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NEW EXAMPLES OF SASAKI-EINSTEIN MANIFOLDS

Dedicated to the memory of Professor Shoshichi Kobayashi

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Abstract. In this note, stimulated by the existence result by Futaki, Ono and Wang for toric Sasaki-Einstein metrics, we obtain new examples of Sasaki-Einstein metrics on S^1-bundles associated to canonical line bundles of $\boldsymbol{P}^1(\boldsymbol{C})$-bundles over Kähler-Einstein Fano manifolds, even though the Futaki's obstruction does not vanish. Here our examples include non-toric Sasaki-Einstein manifolds.

1. Introduction. Sasaki-Einstein manifolds were studied not only by mathematicians but also by physicists, as Sasaki-Einstein manifolds have various interesting phenomena such as "AdS/CFT correspondence" in theoretical physics (cf. [1], [2], [3], [4], [5], [12], [19], [20], [21], [22]). Recently in [6] and [10], classification of toric Sasaki-Einstein manifolds was given.

A Sasaki manifold is a $(2m+1)$-dimensional Riemannian manifold (S, g) whose cone manifold $(C(S), \overline{g})$ is a Kähler manifold with

$$C(S) := S \times \boldsymbol{R}_{>0} \quad \text{and} \quad \overline{g} := (dr)^2 + r^2 g\,,$$

where r is the standard coordinate on the set $\boldsymbol{R}_{>0} = \{r > 0\}$ of positive real numbers. Then S is a contact manifold with the contact form

$$\eta := \big(\sqrt{-1}(\overline{\partial} - \partial)\log r\big)|_{r=1}.$$

Here S is viewed as the submanifold of $C(S)$ defined by the equation $r = 1$. We further consider the *Reeb field* ξ characterized by

$$i(\xi)\eta = 1 \quad \text{and} \quad i(\xi)d\eta = 0\,,$$

where $i(\xi)$ is the interior product by ξ. The Reeb field ξ is a Killing vector field on (S, g) with a lift to a holomorphic Killing vector field on $(C(S), \overline{g})$. This generates a 1-dimensional foliation on S, called the *Reeb foliation*. The Sasaki metric g naturally induces a transverse Kähler metric g^{T} for the Reeb foliation on S. A Sasaki manifold (S, g) is *toric*, if $C(S)$ is a toric manifold.

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The following well-known fact allows us to reduce the existence of Sasaki-Einstein metrics to that of transverse Kähler-Einstein metrics:

FACT 1.1 (cf. [3, Chapter 11]). *A Sasaki manifold* (S, g) *is Einstein with positive scalar curvature* $2m$ *if and only if the transverse Kähler metric* g^{T} *is Einstein with positive scalar curvature* $2(m+1)$.

We now pose the following conjecture:

CONJECTURE 1.2. *Let* M *be a Fano manifold. If there exists a Kähler-Ricci soliton (see for instance* [28] *for Kähler-Ricci solitons) on* M, *then the* S^1*-bundle associated to the canonical line bundle* K_M *of* M *admits a Sasaki-Einstein metric with a suitable choice of the Reeb field.*

By the results of Wang and Zhu [28], the existence of Kähler-Ricci solitons is known for toric Fano manifolds. Hence, the results in [10] shows that Conjecture 1.2 is affirmative for toric Fano manifolds.

We now consider Koiso-Sakane's examples (cf. [23], [16], [17]) of $\boldsymbol{P}^1(\boldsymbol{C})$-bundles over Kähler-Einstein Fano manifolds. To fix our notation, recall the paper [18]. Under the assumption below, we fix once for all a compact connected n-dimensional complex manifold W with $c_1(W) > 0$ and an Hermitian holomorphic line bundle (L, h) over W.

ASSUMPTION 1.3. (1) There exists a Kähler-Einstein form ω_0 on W, i.e., $\mathrm{Ric}(\omega_0) = \omega_0$, where $\mathrm{Ric}(\omega_0)$ is the Ricci form for ω_0.

(2) $2\pi c_1(L; h) := \sqrt{-1}\,\overline{\partial}\partial \log h$ has constant eigenvalues

$$\mu_1 \leqq \mu_2 \leqq \cdots \leqq \mu_n$$

with respect to ω_0 satisfying $-1 < \mu_k < 1$ for $k = 1, 2, \dots, n$.

By this assumption, the compactification $M_W^L := \boldsymbol{P}(L \oplus \mathcal{O}_W)$ of L is a $\boldsymbol{P}^1(\boldsymbol{C})$-bundle over W with $c_1(M_W^L) > 0$. Then M_W^L admits a Kähler-Einstein metric if and only if its Futaki's obstruction (cf. [8]) vanishes:

$$\int_{-1}^{1} x \prod_{k=1}^{n} (1 + \mu_k x)\, dx = 0. \tag{1.4}$$

Let S_W^L be the S^1-bundle over M_W^L associated to the canonical line bundle $K_{M_W^L}$ of M_W^L. In [15], Koiso showed that a Kähler-Ricci soliton exists on M_W^L, whether or not equality (1.4) holds. Hence by Conjecture 1.2, a Sasaki-Einstein metric is expected to exist on S_W^L. The purpose of this note is to give the following affirmative result:

THEOREM 1.5. *Under the Assumption* 1.3, *whether or not the equality* (1.4) *holds,* S_W^L *always admits a Sasaki-Einstein metric for a suitable choice of the Reeb field. Furthermore,* $K_{M_W^L}$ *admits a complete Ricci-flat Kähler metric in every Kähler class.*

REMARK 1.6. Kobayashi [14] (see also Jensen [13], Wang and Ziller [27]) constructed Einstein metrics on S^1-bundles over Einstein manifolds. Our theorem above shows that S_W^L always admits an Einstein metric, even though M_W^L admits no Kähler-Einstein metrics.

2. Transverse holomorphic structures on S_W^L. For an open cover $\{U_\alpha\,;\ \alpha \in A\}$ of W, we choose a system of holomorphic local coordinates $(w_\alpha^1, w_\alpha^2, \dots, w_\alpha^n)$ on each U_α, and by taking a holomorphic local frame e_α for L, we have the fiber coordinate ζ_α^+ for L over U_α with respect to e_α. Then $(w_\alpha^1, w_\alpha^2, \dots, w_\alpha^n; \zeta_\alpha^+)$ forms a system of holomorphic local coordinates for $U_\alpha^+ := L|_{U_\alpha}$. Let f_α be the frame for L^{-1} dual to e_α, and let ζ_α^- be the fiber coordinate for L^{-1} over U_α with respect to f_α. Then $(w_\alpha^1, w_\alpha^2, \dots, w_\alpha^n; \zeta_\alpha^-)$ form a system of holomorphic local coordinates on $U_\alpha^- := L^{-1}|_{U_\alpha}$. Then U_α^+ and U_α^- are glued together by the relation

$$\zeta_\alpha^+ = (\zeta_\alpha^-)^{-1}$$

to form $M_W^L = \boldsymbol{P}(L \oplus \mathcal{O}_W) = \bigcup_{\alpha \in A}(U_\alpha^+ \cup U_\alpha^-)$. Here,

$$\pm dw_\alpha^1 \wedge dw_\alpha^2 \wedge \cdots \wedge dw_\alpha^n \wedge d\zeta_\alpha^\pm$$

is a holomorphic local frame for $K_{M_W^L}$ over $U_\alpha^\pm$, and with respect to this local frame, we have the fiber coordinate $\tau_\alpha^\pm$ for $K_{M_W^L}$, respectively, i.e., all $(+)$-signs and all $(-)$-signs should be chosen respectively. Note that

$$\begin{aligned}
&\tau_\alpha^+ dw_\alpha^1 \wedge dw_\alpha^2 \wedge \cdots \wedge dw_\alpha^n \wedge d\zeta_\alpha^+ \\
&\quad = \tau_\beta^+ dw_\beta^1 \wedge dw_\beta^2 \wedge \cdots \wedge dw_\beta^n \wedge d\zeta_\beta^+ \\
&\quad = \tau_\beta^+ \phi_{\beta\alpha}(w)\psi_{\beta\alpha}(w)^{-1} dw_\alpha^1 \wedge dw_\alpha^2 \wedge \cdots \wedge dw_\alpha^n \wedge d\zeta_\alpha^+
\end{aligned}$$

for $w \in U_\alpha \cap U_\beta$. Here $\{\psi_{\beta\alpha}\,;\ \alpha, \beta \in A\}$ are the transition functions for L with respect to the local frames $\{e_\alpha\,;\ \alpha \in A\}$ for L, while $\{\phi_{\beta\alpha}\,;\ \alpha, \beta \in A\}$ are the transition functions for K_W with respect to the local frames $\{dw_\alpha^1 \wedge \cdots \wedge dw_\alpha^n\,;\ \alpha \in A\}$ for K_W, i.e.,

$$\begin{aligned}
&e_\beta = \psi_{\beta\alpha}(w) e_\alpha\,, \qquad f_\beta = \psi_{\beta\alpha}(w)^{-1} f_\alpha\,, \\
&dw_\beta^1 \wedge dw_\beta^2 \wedge \cdots \wedge dw_\beta^n = \phi_{\beta\alpha}(w) dw_\alpha^1 \wedge dw_\alpha^2 \wedge \cdots \wedge dw_\alpha^n
\end{aligned}$$

for $w \in U_\alpha \cap U_\beta$. Hence τ_α^+ can be viewed as the fiber coordinate for $K_W \otimes L^{-1}$ over U_α with respect to the local frame $\left(dw_\alpha^1 \wedge \cdots \wedge dw_\alpha^n\right) \otimes f_\alpha$. Similarly, τ_α^- is also viewed as the fiber coordinate for $K_W \otimes L$ over U_α with respect to the local frame $\left(dw_\alpha^1 \wedge \cdots \wedge dw_\alpha^n\right) \otimes e_\alpha$. Moreover, since $\tau_\alpha^+ \zeta_\alpha^+ = \tau_\alpha^- \zeta_\alpha^-$ on $U_\alpha^+ \cap U_\alpha^-$, it follows that

$$\tau_\alpha^+ (\zeta_\alpha^+)^2 = \tau_\alpha^-\,.$$

Now, for $-1/2 < a \in \boldsymbol{R}$, we consider holomorphic vector fields

$$\begin{aligned}
&a\sqrt{-1}\zeta_\alpha^+ \frac{\partial}{\partial \zeta_\alpha^+} + \sqrt{-1}\tau_\alpha^+ \frac{\partial}{\partial \tau_\alpha^+} \quad \text{on } \tilde{p}^{-1}(U_\alpha^+)\,, \\
&-a\sqrt{-1}\zeta_\alpha^- \frac{\partial}{\partial \zeta_\alpha^-} + (1+2a)\sqrt{-1}\tau_\alpha^- \frac{\partial}{\partial \tau_\alpha^-} \quad \text{on } \tilde{p}^{-1}(U_\alpha^-)\,,
\end{aligned}$$

where $\tilde{p}\colon K_{M_W^L} \to M_W^L$ is the natural projection. Then these are glued together to define a well-defined global holomorphic vector field ξ_a on $K_{M_W^L}$. We choose $\xi_a + \bar{\xi}_a$ as the Reeb field on S_W^L. However, we call ξ_a also as the Reeb field by abuse of terminology. Put

$$z_\alpha^+ := (\tau_\alpha^+)^{-a}\zeta_\alpha^+ \quad \text{and} \quad z_\alpha^- := (\tau_\alpha^-)^{a/(1+2a)}\zeta_\alpha^- .$$

Then $(w_\alpha^1, w_\alpha^2, \dots, w_\alpha^n; z_\alpha^+)$ and $(w_\alpha^1, w_\alpha^2, \dots, w_\alpha^n; z_\alpha^-)$ are transverse holomorphic local coordinates on $\widetilde{U}_\alpha^+ := p^{-1}(U_\alpha^+)$ and $\widetilde{U}_\alpha^- := p^{-1}(U_\alpha^-)$, respectively, with respect to the Reeb field ξ_a, in view of the identities

$$dz_\alpha^+(\xi_a) = 0 \quad \text{and} \quad dz_\alpha^-(\xi_a) = 0 ,$$

where $p\colon S_W^L \to M_W^L$ is the natural projection. Note that z_α^+ and z_α^- satisfy the relation

$$z_\alpha^+ = (\tau_\alpha^+)^{-a}\zeta_\alpha^+ = (\tau_\alpha^-)^{-a}(\zeta_\alpha^-)^{-(1+2a)} = (z_\alpha^-)^{-(1+2a)} .$$

For the natural projection $q\colon S_W^L \to W$, the fiber $q^{-1}(w)$ over each $w \in U_\alpha$ has a transverse holomorphic structure defined by the transverse holomorphic coordinate $z_\alpha^\pm$. Then on $q^{-1}(w)$,

$$G := \begin{cases} \left(\dfrac{1}{|z_\alpha^+|^{-2} + 1 + |z_\alpha^+|^{2/(1+2a)}}\right)\dfrac{|dz_\alpha^+|^2}{|z_\alpha^+|^2} & \text{on } q^{-1}(w) \cap \widetilde{U}_\alpha^+, \\[2ex] \left(\dfrac{(1+2a)^2}{|z_\alpha^-|^{2(1+2a)} + 1 + |z_\alpha^-|^{-2}}\right)\dfrac{|dz_\alpha^-|^2}{|z_\alpha^-|^2} & \text{on } q^{-1}(w) \cap \widetilde{U}_\alpha^- \end{cases}$$

defines an Hermitian metric for the transverse anti-canonical line bundle of the fiber $q^{-1}(w)$, which is invariant under the standard S^1-action $z_\alpha^+ \overset{t}{\longmapsto} tz_\alpha^+$, $t \in S^1$, for each $w \in U_\alpha$, where $S^1 := \{z \in \boldsymbol{C}\,;\ |z| = 1\}$. By setting $x := -2\log|z_\alpha^+|$, we define

$$v(x) := \log\left\{\exp(x) + 1 + \exp\left(-\frac{x}{1+2a}\right)\right\} .$$

Then its derivative $v'(x)$ defines a moment map whose image is the closed interval $[-1/(1+2a),\ 1]$.

3. Sasaki-Einstein metrics on S_W^L. In this section, by an argument as in [18], we construct a Sasaki-Einstein metric on S_W^L by reducing the Sasaki-Einstein equation to the transverse Einstein equation (3.1) below. For $a > -1/2$, define a polynomial $A_a(x)$ in x by

$$A_a(x) := -\int_{-1/(1+2a)}^{x} s\prod_{k=1}^{n}\left(1 + \mu_{k,a}s\right) ds ,$$

where $\mu_{k,a} := \mu_k + a(1+\mu_k)$ for $k = 1, 2, \dots, n$. Now, we assume that $A_a(1) = 0$. Since $a > -1/2$, it follows from Assumption 1.3 that

$$0 < A_a(x) \leqq A_a(0) \quad \text{and} \quad \frac{A_a'(x)}{x} < 0$$

for $-1/(1+2a) < x < 1$. In particular, the rational function

$$\frac{A_a'(x)}{xA_a(x)}$$

is free from poles and zeros over the open interval $(-1/(1+2a),\ 1)$ and has a pole of order 1 at both $x = -1/(1+2a)$ and $x = 1$. Hence,

$$B_a(x) := -\int_0^x \frac{A_a'(s)}{sA_a(s)}ds$$

is monotone increasing over the interval $(-1/(1+2a),\ 1)$ and moreover, B_a maps $(-1/(1+2a),\ 1)$ diffeomorphically onto $\boldsymbol{R}$. Let

$$B_a^{-1}\colon \boldsymbol{R} \to \left(-\frac{1}{1+2a},\ 1\right)$$

be the inverse function of $B_a\colon (-1/(1+2a),\ 1) \to \boldsymbol{R}$, and define C^∞ functions $x_a(\rho)$ and $u_a(\rho)$ in $\rho \in \boldsymbol{R}$ by $x_a(\rho) := B_a^{-1}(\rho)$ and $u_a(\rho) := -\log(A_a(x_a(\rho)))$, respectively. Then $u_a'(\rho) = x_a(\rho)$ and hence

$$u_a''(\rho)\prod_{k=1}^{n}\big(1+\mu_{k,a}u_a'(\rho)\big) = e^{-u_a(\rho)}. \tag{3.1}$$

On $\widetilde{U}_\alpha := \widetilde{U}_\alpha^+ \cup \widetilde{U}_\alpha^-$, we define

$$\rho_\alpha := \begin{cases} -\log|z_\alpha^+|^2 - \log\big(\kappa_\alpha^{-a}h_\alpha^{1+a}\big) & \text{on } \widetilde{U}_\alpha^+\,, \\ (1+2a)\log|z_\alpha^-|^2 - \log\big(\kappa_\alpha^{-a}h_\alpha^{1+a}\big) & \text{on } \widetilde{U}_\alpha^-\,, \end{cases} \tag{3.2}$$

by setting $\kappa_\alpha := h_{K_W}\big(dw_\alpha^1\wedge\cdots\wedge dw_\alpha^n, dw_\alpha^1\wedge\cdots\wedge dw_\alpha^n\big)$ and $h_\alpha := h(e_\alpha, e_\alpha)$, that is, on $\widetilde{U}_\alpha^+$, $\exp(-\rho_\alpha/2)$ can be formally viewed as the norm of

$$z_\alpha^+\left(\left(\frac{\partial}{\partial w_\alpha^1}\wedge\cdots\wedge\frac{\partial}{\partial w_\alpha^n}\right)^a\otimes e_\alpha^{1+a}\right)$$

with respect to the Hermitian metric $h_{K_W}^{-a}\otimes h^{1+a}$ for $K_W^{-a}\otimes L^{1+a}$. Here h_{K_W} denotes the Hermitian metric for K_W induced by ω_0. Then we have $\rho_\alpha = \rho_\beta$ on $\widetilde{U}_\alpha \cap \widetilde{U}_\beta$. Now we consider the following transverse $(n+1, n+1)$-form $\varPhi_\alpha$, with respect to ξ_a, on $\widetilde{U}_\alpha$:

$$\varPhi_\alpha := \begin{cases} \sqrt{-1}\,(n+1)e^{-u_a(\rho_\alpha)}\,(q^*\omega_0)^n \wedge \dfrac{dz_\alpha^+\wedge d\overline{z_\alpha^+}}{|z_\alpha^+|^2} & \text{on } \widetilde{U}_\alpha^+\,, \\ \sqrt{-1}\,(n+1)e^{-u_a(\rho_\alpha)}\,(q^*\omega_0)^n \wedge (1+2a)^2\dfrac{dz_\alpha^-\wedge d\overline{z_\alpha^-}}{|z_\alpha^-|^2} & \text{on } \widetilde{U}_\alpha^-\,. \end{cases}$$

Then $\{\varPhi_\alpha\,;\ \alpha\in A\}$ define the transverse $(n+1, n+1)$-form $\varPhi$ on S_W^L. Note that $\mathrm{Ric}(\omega_0) = \sqrt{-1}\,\overline{\partial}\partial\log\omega_0^n = \omega_0$ and that, for each fixed $w_0 \in U_\alpha$, we can choose a local frame e_α for L and a system $(w_\alpha^1, w_\alpha^2, \dots, w_\alpha^n)$ of holomorphic local coordinates on U_α satisfying

$$d(\kappa_\alpha^{-a}h_\alpha^{1+a})(w_0) = 0\,,\quad \omega_0(w_0) = \sqrt{-1}\sum_{k=1}^{n} dw_\alpha^k\wedge d\overline{w_\alpha^k}\,,$$

$$(\sqrt{-1}\,\bar\partial\partial\log h_\alpha)(w_0) = \sqrt{-1}\sum_{k=1}^{n}\mu_k dw_\alpha^k \wedge d\overline{w_\alpha^k}\,.$$

Then, along $q^{-1}(w_0)\cap\widetilde{U}_\alpha$, we write $\omega_\alpha^{\mathrm{T}} := \{\sqrt{-1}/(2n+4)\}\bar\partial\partial\log\Phi_\alpha$ as a sum

$$\frac{1}{2n+4}\sum_{k=1}^{n}\Big\{\big(1+\mu_{k,a}u_a'(\rho_\alpha)\big)\sqrt{-1}dw_\alpha^k\wedge d\overline{w_\alpha^k}\Big\} + \frac{1}{2n+4}u_a''(\rho_\alpha)\frac{\sqrt{-1}dz_\alpha^+\wedge d\overline{z_\alpha^+}}{|z_\alpha^+|^2}$$

on $\widetilde{U}_\alpha^+$, and

$$\frac{1}{2n+4}\sum_{k=1}^{n}\Big\{\big(1+\mu_{k,a}u_a'(\rho_\alpha)\big)\sqrt{-1}dw_\alpha^k\wedge d\overline{w_\alpha^k}\Big\} + \frac{1}{2n+4}(1+2a)^2u_a''(\rho_\alpha)\frac{\sqrt{-1}dz_\alpha^-\wedge d\overline{z_\alpha^-}}{|z_\alpha^-|^2}$$

on $\widetilde{U}_\alpha^-$. Since $a > -1/2$ and $-1 < \mu_k < 1$ $(k = 1, 2, \dots, n)$, $\omega_\alpha^{\mathrm{T}}$ is a transverse Kähler form, with respect to ξ_a, on $\widetilde{U}_\alpha^+\setminus\{z_\alpha^+ = 0\} = \widetilde{U}_\alpha^-\setminus\{z_\alpha^- = 0\}$. Furthermore, by (3.1), we have $\big\{(2n+4)\omega_\alpha^{\mathrm{T}}\big\}^{n+1} = \Phi_\alpha$. Therefore, $\omega_\alpha^{\mathrm{T}}$ defines a transverse Kähler-Einstein metric, with respect to ξ_a, on $\widetilde{U}_\alpha^+\setminus\{z_\alpha^+ = 0\} = \widetilde{U}_\alpha^-\setminus\{z_\alpha^- = 0\}$. Since $\rho_\alpha = B_a(x_a(\rho_\alpha))$, we have

$$\rho_\alpha = \begin{cases} -\log(1-x_a(\rho_\alpha)) \\ \quad + \text{real analytic function in } x_a(\rho_\alpha) \quad \text{near } x_a(\rho_\alpha) = 1\,, \\ (1+2a)\log\Big(\dfrac{1}{1+2a} + x_a(\rho_\alpha)\Big) \\ \quad + \text{real analytic function in } x_a(\rho_\alpha) \quad \text{near } x_a(\rho_\alpha) = \dfrac{-1}{1+2a}\,, \end{cases}$$

while we see from (3.2) that

$$\begin{cases} |z_\alpha^+|^{-2} = (1-x_a(\rho_\alpha))^{-1}\exp\sigma^+ & \text{near } x_a(\rho_\alpha) = 1\,, \\ |z_\alpha^-|^{-2} = \Big(\dfrac{1}{1+2a} + x_a(\rho_\alpha)\Big)^{-1}\exp\sigma^- & \text{near } x_a(\rho_\alpha) = \dfrac{-1}{1+2a}\,. \end{cases}$$

Here σ^+ and σ^- are real analytic functions on $\tilde{U}_\alpha^+$ and $\tilde{U}_\alpha^-$, respectively. Hence the argument as in Step 2 in the proof of [18, Theorem 10.3] is valid for transverse Kähler cases even when the Reeb field is irregular. Therefore, the condition $A_a(1) = 0$ implies that $\big\{\omega_\alpha^{\mathrm{T}}\,;\,\alpha\in A\big\}$ are glued together to define a well-defined global transverse Kähler-Einstein form ω^{T} on S_W^L with the Reeb field ξ_a.

REMARK 3.3. Let $a \in \boldsymbol{R}$ be such that $A_a(1) = 0$. On $\widetilde{U}_\alpha^+$ (resp. $\widetilde{U}_\alpha^-$), Φ_α is formally viewed as a Hermitian metric for

$$K_W^{-1}\otimes\big(K_W^a\otimes L^{-(1+a)}\big)^{-1} = \big(K_W\otimes L^{-1}\big)^{-(1+a)}$$

$$\big(\text{resp. } (K_W \otimes L)^{-(1+a)/(1+2a)}\big)\,.$$

Then, on $\widetilde{U}^+_\alpha$ (resp. $\widetilde{U}^-_\alpha$), we put

$$r := \left\{ \big(|\tau^+_\alpha|\big)^{2(1+a)} \,\frac{\exp(u_a(\rho_\alpha))\,\kappa_\alpha |z^+_\alpha|^2}{n+1} \right\}^{1/(2n+4)}$$

$$\left(\text{resp. } r := \left\{ \big(|\tau^-_\alpha|\big)^{(2(1+a))/(1+2a)} \,\frac{\exp(u_a(\rho_\alpha))\,\kappa_\alpha |z^-_\alpha|^2}{n+1} \right\}^{1/(2n+4)} \right),$$

and $\eta := \big(\sqrt{-1}(\overline{\partial} - \partial)\log r\big)|_{r=1}$. On $\widetilde{U}^+_\alpha$ (resp. $\widetilde{U}^-_\alpha$), r^{n+2} is regarded as the norm of

$$\big(\tau^+_\alpha\big)^{1+a}\big(\big(dw^1_\alpha \wedge \cdots \wedge dw^n_\alpha\big) \otimes dz^+_\alpha\big)$$
$$\big(\text{resp. } -(1+2a)\big(\tau^-_\alpha\big)^{(1+a)/(1+2a)}\big(\big(dw^1_\alpha \wedge \cdots \wedge dw^n_\alpha\big) \otimes dz^-_\alpha\big)\big)\,,$$

with respect to the Hermitian metric $(\Phi_\alpha)^{-1}$ for

$$(K_W \otimes L^{-1})^{1+a} \quad \big(\text{resp. } (K_W \otimes L)^{(1+a)/(1+2a)}\big)\,.$$

Hence r defines a well-defined C^∞ function on $K_{M^L_W} \setminus \{\text{zero section}\}$, and in particular S^L_W is identified with the submanifold of $K_{M^L_W}$ defined by the equation $r = 1$. Here, we note that, on $\widetilde{U}^+_\alpha \cap \widetilde{U}^-_\alpha$,

$$\begin{aligned}\big(\tau^+_\alpha\big)^{1+a}&\big(\big(dw^1_\alpha \wedge \cdots \wedge dw^n_\alpha\big) \otimes dz^+_\alpha\big) \\ &= -(1+2a)(\tau^-_\alpha)^{(1+a)/(1+2a)}\big(\big(dw^1_\alpha \wedge \cdots \wedge dw^n_\alpha\big) \otimes dz^-_\alpha\big)\,.\end{aligned}$$

Moreover, $g := (\eta)^2 + g^{\mathrm{T}}$ is a Riemannian metric on S^L_W and η is a contact form on S^L_W, where g^{T} is the transverse Kähler metric associated to ω^{T}. Furthermore, the fundamental form $\overline{\omega}$ of the cone metric $\overline{g}$ associated to g is given by

$$\overline{\omega} := r dr \wedge \eta + r^2 \omega^{\mathrm{T}}\,.$$

In view of $d\eta = 2\omega^{\mathrm{T}}$, we obtain $d\overline{\omega} = 0$, and hence (S^L_W, g) is a Sasaki manifold with the Reeb field ξ_a.

Now by Fact 1.1, we obtain the following criterion on the existence of Sasaki-Einstein metrics on S^L_W:

PROPOSITION 3.4. *Under the Assumption* 1.3, *if*

$$A_a(1) = -\int_{-1/(1+2a)}^{1} x \prod_{k=1}^{n} (1 + \mu_{k,a} x)dx = 0\,, \tag{3.5}$$

then S^L_W admits a Sasaki-Einstein metric with the Reeb field ξ_a.

REMARK 3.6. In the special case $a = 0$, we easily see that (3.5) is nothing but the condition (1.4) in the introduction.

Next, we shall show the existence of $a \in \boldsymbol{R}$ such that both $a > -1/2$ and $A_a(1) = 0$ hold. We now put

$$f(x; a) := x \prod_{k=1}^{n} \left(1 + \mu_{k,a} x\right) ,$$

$$F(a) := \int_{-1/(1+2a)}^{1} f(x; a)\, dx \quad (= -A_a(1)) .$$

Since $\lim_{a \to +\infty} F(a) = +\infty$ and $\lim_{a \to -1/2+0} F(a) = -\infty$, the continuity of F allows us to find $a_0 > -1/2$ such that $F(a_0) = 0$. Moreover,

$$F'(a) = \int_{-1/(1+2a)}^{1} \frac{\partial}{\partial a} f(x; a) dx + \frac{-2}{(1+2a)^2} f\left(-\frac{1}{1+2a}; a\right) .$$

Note also that $\mu_{k,a} = \mu_k + a(1 + \mu_k)$. Hence for $-1/(1+2a) \leqq x \leqq 1$,

$$\frac{\partial}{\partial a} f(x; a) = x^2 \sum_{j=1}^{n} \left\{ (1 + \mu_j) \prod_{k \neq j} (1 + \{\mu_k + a(1 + \mu_k)\}\, x) \right\} \geqq 0 ,$$

$$f\left(-\frac{1}{1+2a}; a\right) = -\left(\frac{1}{1+2a}\right)^{n+1} \prod_{k=1}^{n} \{(1+a)(1-\mu_k)\} < 0 .$$

Now in the expression of $F'(a)$, the first term is nonnegative and the second term is positive. Therefore $F'(a) > 0$. Hence we obtain the following lemma.

LEMMA 3.7. *Under the Assumption* 1.3, *there exists a unique real number* $a_0 > -1/2$ *such that* $F(a_0) = 0$.

Therefore, by Proposition 3.4 and Lemma 3.7, if Assumption 1.3 is satisfied, then S_W^L always admits a Sasaki-Einstein metric with the Reeb field ξ_{a_0}. On the other hand, in view of [9], [11](see also [26]), we now conclude that $K_{M_W^L}$ admits a complete Ricci-flat Kähler metric in every Kähler class. The proof of Theorem 1.5 is now complete.

4. Examples. In this section, we shall give a couple of examples of Sasaki-Einstein manifolds as an application of Theorem 1.5.

EXAMPLE 4.1. We first put

$$W := \prod_{i=1}^{l} \boldsymbol{P}^{n_i}(\boldsymbol{C}) ,$$

$$L := \bigotimes_{i=1}^{l} p_i^* \left(\mathcal{O}_{\boldsymbol{P}^{n_i}(\boldsymbol{C})}(\nu_i)\right) ,$$

where $p_i \colon W \to \boldsymbol{P}^{n_i}(\boldsymbol{C})$ is the natural projection to the i-th factor ($i = 1, 2, \dots, l$). In view of the isomorphism $K^{-1}_{\boldsymbol{P}^k(\boldsymbol{C})} \cong \mathcal{O}_{\boldsymbol{P}^k(\boldsymbol{C})}(k+1)$, if

$$-(n_i + 1) < \nu_i < n_i + 1 , \quad (i = 1, 2, \dots, l) ,$$

then the pair (W, L) satisfies Assumption 1.3. Hence by Theorem 1.5, S_W^L admits a Sasaki-Einstein metric, though this is toric. Then $F(a)$ in Section 3 is given by

$$F(a) = \int_{-1/(1+2a)}^{1} x \prod_{i=1}^{l} \left(1 + \left\{\frac{\nu_i}{n_i + 1} + a\left(1 + \frac{\nu_i}{n_i + 1}\right)\right\} x\right)^{n_i} dx\,.$$

For instance, we consider the simplest case, that is, $W = \boldsymbol{P}^1(\boldsymbol{C})$ and $L = \mathcal{O}_{\boldsymbol{P}^1(\boldsymbol{C})}(1)$. In this case, M_W^L is a del Pezzo surface obtained from $\boldsymbol{P}^2(\boldsymbol{C})$ by blowing up one point, and we see the irregularity of (S_W^L, ξ_{a_0}) by

$$a_0 = \frac{-5 + \sqrt{13}}{12}\,.$$

EXAMPLE 4.2. Next, let $W := \mathrm{Gr}(k, p)$ be the complex Grassmannian manifold of all p-dimensional subspaces of $\boldsymbol{C}^k$, which is a complex manifold of dimension $p(k - p)$. Then there exists an ample line bundle $A(k, p)$ over $\mathrm{Gr}(k, p)$ such that $K_{\mathrm{Gr}(k,p)}^{-1} \cong A(k, p)^k$ (see for instance [24, p. 205]). We put $L := A(k, p)^\nu$. If $-k < \nu < k$, then the pair (W, L) satisfies Assumption 1.3. Hence by Theorem 1.5, S_W^L admits a Sasaki-Einstein metric, and if $2 \leqq p \leqq k - 2$, then S_W^L is non-toric.

EXAMPLE 4.3. Let $\mathcal{M}_n$ be the moduli space of smooth hypersurfaces of degree n in $\boldsymbol{P}^{n+1}(\boldsymbol{C})$. For the Fermat type hypersurface

$$W_0 := \left\{[X_0, X_1, \ldots, X_{n+1}] \in \boldsymbol{P}^{n+1}(\boldsymbol{C})\ ;\ \sum_{i=0}^{n+1} (X_i)^n = 0\right\} \in \mathcal{M}_n\,,$$

a theorem of Tian [25] shows that W_0 admits a Kähler-Einstein metric, and in particular

$$\mathcal{M}_n^{\mathrm{KE}} := \{W \in \mathcal{M}_n\ ;\ W \text{ admits a Kähler-Einstein metric}\}$$

is a non-empty open subset of $\mathcal{M}_n$. For every $W \in \mathcal{M}_n^{\mathrm{KE}}$, we have $K_W \cong \mathcal{O}_{\boldsymbol{P}^{n+1}(\boldsymbol{C})}(-2)|_W$ by adjunction formula. Put $L := \mathcal{O}_{\boldsymbol{P}^{n+1}(\boldsymbol{C})}(1)|_W$. Then the pair (W, L) satisfies Assumption 1.3, and Theorem 1.5 shows that S_W^L admits a Sasaki-Einstein metric. If $n = 3$, W is a well-known cubic threefold, and in this case by [7, Theorem 13.12], W is not birationally equivalent to $\boldsymbol{P}^3(\boldsymbol{C})$, and S_W^L is again non-toric.

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