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# NEW EXAMPLES OF CONVOLUTIONS AND NON-COMMUTATIVE CENTRAL LIMIT THEOREMS

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**Abstract.** A family of transformations on the set of all probability measures on the real line is introduced, which makes it possible to define new examples of convolutions. The associated central limit theorems are studied, and examples of the limit measures, related to the classical, free and boolean convolutions, are shown.

**1. Introduction.** By studying series of examples of known limit theorems we came up with an idea of new convolutions of measures. We found a family $\mathcal{U}_t$, $t \geq 0$, of continuous transformations, acting on probability measures on the real line, which allowed us to define the convolutions and study associated central limit theorems. The most instructive way of describing the transformations is given by observing their action on the Cauchy transform of a measure, given in the form of a continued fraction by a theorem of Stieltjes. The crucial ideological role in our construction is played by relations between moments and cumulants of a given measure; though the construction itself does not require that the measure has finite moments.

**2. General form of the non-commutative central limit theorem and examples of limit measures.** In [BSp1] the following general form of the non-commutative central limit theorem was obtained:

THEOREM 1. *Let $\mathcal{B}$ be a unital $^*$-algebra with a state $\varphi$ and let $b_i = b_i^*$, $i = 1, 2, 3, \ldots$ be a sequence of self-adjoint elements in $\mathcal{B}$, satisfying the following assumptions*:

*1. for all positive integers $n$ and all sequences $i(1), i(2), \ldots, i(n)$ of indices*,

$$\varphi(b_{i(1)}b_{i(2)}\ldots b_{i(n)}) = 0$$

*whenever there exists an index $i(k), 1 \leq k \leq n$, which is distinct from all other indices*;

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*2. the expression* $\varphi(b_{i(1)}b_{i(2)}\dots b_{i(n)})$ *is invariant under all permutations of the indices*: *for every permutation* $\pi$ *of the set* $\mathbb{N}$ *of positive integers*

$$\varphi(b_{i(1)}b_{i(2)}\dots b_{i(n)}) = \varphi(b_{\pi(i(1))}b_{\pi(i(2))}\dots b_{\pi(i(n))}).$$

*Moreover*, *for each* $n \in \mathbb{N}$ *let* $S_n = \frac{1}{\sqrt{n}} \cdot (b_1 + b_2 + \dots b_n)$; *then for each power* $k$ *there exists the limit*

$$\lim_{n\to\infty} \varphi(S_n^k) = \begin{cases} 0, & \text{if } k \text{ is odd}, \\ \sum_{\pi\in P_2(1,2,\dots k)} \mathbf{t}(\pi), & \text{if } k \text{ is even}, \end{cases} \tag{1}$$

*where* $P_2(1,2,\dots,2r)$ *is the set of all pair partitions of the set* $\{1,2,\dots,2r\}$, *and the function* $\mathbf{t}$ *is given by the value* $\mathbf{t}(\pi) = \varphi(b_{i(1)}b_{i(2)}\dots b_{i(2r)})$ *common for all these sequences* $i(1), i(2), \dots i(2r)$, *which satisfy the condition*: $i(s) = i(m)$ *if and only if* $s$ *and* $m$ *belong to the same block of the partition* $\pi$.

For special choices of functions $\mathbf{t}$ one can obtain, in the limit, moments of various known measures. This inspired the question of what measures may appear as the limit in a non-commutative central limit theorem. A contribution to this problem is contained in the next sections.

Examples 1.

1. For $\mathbf{t}(\pi) \equiv 1$ the limit is the gaussian measure $\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}dx$, as the limit moments are $m_\mu(2n) = 1\cdot 3\cdot\ldots\cdot(2n-1) = (2n-1)!!$.

2. For $\mathbf{t}(\pi) = (-1)^{i(\pi)}$ one gets the fermionic case with the limit measure $\frac{1}{2}(\delta_1 + \delta_{-1})$; here $i(\pi)$ is the number of inversions of a partition $\pi$.

3. For

$$\mathbf{t}(\pi) = \begin{cases} 1, & \text{if } \pi \text{ has no inversions}, \\ 0, & \text{if } \pi \text{ has inversions}, \end{cases}$$

one gets the free case, i.e. the semi-circular Wigner measure $\frac{1}{2\pi}\chi_{[-2,2]}(x)\sqrt{4-x^2}dx$ with (even) moments being the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$.

4. For $\mathbf{t}_q(\pi) = q^{i(\pi)}$, where $-1 \le q \le 1$, the resulting q-gaussian measure has the Jacobi $\theta_1$ function as its density, see [BSp2].

5. In the case of $\mathbf{t}_r(\pi) = r^{n-b(\pi)}$ where $b(\pi)$ is the number of blocks of $\pi$ and $0 \le r \le 1$ the limit measures were obtained in [BSp1]. For $r = \frac{1}{N}$ the measure $\mu_r$ is the free product of $N$ copies of a dilation of the gaussian measure.

6. For negative $r$ with $-1 \le r \le 0$, by a slight modification of the previous case, namely by putting $\mathbf{t}_r(\pi) = (-r)^{n-b(\pi)} \cdot (-1)^{i(\pi)}$ one can also compute the limit measure using the results of [BSp1], Theorem 7. The even moments of the limit measure $\mu_r$ are given by the formula

$$m_{\mu_r}(2n) = \sum_{\pi\in NC_2(2n)} (1+r)^{\mathbf{inn}(\pi)} \tag{2}$$

where $\mathbf{inn}(\pi)$ is the number of inner blocks in a partition $\pi$. In particular for $r = -\frac{1}{N}$ the limit measure $\mu_r$ is the N-fold free convolution of a dilation of the two-point measure $\mu_{-1} = \frac{1}{2}(\delta_{-1} + \delta_1)$.

Another series of examples come from the group case. The general scheme in these cases is the following. Given a (discrete) group $\mathcal{G}$ and an infinite set $\mathcal{S}$ of its generators consider a random walk associated with the Cayley's graph of the pair $(\mathcal{G},\mathcal{S})$. In other words, define

$$S_n = \frac{1}{\sqrt{2n}}\sum_{j=1}^{n}(\lambda(s_j)+\lambda(s_j^{-1}))$$

where $\lambda(s)$ is the left translation operator by $s$, and let $\varphi(Y) =< Y\delta_e,\delta_e >$ be the standard state on the algebra of all bounded operators on $\ell^2(\mathcal{G})$. We study the limit $\lim_{n\to\infty}\varphi(S_n^k)$, which is of the form established above, with $b_j = \frac{1}{\sqrt{2}}(\lambda(s_j)+\lambda(s_j^{-1}))$, provided the assumptions (1) and (2) are satisfied.

Examples 2.

1. Let $\mathcal{G} = \mathcal{F}_\infty$ be a free group on infinitely many generators $\mathcal{S} = \{s_j :\ j = 1,2,3,\ldots\}$; then one gets

$$\mathbf{t}(\pi) = \begin{cases} 1, & \text{if } \pi \text{ has no inversions,}\\ 0, & \text{if } \pi \text{ has inversions,}\end{cases}$$

and $S_n$ tends $*$-weakly to the Wigner semi-circular distribution.

2. If $\mathcal{G}$ is a free abelian (Coxeter) group with the set of generators $\mathcal{S} = \{s_j :\ j = 1,2,3,\ldots\}$ which satisfy the relations $s_js_k = s_ks_j$ and $s_k = s_k^{-1}$; then $\mathbf{t}(\pi)\equiv 1$ and $S_n$ tends to the gaussian measure.

3. Let $\mathcal{G}$ be a free nilpotent group of class 2 or more; then

$$\mathbf{t}(\pi) = \begin{cases} 1, & \text{if } \pi \text{ has no inversions,}\\ 0, & \text{otherwise,}\end{cases}$$

and $S_n = \frac{1}{\sqrt{2n}}\sum_{j=1}^{n}(\lambda(s_j)+\lambda(s_j^{-1}))$ tends to the Wigner semi-circular distribution.

**3. $t$-transformation and $t$-convolution.** For a given probability measure $\mu$ with compact support on the real line $\mathbb{R}$, its Cauchy transform $G_\mu$ is defined for $z\in\mathbb{C}^+ = \{z\in\mathbb{C} :\ \Im z > 0\}$, by:

$$G_\mu(z) = \int_{-\infty}^{+\infty}\frac{d\mu(x)}{z-x}$$

and, by a theorem of Stieltjes (see [AkG]), can be expressed as a continued fraction:

$$G_\mu(z) = \cfrac{1}{z-a_1-\cfrac{b_1}{z-a_2-\cfrac{b_2}{\ddots}}} \tag{3}$$

where the numbers $a_1,a_2,\ldots$ disappear if the measure is symmetric, and they, together with the numbers $b_1,b_2,b_3,\ldots$ come from a recurrence formula for the polynomials orthogonal with respect to the measure $\mu$.

The Cauchy transform of a measure $\mu$ can be also expressed by the moments

$$m_\mu(k) = \int_{-\infty}^{+\infty}x^k d\mu(x)$$

of the measure as the series

$$G_\mu(z) = \sum_{k=0}^{\infty} m_\mu(k) z^{-k-1}$$

The Voiculescu's R-transform $r_\mu$ of the measure is then defined by the equality

$$(G_\mu)^{-1}(z) = z^{-1} + r_\mu(z)$$

which involves the inverse of the Cauchy transform. The R-transform is nothing else but the generating function of the sequence of free cumulants $(r_\mu(k))_{k=0}^{\infty}$ of the measure. The free (additive) convolution of a pair of measures $\mu_1$ and $\mu_2$ is defined, following Voiculescu, to be the measure $\mu$ for which $r_\mu(k) = r_{\mu_1}(k) + r_{\mu_2}(k)$. The relation between the sequence of moments $(m_\mu(k))$ and the sequence of free cumulants $(r_\mu(k))$, found by Speicher (see [Sp]), is the following:

$$m_\mu(n) = \sum_{\substack{s_1,\ldots,s_k \\ s_1+\ldots+s_k=n-k}} r_\mu(k) \cdot m_\mu(s_1) \cdot \ldots \cdot m_\mu(s_k) \tag{4}$$

Remark. Our notation of $n$-th moment $m_\mu(n)$ of a measure $\mu$ and of $n$-th cumulant $r_\mu(n)$ of a measure $\mu$ slightly differs from that used elsewhere, which is $m_n(\mu)$ for $n$-th moment and $r_n(\mu)$ form $n$-th cumulant.

Now we define the $t$-transform. Let $t$ be a positive real number and let $\mu$ be a compactly supported measure on the real line. Then the function $G_{\mu_t}(z)$ defined for $z \in \mathbb{C}^+$ by the formula:

$$\frac{1}{G_{\mu_t}(z)} = \frac{t}{G_\mu(z)} + (1-t)z \tag{5}$$

turns out to be the Cauchy transform of a probability measure denoted by $\mu_t$. This is a consequence of the following theorem, that can be found, for example, in ([Ma]):

THEOREM 2 (Nevanlinna). *A function $F(z)$ is the reciprocal of the Cauchy transform of a probability measure on the real line if and only if there exists a positive measure $\rho$ and a real number $a$ such that for $\Im z > 0$*

$$F(z) = a + z + \int_{-\infty}^{+\infty} \frac{1+xz}{x-z} d\rho(x).$$

COROLLARY 3. *For a pair of probability measures $\rho$ and $\nu$ on the real line, and a real number $0 \le t \le 1$ there exists a probability measure $\mu$ such that*

$$\frac{1}{G_{\mu_t}(z)} = \frac{t}{G_\nu(z)} + \frac{(1-t)}{G_\rho(z)}$$

*where $z \in \mathbb{C}^+$.*

This follows directly from the Nevanlinna's theorem. For our special choice of the measure $\rho = \delta_0$, we get a little more:

COROLLARY 4. *For a given probability measure $\mu$ on the real line and a non-negative number $t \geq 0$, there exists a (unique) probability measure $\mu_t$ such that*

$$\frac{1}{G_{\mu_t}(z)} = \frac{t}{G_\mu(z)} + (1-t)z$$

*where $z \in \mathbb{C}^+$.*

Proof. It follows from the Nevanlinna's theorem that

$$\frac{t}{G_\mu(z)} + (1-t)z = tF(z) + (1-t)z = ta + z + \int_{-\infty}^{+\infty} \frac{1+xz}{x-z} d(t\rho)(x)$$

is the reciprocal of the Cauchy transform of the probability measure denoted by $\mu_t$. ∎

DEFINITION 1. The measure $\mu_t$ is called the *$t$-transform of a measure $\mu$* and the transformation $\mathcal{U}_t : \mu \mapsto \mu_t$ is called the *$t$-transformation*.

The following properties of $t$-transformation are direct consequences of the definition:

PROPOSITION 5. *For a probability measure $\mu$ and real numbers $t, s \geq 0$, the following properties are satisfied:*

*1. $(\mathcal{U}_t)_{t\geq 0}$ is a multiplicative semigroup: $\mathcal{U}_s(\mathcal{U}_t(\mu)) = \mathcal{U}_{st}(\mu)$;*
*2. dilations of a measure commute with $\mathcal{U}_t$: $D_\lambda(\mathcal{U}_t(\mu)) = \mathcal{U}_t(D_\lambda(\mu))$;*
*3. $\mathcal{U}_t(\mu) \to \mu$ in the $\star$-weak topology, if $t \to 1$;*
*4. $\mathcal{U}_t$ and $\mathcal{U}_{\frac{1}{t}}$ are inverses of each other;*
*5. $\mathcal{U}_t(\delta_a) = \delta_{ta}$ for any real number $a$;*
*6. The $t$-transformation is continuous in the $*$-weak topology of measures: if $\mu_n \to \mu$ then $\mathcal{U}_t\mu_n \to \mathcal{U}_t\mu$.*

Therefore, the mapping $\mathcal{U}_t$ is a multiplicative $*$-weakly continuous transformation, which commutes with dilations of measures. The $t$-transform of a measure is the inverse of the $1/t$-transform of the measure.

It is instructive to identify the action of $t$-transformation on the Cauchy transformation of a measure, given in the form of a continued fraction:

$$G_{\mu_t}(z) = \cfrac{1}{z - t\cdot a_1 - \cfrac{t\cdot b_1}{z - a_2 - \cfrac{b_2}{z - a_3 - \cfrac{b_3}{\ddots}}}} \tag{6}$$

so the action looks quite simple: only the "first" level (i.e. $a_1, b_1$) is multiplied by $t$.

The $t$-convolution $\oplus_t$ is defined in the following way. Given two probability measures $\mu$ and $\nu$ on the real line, a non-negative number $t$, and a convolution $\oplus$ (for which the classical convolution, free Voiculescu convolutions, boolean free convolution, and other convolutions may serve) one defines:

$$\mu \oplus_t \nu = (\mu_t \oplus \nu_t)_{1/t} = \mathcal{U}_{\frac{1}{t}}(\mathcal{U}_t(\mu) \oplus \mathcal{U}_t(\nu)) \tag{7}$$

The $t$-convolution provides a large new class of convolutions, which can be studied from the point of view of non-commutative central limit theorem. In the case of the classical multiplicative convolution (i.e. on the multiplicative group of positive numbers) of two point-mass measures $\delta_a$ and $\delta_b$, with $a, b$ positive, their $t$-(classical) convolution is not $\delta_{ab}$ but $\delta_{abt}$. On the other hand the $t$-(free) boolean convolution commutes with $t$-transform, which seems to be an exceptional case.

In [BLS] the concept of c-free (i.e. conditionally free) convolution was developed, in which c-free cumulants played crucial role. Let us recall that if $r_\nu(z)$ is the free cumulant (generating) function of a measure $\nu$, then it is related to the Cauchy transform of $\nu$ by the formula:

$$\frac{1}{G_\nu(z)} = z - r_\nu(G_\nu(z)). \tag{8}$$

Let us also recall that the c-free convolution is defined for pairs of measures (and more general, for pairs of states on unital *-algebras), say $(\mu_1, \nu_1)$ and $(\mu_2, \nu_2)$, where for the second terms one applies just the free convolution. The c-free cumulant (generating) function $\mathbf{T}_{(\mu,\nu)}(z) = \sum_{n=0}^{\infty} T_{(\mu,\nu)}(n) z^n$, which depends on the given pair of measures $(\mu, \nu)$, is related to the Cauchy transform of the first measure $\mu$ by the formula:

$$\frac{1}{G_\mu(z)} = z - \mathbf{T}_{(\mu,\nu)}(G_\nu(z)). \tag{9}$$

As the first formula is equivalent to the formula relating moments with free cumulants, the second one is equivalent to the following relation between moments and c-free cumulants:

$$m_\mu(n) = \sum_{\substack{s_1,\ldots,s_k \\ s_1+\ldots+s_k=n-k}} T_{(\mu,\nu)}(k) \cdot m_\nu(s_1) \cdot \ldots \cdot m_\nu(s_{k-1}) m_\mu(s_k). \tag{10}$$

We shall consider a special case of the above construction, namely when $\nu = \mu_t$. In this case the two formulas above combined with the definition of the $t$-transform give the relation:

$$r_{\mu_t}(n) = t \cdot T_{(\mu,\nu)}(n). \tag{11}$$

Therefore one gets the following special case of the formula for c-free cumulants:

$$m_\mu(n) = \sum_{\pi \in NC(n)} \prod_{\substack{B\in\pi \\ B-inner}} r_{\mu_t}(|B|) \cdot \prod_{\substack{B\in\pi \\ B-outer}} T_{(\mu,\nu)}(|B|) = \sum_{\pi\in NC(n)} R^t(\pi) \cdot t^{\mathbf{inn}(\pi)}$$

where $|B|$ is the number of elements in a block $B$, and

$$R^t(\pi) = R^t_{(\mu,\nu)}(\pi) = \prod_{B\in\pi} T_{(\mu,\nu)}(|B|).$$

This formula is an extension to all $q \geq -1$ of the formula obtained for $-1 \leq q \leq 0$ in [BSp1] in the proof of Theorem 7, with $t = 1 + q$, since in our case $t$ can be any non-negative number (cf. Example 1.6).

**4. Central limit theorems for $t$-convolutions.** As a preparation to the formulation of central limit theorems we start with the following

PROPOSITION 6. *If a given convolution $\oplus$ is associative, then for any positive number $t$ the associated $t$-convolution is also associative.*

Proof. By definition, for arbitrary measures $\mu, \nu, \rho$

$$(\mu \oplus_t \nu) \oplus_t \rho = \mathcal{U}_{\frac{1}{t}}(\mathcal{U}_t(\mathcal{U}_{\frac{1}{t}}(\mathcal{U}_t\mu \oplus \mathcal{U}_t\nu)) \oplus \mathcal{U}_t\rho) = \mathcal{U}_{\frac{1}{t}}((\mathcal{U}_t\mu \oplus \mathcal{U}_t\nu) \oplus \mathcal{U}_t\rho).$$

From these equalities the result easily follows. ∎

Let $D_\lambda\mu$ be the dilation of a measure $\mu$ by a number $\lambda$, defined as

$$D_\lambda\mu(A) = \mu(\lambda^{-1}A)$$

for an arbitrary measurable set $A$. In central limit theorems one studies the limit of a sequence of measures of the form $D_\lambda\mu \otimes \ldots \otimes D_\lambda\mu$, which is the $n$-th $\otimes$-convolution power of the dilation of a measure $\mu$ by an appropriate number $\lambda$. It is therefore essential to know that the expression makes sense, i.e. that the convolution $\otimes$ is associative.

Our central limit theorem has the following form

THEOREM 7. *Let $\mu$ be an arbitrary probability measure on the real line, with mean zero and with second moment equal to 1. Let also $t$ be a non-negative number and $\oplus$ be a given convolution. Then the sequence of measures $D_{\frac{1}{\sqrt{n}}}\mu \oplus_t \ldots \oplus_t D_{\frac{1}{\sqrt{n}}}\mu$ tends in the $*$-weak topology to a measure $\nu^{(t)}$, which is a transformation of the central limit measure $\nu$ for the convolution $\oplus$.*

Proof. For a fixed $n$ let $\lambda = \frac{1}{\sqrt{n}}$. Then the sequence of the $n$-fold $t$-convolution of the measure $\mu$ is of the form

$$D_\lambda\mu \oplus_t \ldots \oplus_t D_\lambda\mu = \mathcal{U}_{\frac{1}{t}}(\mathcal{U}_t D_\lambda\mu \oplus \ldots \oplus \mathcal{U}_t D_\lambda\mu) = \mathcal{U}_{\frac{1}{t}}(D_\lambda\mu_t \oplus \ldots \oplus D_\lambda\mu_t).$$

Since the measure $\mu_t = \mathcal{U}_t\mu$ has the second moment $t$, the sequence $D_\lambda\mu_t \oplus \ldots \oplus D_\lambda\mu_t$ has the $*$-weak limit $D_{\sqrt{t}}(\nu)$, where $\nu$ is the central limit measure for the $\oplus$-convolution, with $m_\nu(2) = 1$. Since the Cauchy transform of $\nu$ is given by a continued fraction of the form

$$G_\nu(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{b_2}{z - \cfrac{b_3}{\ddots}}}}$$

we obtain

$$G_{D_{\sqrt{t}}(\nu)}(z) = \cfrac{1}{z - \cfrac{t}{z - \cfrac{tb_2}{z - \cfrac{tb_3}{\ddots}}}}$$

and therefore the application of the transformation $\mathcal{U}_{\frac{1}{t}}$ to the measure $D_{\sqrt{t}}(\nu)$ gives the measure $\nu^{(t)}$, which has the following Cauchy transform:

$$G_{\nu^{(t)}}(z) = G_{\mathcal{U}_{\frac{1}{t}}(D_{\sqrt{t}}(\nu))}(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{tb_2}{z - \cfrac{tb_3}{\ddots}}}}. \tag{12}$$

Hence the sequence under consideration has the $*$-weak limit $\mathcal{U}_{\frac{1}{t}}(D_{\sqrt{t}}(\nu)) = \nu^{(t)}$, by the continuity of the $t$-transformation, since $m_{\mu_t}(1) = m_\mu(1) = 0$ and $m_{\mu_t}(2) = t \cdot m_\mu(2) = t$. ∎

R e m a r k. The above continued fractions combined may serve as a definition of the transformation $\nu \mapsto \nu^{(t)}$. Since the coefficients $b_k$ are non-negative, $t$ also has to be non-negative in this setting.

EXAMPLES 3. Our three basic examples are the following.

1. If $\oplus$ is the classical convolution of measures, then the limit measure is the transformation $\nu \to \nu^{(t)}$ of the gaussian measure $d\nu(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$. As the gaussian measure has the Cauchy transform

$$G_\nu(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{2}{z - \cfrac{3}{\ddots}}}}$$

the limit measure has the Cauchy transform

$$G_{\nu^{(t)}}(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{2t}{z - \cfrac{3t}{\ddots}}}}. \tag{13}$$

2. If $\oplus$ is the free convolution, then the limit is the transformation $\nu \to \nu^{(t)}$ of the Wigner semi-circular distribution $d\nu(x) = \frac{1}{2\pi} \chi_{[-2,2]}(x) \sqrt{4 - x^2} dx$. As the Wigner measure has the Cauchy transform

$$G_\nu(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{1}{\ddots}}}$$

the limit measure in this case has the Cauchy transform

$$G_{\nu^{(t)}}(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{t}{z - \cfrac{t}{\ddots}}}}. \tag{14}$$

3. If $\oplus$ is the boolean convolution, then we have a little more. In this case the transformation $\mathcal{U}_t$ commutes with the convolution, so the limit measure $\frac{1}{2}(\delta_1 + \delta_{-1})$ remains unchanged. The Cauchy transform of this measure is

$$G_\mu(z) = \cfrac{1}{z - \cfrac{1}{z}}. \tag{15}$$

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