \sum_{i=0}^{\infty}\int_{-\infty}^{\infty}\left[[V^{(i)}(x)]^3 + \frac{1}{2}\left(\frac{dV^{(i)}(x)}{dx}\right)^2\right]dx, \tag{5.8}$$

where

$$V^i(x) = -2k_i^2 \text{sech}^2 k_i x, \tag{5.9}$$

with $k_i = k_0 p^i$ and $k_0^2 = C_1/(1-p^2)$. Using Eq. (5.9) it is straightforward to evaluate the right hand sides of Eqs. (5.6)–(5.8) and we find

$$\int_{-\infty}^{\infty} V_0(x)dx = -4\sum_{i=0}^{\infty} k_i = \frac{-4k_0}{1-p}, \tag{5.10}$$

$$\int_{-\infty}^{\infty} V_0^2(x)dx = \frac{16}{3}\sum_{i=0}^{\infty} k_i^3 = \frac{16}{3}\frac{k_0^3}{(1-p^3)}, \tag{5.11}$$

$$\int_{-\infty}^{\infty}\left[V_0^3(x) + \frac{1}{2}\left(\frac{dV_0}{dx}\right)^2\right] = -\frac{32}{5}\sum_{i=0}^{\infty} k_i^5 = -\frac{32}{5}\frac{k_0^5}{(1-p^5)}, \tag{5.12}$$

thereby providing strong constraints on the potential $V_0(x)$.

All deformations of potentials considered so far have been such that the spectra obtained are $q$ (or $p$) dependendent. Before ending this section, it is worth remarking that, as with any potential, there are also distortions of the $V_n(x)$, with deformation parameter $\lambda$, which leave the spectra unchanged. Using the techniques of supersymmetric quantum mechanics, one can construct a large class of strictly isospectral potentials. For example, using any one of the $W_i(x)$ as given by Eqs. (2.5), (2.8), and (2.14)–(2.17) one can immediately obtain a one parameter family of strictly isospectral reflectionless potentials $V_n(x, \lambda)$ by using the formula [18]

$$V_n(x,\lambda) = V_n(x) - 2\frac{d^2}{dx^2}\ln[I_n(x) + \lambda], \tag{5.13}$$

where $V_n(x)$ can easily be obtained using Eqs. (2.1)–(2.3), (2.8), and (2.14)–(2.17), $\lambda$ is any arbitrary parameter ($\lambda > 0$ or $\lambda < -1$), and

$$I_n(x) = \int_{-\infty}^{x} [\psi_0^{(n)}(y)]^2 dy. \tag{5.14}$$

Here $\psi_0^{(n)}$ is as given by Eq. (2.5) which can be explicitly obtained by using Eqs. (2.8) and (2.14)–(2.17). As an illustration, we give graphs of the $V_0(x, \lambda)$ obtained from the SS potential with $p = 0.5$ for various values of $\lambda$ in Fig. 4. Large values of $\lambda$ correspond to the original SS potential [see Fig. 4(a)]. As $\lambda$ takes on values closer to zero, the potential gradually breaks into two pieces [Figs. 4(b) and 4(c)], one corresponding to the $E = 0$

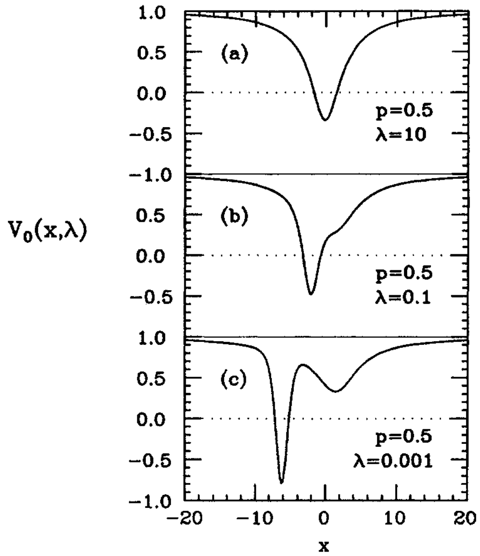


FIG. 4. Selected members of the one-parameter family of isospectral potentials $V_0(x,\lambda)$ which includes the self-similar potential with $p=0.5$. Note that a different choice of $C_1$ (i.e., $r_1$) has been made compared to Figs. 1 and 2.

state only and the other containing the remaining energy levels [8].

## VI. SHAPE INVARIANCE IN MORE THAN ONE STEP

Having obtained potentials which are multiparameter deformations of the one-soliton solution of the Rosen-Morse potential, an obvious question to ask is if one can also obtain deformations of the multisoliton solutions. The answer is yes and as an illustration we now explicitly obtain multiparameter deformations of the two-soliton case. The desired deformation is achieved by extending the usual shape-invariance ideas to the more general concept of shape invariance in two steps.

Consider the unbroken SUSY case of two superpotentials $W_0(x,a_0)$ and $W_1(x,a_0)$ such that $V_0^{(+)}(x,a_0)$ and $V_1^{(-)}(x,a_0)$ are the same up to an additive constant.

$$V_0^{(+)}(x,a_0)=V_1^{(-)}(x,a_0)+R(a_0), \tag{6.1}$$

or equivalently,

$$W_0^2(x,a_0)+W_0'(x,a_0)$$
$$=W_1^2(x,a_0)-W_1'(x,a_0)+R(a_0). \tag{6.2}$$

Shape invariance in two steps means that

$$V_1^{(+)}(x,a_0)=V_0^{(-)}(x,a_1)+\tilde{R}(a_0), \tag{6.3}$$

that is,

$$W_1^2(x,a_0)+W_1'(x,a_0)$$
$$=W_0^2(x,a_1)-W_0'(x,a_1)+\tilde{R}(a_0). \tag{6.4}$$

For the above situation, the energy eigenvalues and eigenfunctions of the potential $V_0^{(-)}(x,a_0)$ can be algebraically calculated as shown below.

Unbroken SUSY implies zero-energy ground states for the potentials $V_0^{(-)}(x,a_0)$ and $V_1^{(-)}(x,a_0)$:

$$E_0^{(-)0}=0, \qquad E_0^{(-)1}=0. \tag{6.5}$$

The degeneracy of energy levels for supersymmetric partner potentials yields

$$E_n^{(+)0}(a_0)=E_{n+1}^{(-)0}(a_0), \quad E_n^{(+)1}(a_0)=E_{n+1}^{(-)1}(a_0). \tag{6.6}$$

From Eq. (6.1) it follows that

$$E_n^{(+)0}(a_0)=E_n^{(-)1}(a_0)+R(a_0). \tag{6.7}$$

For the special case $n=0$, Eqs. (6.6) and (6.7) give

$$E_1^{(-)0}=R(a_0). \tag{6.8}$$

Also, the shape-invariance constraint (6.3) gives

$$E_n^{(+)1}(a_0)=E_n^{(-)0}(a_1)+\tilde{R}(a_0). \tag{6.9}$$

Using Eqs. (6.6), (6.7), (6.9), and some algebra, one gets

$$E_{n+1}^{(-)0}(a_0)=E_{n-1}^{(-)0}(a_1)+R(a_0)+\tilde{R}(a_0). \tag{6.10}$$

These equations can be solved recursively to get

$$E_{2n}^{(-)0}=\sum_{k=0}^{n-1}\left[R(a_k)+\tilde{R}(a_k)\right],$$

$$E_{2n+1}^{(-)0}=\sum_{k=0}^{n-1}\left[R(a_k)+\tilde{R}(a_k)\right]+R(a_n). \tag{6.11}$$

The above discussion has been completely general and is valid for any change of parameters, $a_1=f(a_0)$. Following the treatment of Sec. III, we now take the scaling *Ansatz* $a_1=qa_0$ and expand the superpotentials $W_0$ and $W_1$ in powers of $a_0$,

$$W_0(x,a_0)=\sum_{j=0}^{\infty} g_j(x)a_0^j, \tag{6.12}$$

$$W_1(x,a_0)=\sum_{j=0}^{\infty} h_j(x)a_0^j. \tag{6.13}$$

Further, write $R$ and $\tilde{R}$ in the form

$$R(a_0)=\sum_{j=0}^{\infty}R_ja_0^j, \qquad \tilde{R}(a_0)=\sum_{j=0}^{\infty}\tilde{R}_ja_0^j. \tag{6.14}$$

Using these in Eqs. (6.2) and (6.4) and equating powers of $a_0$ yields ($n=0,1,2,\ldots$)

$$g_n'+\sum_{j=0}^{n}g_jg_{n-j}=\sum_{j=0}^{n}h_jh_{n-j}-h_n'+R_n, \tag{6.15}$$

$$h'_n + \sum_{j=0}^{n} h_j h_{n-j} = q^n \sum_{j=0}^{n} g_j g_{n-j} - q^n g'_n + \tilde{R}_n. \tag{6.16}$$

This set of linear differential equations is easily solvable in succession. Let us first discuss the special case

$$g_0(x) = h_0(x) = 0, \tag{6.17}$$

which implies that $R_0 = \tilde{R}_0 = 0$, and further assume that $R_n = \tilde{R}_n = 0$, $n \geq 3$. In this case one can readily calculate all $g_n(x)$ and $h_n(x)$; the first two of each are

$$\begin{aligned} g_1 &= \frac{(R_1 - \tilde{R}_1)}{(1-q)} x, \quad g_2 = \frac{(R_2 - \tilde{R}_2)}{(1-q^2)} x + \frac{x^3}{3(1-q)^3}[(1-q)(\tilde{R}_1^2 - R_1^2) - 2(1+q)R_1\tilde{R}_1], \\ h_1 &= \frac{(\tilde{R}_1 - qR_1)}{(1-q)} x, \\ h_2 &= \frac{R_2 x}{(1-q^2)} - \frac{x^3}{3(1+q)(1-q)^2}[(1+q)\tilde{R}_1^2 + (1+q)(1-q^2)R_1^2 - 2q(1-q)R_1\tilde{R}_1]. \end{aligned} \tag{6.18}$$

It may be noted that both $W_0$ and $W_1$ contain only odd powers of $x$ so that the potentials $V_0^{(\pm)}$ and $V_1^{(\pm)}$ are all symmetric in $x$ and SUSY is unbroken. The energy eigenvalues can be obtained from Eqs. (6.8) and (6.11).

$$E_1^{(-)0}(a_0) = R_1 a_0 + R_2 a_0^2, \tag{6.19}$$

$$E_{2n}^{(-)0}(a_0) = \sum_{j=1}^{2} (R_j + \tilde{R}_j) a_0^j \left(\frac{1 - q^{jn}}{1 - q^j}\right), \tag{6.20}$$

and

$$E_{2n+1}^{(-)0}(a_0) = \sum_{j=1}^{2} R_j a_0^j \left(\frac{1 - q^{j(n+1)}}{1 - q^j}\right) + \sum_{j=1}^{2} \tilde{R}_j a_0^j \left(\frac{1 - q^{jn}}{1 - q^j}\right). \tag{6.21}$$

For the special case when $R_2 = \tilde{R}_2 = 0$, the spectrum has been obtained previously by Spiridonov from consideration of self-similar potentials [3]. However, the spectrum in the general case given by Eqs. (6.20) and (6.21) cannot be obtained in such a fashion. The energy eigenvalues $E_n^{(-)1}$ are now immediately obtained from Eq. (6.9) and the energy eigenfunctions and transmission coefficient for these reflectionless potentials can also be found using Eqs. (3.5), (3.27), (3.30), and (3.31). Further, the above discussion can be readily generalized to an arbitrary number of nonzero $R_j, \tilde{R}_j$.

The limit $q \to 0$ of the above equations is particularly simple and yields the two-soliton solution of the Rosen-Morse potential, i.e.,

$$W_0 = 2\sqrt{\tilde{R}} \tanh \sqrt{\tilde{R}} x, \qquad W_1 = \sqrt{\tilde{R}} \tanh \sqrt{\tilde{R}} x \tag{6.22}$$

provided $R = 3\tilde{R}$. Thus our results can be regarded as multiparameter deformations of this potential.

Finally, it is clear that one can easily generalize this procedure and consider shape invariance with a scaling *Ansatz* in 3,4,. . . , $p$ steps and thereby obtain multiparameter deformations of the 3,4,. . . , $p$ soliton Rosen-Morse solution.

## VII. SHAPE INVARIANCE WITH A NONSCALING CHANGE OF PARAMETERS

We have so far obtained new shape-invariant potentials for $a_1$ and $a_0$ related by the scaling *Ansatz* ($a_1 = qa_0$). Are there shape-invariant potentials where $a_1$ and $a_0$ are neither related by scaling nor by translation ($a_1 = a_0 + \alpha$)? We now demonstrate the existence of yet other possibilities by obtaining potentials for $a_1 = qa_0^p$ and $a_1 = qa_0/(1 + pa_0)$.

First consider the case when

$$a_1 = qa_0^p, \tag{7.1}$$

where $p$ could be any integer. Again consider the expansions of the superpotential $W$ and $R(a_0)$ given by Eqs. (3.13) and (3.14), respectively. On using Eqs. (3.12), (3.13), and (7.1) in the shape-invariance condition (3.9) and equating powers of $a_0$ one finds two sets of equations: (i) $n = pm$, $m = 0, 1, 2, \ldots$,

$$g'_{pm}(x) + \sum_{j=0}^{pm} g_j(x) g_{pm-j}(x) = q^m \sum_{j=0}^{m} g_j(x) g_{m-j}(x) - q^m g'_m(x) + R_{pm}, \tag{7.2}$$

(ii) $n = pm + q$, $q = 1, 2 \ldots, (p-1)$,

$$g'_{pm+q} + \sum_{j=0}^{pm+q} g_j(x) g_{pm+q-j}(x) = R_{pm+q}. \tag{7.3}$$

These sets of equations are easily solved in succession to produce more solutions of Eq. (3.9). Further, the energy eigenvalue spectrum can be easily obtained from Eqs. (3.10), (3.14), and (7.1). For example, in the case where only $R_1$ and $R_2$ are nonzero the spectrum can be shown to be ($E_0^{(-)} = 0$)

$$E_n^{(-)} = \frac{R_1}{q^{1/(p-1)}} \sum_{j=1}^{n} (q^{\frac{1}{p-1}} a_0)^{p^{(j-1)}} + \frac{R_2}{q^{2/(p-1)}} \sum_{j=1}^{n} (q^{\frac{1}{p-1}} a_0)^{2p^{(j-1)}}, \quad n = 1, 2, \ldots . \tag{7.4}$$

The energy eigenfunctions and the transmission coefficient for these reflectionless potentials can be written down immediately using Eqs. (3.5), (3.27), (3.30), and (3.31).

As an illustration, let us discuss the case $p = 2$ explicitly. The set of equations which follows from Eqs. (7.2) and (7.3) is

$$g'_{2m}(x) + \sum_{j=0}^{2m} g_j(x) g_{2m-j}(x) = q^m \sum_{j=0}^{m} g_j(x) g_{m-j}(x) - q^m g'_m(x) + R_{2m}, \tag{7.5}$$

$$g'_{2m+1}(x) + \sum_{j=0}^{2m+1} g_j(x) g_{2m+1-j}(x) = R_{2m+1}, \tag{7.6}$$

and one can thus readily calculate all the $g_n(x)$. For example, in the case when only $R_1$ and $R_2$ are nonzero it is easily shown that the first three $g(x)$'s are

$$g_1(x) = R_1 x, \qquad g_2(x) = (R_2 - qR_1)x - \tfrac{1}{3}R_1^2 x^3, \qquad g_3(x) = \tfrac{2}{3}R_1(qR_1 - R_2)x^3 + \tfrac{2}{15}R_1^3 x^5. \tag{7.7}$$

Notice that we have chosen $g_0(x) = 0$ so that again $W(x)$ contains only odd powers of $x$, $V_\pm(x)$ are symmetric in $x$, and SUSY is unbroken. The spectrum which follows from (7.4) (for $p = 2$) is $E_0^- = 0$ and

$$E_n^{(-)} = \frac{R_1}{q} \sum_{j=1}^{n} (a_0 q)^{2^{j-1}} + \frac{R_2}{q^2} \sum_{j=1}^{n} (a_0 q)^{2^j}, \quad n = 1, 2, \ldots . \tag{7.8}$$

The $q \to 0$ limits of the equations above again correspond to the one-soliton solution of the Rosen-Morse potential, so that our results for $a_1 = q a_0^p$ can be regarded as multiparameter deformations of this potential. Generalization to the case when an arbitrary number of $R_j$ are nonzero is straightforward. Similarly, one can also consider shape invariance in multisteps along with the *Ansatz* (7.1), thereby obtaining deformations of the multisoliton solutions.

Finally, consider solutions to the shape-invariance condition (3.9) for

$$a_1 = \frac{q a_0}{1 + p a_0}, \tag{7.9}$$

where $0 < q,\ p < 1$. We also assume that $p a_0 \ll 1$ so that one can expand $(1 + p a_0)^{-1}$ in powers of $a_0$. Further, assume that in Eqs. (3.13) and (3.14)

$$R(a_0) = R_1 a_0 + R_2 a_0^2 \tag{7.10}$$

and $g_0(x) = 0$ so that $W$ is again an odd function of $x$. On using Eqs. (3.13), (3.14), (7.9), and (7.10) in the shape-invariance condition (3.9), expanding negative powers of $(1 + p a_0)$ in powers of $a_0$, and finally equating powers of $a_0$, one again obtains a set of linear differential equations. For example, to order $a_0^2$, the shape-invariance condition looks like

$$a_0 g'_1(x) + a_0^2[g_1^2 + g'_2(x)] = -\left(\frac{q a_0}{1 + p a_0}\right) g'_1(x) + \left(\frac{q a_0}{1 + p a_0}\right)^2 [g_1^2 - g'_2(x)] + R_1 a_0 + R_2 a_0^2. \tag{7.11}$$

Expanding the denominators in powers of $a_0$ and equating terms of order $a_0$ and $a_0^2$ yields equations for the functions $g_1(x)$ and $g_2(x)$ which give

$$g_1(x) = \frac{R_1 x}{1 + q}, \qquad g_2(x) = \left[R_2 + \left(\frac{p q R_1}{(1 + q)}\right)\right] x - \frac{(1 - q)}{(1 + q)^2 (1 + q^2)} \frac{x^3}{3}. \tag{7.12}$$

The energy spectrum which follows from Eqs. (3.10), (7.9), and (7.10) is $E_0^- = 0$ and

$$E_n^{(-)} = R_1 \sum_{j=1}^{n} \frac{q^{j-1} a_0}{\left[1 + p a_0 \left(\frac{1 - q^{j-1}}{1 - q}\right)\right]} + R_2 \sum_{j=1}^{n} \frac{(q^{j-1} a_0)^2}{\left[1 + p a_0 \left(\frac{1 - q^{j-1}}{(1 - q)}\right)\right]^2}, \qquad n = 1, 2, \ldots . \tag{7.13}$$

As usual, $\psi_n^{(-)}$ and $T_-$ can be found using Eqs. (3.5), (3.27), (3.30), and (3.31). Generalization to the case when arbitrary numbers of the $R_j$ are nonzero is straightforward. Similarly, one can also consider shape invariance in multisteps along with the *Ansatz* (7.9) and obtain deformations of the multisoliton Rosen-Morse solutions.

## VIII. SUMMARY AND OPEN PROBLEMS

Until now, the only known shape-invariant potentials were such that the parameters $a_1$ and $a_0$ which appear in shape-invariance condition (1.1) were related by a translation. In this paper, we have discovered a wider class of new shape-invariant potentials for which $a_1$ and $a_0$ are related by scaling, as well as in a variety of other ways. All these new potentials are reflectionless and have an infinite number of bound states. They can be considered to be $q$ deformations of the multisoliton solutions corresponding to the Rosen-Morse potential. We were able to obtain the energy eigenvalues, eigenfunctions, and transmission coefficients for these potentials algebraically. It was also possible to obtain analytical answers for the moments of these potentials, which should be useful since these potentials could not be explicitly expressed in a closed analytic form. The recently discovered self-similar potentials of Shabat and Spiridonov were shown to be a very special case of our shape-invariant potentials. We were also able to obtain $q$ deformations of the one-dimensional harmonic oscillator potential. This work has raised several questions which need to be looked into. Some of these follow.

(i) Just as we have obtained $q$ deformations of the reflectionless Rosen-Morse and harmonic oscillator potentials, can one also obtain deformations of the other simple shape-invariant potentials? In particular, can one obtain deformations of potentials which are not reflectionless, say nonsolitonic Rosen-Morse potentials of the form $V(x) = -A(A+1)\text{sech}^2 x$ for noninteger values of $A$?

(ii) What are the various potentials satisfying the shape-invariance condition (3.9)? In this paper, we have significantly expanded that list but it is clear that the possibilities are far from exhausted. In fact it appears that there are an unusually large number of shape-invariant potentials, for all of which the whole spectrum can be obtained algebraically. How does one classify all these potentials? Would such a classification exhaust all the known exactly solvable ones dicussed by Natanzon [19]?

(iii) The shape-invariant potentials have been treated algebraically in this paper. An obvious interesting question is whether one can also solve the Schrödinger equation for these potentials directly. This should be possible, at least in principle. In that case the next question is if the Schrödinger equation gets essentially reduced to a hypergeometric or confluent hypergeometric equation or not. If not, then one would have generalized the concept of solvable potentials as introduced by Natanzon [19].

(iv) Now that a host of new shape-invariant potentials have been discovered, it is worth asking if all the known exactly solvable potentials of Natanzon can be cast in a shape-invariant form. In fact one can ask an even more general question: can any exactly solvable problem in quantum mechanics (i.e., for which the Schrödinger equation need not necessarily reduce to a hypergeometric or confluent hypergeometric equation) be cast in a shape-invariant form? In other words, is shape invariance not only sufficient but even necessary for exact solvability, as first conjectured by Gendenshteĭn [6]?

We hope to answer some of these questions in the future.

## ACKNOWLEDGMENTS

One of us (U.S.) would like to thank the Council of Scientific and Industrial Research, India and the United Nations Development Programme for support under their TOKTEN scheme and the kind hospitality of the Institute of Physics, Bhubaneswar where a part of this work was done. This work was supported in part by the U.S. Department of Energy.

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