New Exponential Estimates for Time-Delay Systems

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***Abstract*—This note considers the problem of exponential stability for time-delay systems. In terms of linear matrix inequalities, a new sufficient condition for exponential stability is obtained. Based on this, an improved upper bound of the decay rate can be easily calculated. When time-varying norm-bounded parameter uncertainties appear, a new sufficient condition for robust exponential stability of uncertain time-delay systems is also provided. The reduced conservatism of the proposed conditions is illustrated via two numerical examples.**

***Index Terms*—Exponential stability, linear matrix inequality (LMI), time-delay systems, uncertain systems.**

I. Introduction

Time-delay systems have been investigated by many researchers since they are encountered in engineering systems, biology, economics, and other areas [7], [11]. Stability analysis of time-delay systems is of both practical and theoretical importance since time delays are frequently the main cause of instability and poor performance of a system. A great number of stability results have been proposed in the literature; see, e.g., [1]–[4], [6], [14]–[16], and the references therein. These results can be classified into two types according to their dependence of the delay size; that is, delay-dependent stability results and delay-independent ones. Delay-dependent stability results are generally less conservative than delay-independent ones.

It is noted that many stability results for time-delay systems are concerned with asymptotic stability. In practical applications, however, it is also important to find estimates of the transient decay rate of a delay system. Therefore, the problem of exponential stability has been studied. For instance, an estimate of the decay rate of a linear stable delay systems was given in [10], which was further improved in [5]. By using the properties of matrix measure, sufficient conditions for the exponential stability of time-delay systems were obtained in [13]. When time-varying delays appear, some robust exponential stability results were proposed in [12]. However, the conditions in both [12] and [13] are not easy to check. Very recently, by a linear matrix inequality (LMI) approach, exponential stability conditions were presented in [8] and [9], respectively. These conditions can be easily checked.

In this note, we provide a new exponential stability condition for time-delay systems by choosing an appropriate Lyapunov–Krasovskii functional and introducing slack variables. Based on this, an upper bound of the decay rate can be easily calculated. When time-varying norm-bounded parameter uncertainties appear, a robust exponential stability condition is also provided. Both the exponential stability and the robust exponential stability conditions are given in terms of LMIs. The proposed conditions in this note are less conservative than some of those in the literature, which is demonstrated via two numerical examples.

Notation: Throughout this note, for real symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semidefinite (respectively, positive definite). I is an identity matrix with appropriate dimension. The superscript "T" represents the transpose. The notations $|\cdot|$ and $\|\cdot\|$ refer to the Euclidean vector norm and the induced matrix two-norm, respectively. We use $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ to denote the minimum and maximum eigenvalue of a symmetric matrix, respectively. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions.

II. Main Results

Consider the following time-delay system:

$$(\Sigma): \quad \dot{x}(t) = Ax(t) + A_1 x(t-h) \tag{1}$$

$$x(t) = \varphi(t) \qquad \forall t \in [-h, 0] \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state, and $\varphi(t)$ is the initial condition. The scalar $h > 0$ is the delay of the system, A and A_1 are known real constant matrices.

Definition 1: System (Σ) is said to be exponentially stable with a decay rate λ if there exist scalars $\sigma \geq 1$ and $\lambda > 0$ such that $|x(t)| \leq \sigma e^{-\lambda t}|\varphi|_h$ where $|\varphi|_h = \sup_{-h \leq \theta \leq 0} |\varphi(\theta)|$.

We provide a new exponential stability test for delay system (Σ) in the following theorem.

Theorem 1: For given scalars $\lambda > 0$ and $h > 0$, the time-delay system (Σ) is exponentially stable with a decay rate λ if there exist matrices $P_1 > 0, P_3 > 0, Q > 0, Z_1 > 0, Z_2 > 0, Y, W, S,$ and P_2 such that the LMIs, as shown in (3) and (4) at the bottom of the next page, hold.

$$\tilde{A}(\lambda) = A + \lambda I \tag{5}$$

$$\tilde{A}_1(\lambda) = e^{\lambda h} A_1 \tag{6}$$

$$\Omega(\lambda) = P_1 \tilde{A}(\lambda) + \tilde{A}(\lambda)^T P_1 + P_2 + P_2^T - Y - Y^T + Q + hZ_2 \tag{7}$$

$$\Psi_1(\lambda) = P_1 \tilde{A}_1(\lambda) - P_2 + Y - W^T \tag{8}$$

$$\Psi_2(\lambda) = \tilde{A}(\lambda)^T P_2 + P_3 - S^T \tag{9}$$

$$\Psi_3(\lambda) = \tilde{A}_1(\lambda)^T P_2 - P_3 + S^T. \tag{10}$$

Proof: Let

$$\xi(t) = e^{\lambda t} x(t). \tag{11}$$

Then, it is easy to see that the delay system (Σ) is transformed to

$$\dot{\xi}(t) = \tilde{A}(\lambda)\xi(t) + \tilde{A}_1(\lambda)\xi(t-h) \tag{12}$$

$$\xi(t) = \phi(t) = e^{\lambda t}\varphi(t) \qquad \forall t \in [-h, 0]. \tag{13}$$

Under the condition of the theorem, we first show the asymptotic stability of the delay system (12). To this end, we define a Lyapunov functional candidate for (12) as follows:

$$V(\xi_t) = V_1(\xi_t) + V_2(\xi_t) + V_3(\xi_t) \tag{14}$$

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where $t \geq h$, and

$$\begin{aligned}
\xi_t &= \xi(t+\theta) \quad -2h \leq \theta \leq 0 \\
V_1(\xi_t) &= \xi(t)^T P_1 \xi(t) \\
V_2(\xi_t) &= 2\xi(t)^T P_2 \int_{t-h}^{t} \xi(\alpha) d\alpha \\
&\quad + \left[\int_{t-h}^{t} \xi(\alpha) d\alpha\right]^T P_3 \left[\int_{t-h}^{t} \xi(\alpha) d\alpha\right] \\
&\quad + \int_{t-h}^{t} \xi(\alpha)^T Q \xi(\alpha) d\alpha \\
V_3(\xi_t) &= \int_{-h}^{0}\int_{t+\beta}^{t} \dot{\xi}(s)^T Z_1 \dot{\xi}(s) ds d\beta \\
&\quad + \int_{-h}^{0}\int_{t+\beta}^{t} \xi(s)^T Z_2 \xi(s) ds d\beta.
\end{aligned}$$

Then, we have the time derivative of $V_i(\xi_t), i = 1,2,3$, along the trajectories of (12) as

$$\dot{V}_1(\xi_t) = 2\xi(t)^T P_1 [\tilde{A}(\lambda)\xi(t) + \tilde{A}_1(\lambda)\xi(t-h)] \tag{15}$$

$$\begin{aligned}
\dot{V}_2(\xi_t) &= 2[\tilde{A}(\lambda)\xi(t) + \tilde{A}_1(\lambda)\xi(t-h)]^T P_2 \int_{t-h}^{t} \xi(\alpha) d\alpha \\
&\quad + 2\xi(t)^T P_2 [\xi(t) - \xi(t-h)] \\
&\quad + 2\left[\xi(t) - \xi(t-h)\right]^T P_3 \int_{t-h}^{t} \xi(\alpha) d\alpha \\
&\quad + \xi(t)^T Q \xi(t) - \xi(t-h)^T Q \xi(t-h)
\end{aligned} \tag{16}$$

$$\begin{aligned}
\dot{V}_3(\xi_t) &= h\dot{\xi}(t)^T Z_1 \dot{\xi}(t) + h\xi(t)^T Z_2 \xi(t) \\
&\quad - \int_{t-h}^{t} \dot{\xi}(\beta)^T Z_1 \dot{\xi}(\beta) d\beta \\
&\quad - \int_{t-h}^{t} \xi(\alpha)^T Z_2 \xi(\alpha) d\alpha.
\end{aligned} \tag{17}$$

By using the Newton-Leibniz formula

$$\int_{t-h}^{t} \dot{\xi}(\beta) d\beta = \xi(t) - \xi(t-h)$$

and (15)–(17), we obtain

$$\begin{aligned}
\dot{V}(\xi_t) &= \xi(t)^T \Big[P_1 \tilde{A}(\lambda) + \tilde{A}(\lambda)^T P_1 \\
&\quad + P_2 + P_2^T + Q + hZ_2\Big] \xi(t) \\
&\quad + 2\xi(t)^T [P_1 \tilde{A}_1(\lambda) - P_2] \xi(t-h) \\
&\quad + 2\xi(t)^T [\tilde{A}(\lambda)^T P_2 + P_3] \int_{t-h}^{t} \xi(\alpha) d\alpha \\
&\quad + 2\xi(t-h)^T [\tilde{A}_1(\lambda)^T P_2 - P_3] \int_{t-h}^{t} \xi(\alpha) d\alpha \\
&\quad - \xi(t-h)^T Q \xi(t-h) + h[\tilde{A}(\lambda)\xi(t) \\
&\quad + \tilde{A}_1(\lambda)\xi(t-h)]^T Z_1 [\tilde{A}(\lambda)\xi(t) + \tilde{A}_1(\lambda)\xi(t-h)]^T \\
&\quad - \int_{t-h}^{t} \dot{\xi}(\beta)^T Z_1 \dot{\xi}(\beta) d\beta - \int_{t-h}^{t} \xi(\alpha)^T Z_2 \xi(\alpha) d\alpha \\
&\quad + 2\xi(t)^T Y \int_{t-h}^{t} \dot{\xi}(\beta) d\beta - 2\xi(t)^T Y [\xi(t) - \xi(t-h)] \\
&\quad + 2\xi(t-h)^T W \int_{t-h}^{t} \dot{\xi}(\beta) d\beta \\
&\quad - 2\xi(t-h)^T W [\xi(t) - \xi(t-h)] \\
&\quad + 2\int_{t-h}^{t} \xi(\alpha)^T d\alpha S \int_{t-h}^{t} \dot{\xi}(\beta) d\beta \\
&\quad - 2\int_{t-h}^{t} \xi(\alpha)^T d\alpha S [\xi(t) - \xi(t-h)] \\
&= \frac{1}{h^2} \int_{t-h}^{t}\int_{t-h}^{t} \zeta(t,\alpha,\beta)^T \Gamma(\lambda) \zeta(t,\alpha,\beta) d\alpha d\beta
\end{aligned} \tag{18}$$

where

$$\zeta(t,\alpha,\beta) = [\,\xi(t)^T \quad \xi(t-h)^T \quad \xi(\alpha)^T \quad \dot{\xi}(\beta)^T\,]^T$$

$$\Gamma(\lambda) = \begin{bmatrix} \Omega(\lambda) & \Psi_1(\lambda) & h\Psi_2(\lambda) & hY \\ \Psi_1(\lambda)^T & -Q - W - W^T & h\Psi_3(\lambda) & hW \\ h\Psi_2(\lambda)^T & h\Psi_3(\lambda)^T & -hZ_2 & h^2 S \\ hY^T & hW^T & h^2 S^T & -hZ_1 \end{bmatrix} + h \begin{bmatrix} \tilde{A}(\lambda)^T Z_1 \\ \tilde{A}_1(\lambda)^T Z_1 \\ 0 \\ 0 \end{bmatrix} Z_1^{-1} \begin{bmatrix} \tilde{A}(\lambda)^T Z_1 \\ \tilde{A}_1(\lambda)^T Z_1 \\ 0 \\ 0 \end{bmatrix}^T.$$

Now, by Schur complement, it follows from (3) that $\Gamma(\lambda) < 0$. By this and (18), it is easy to have

$$\dot{V}(\xi_t) \leq -a|\xi(t)|^2 \tag{19}$$

where $a = \lambda_{\min}(-\Gamma(\lambda)) > 0$. Now, let $k_1 = \max(\|\tilde{A}(\lambda)\|, \|\tilde{A}_1(\lambda)\|)$. Then, for any $t \geq 0$, it follows from (12) that

$$\begin{aligned}
|\xi(t)| &= \left|\xi(0) + \int_0^t [\tilde{A}(\lambda)\xi(s) + \tilde{A}_1(\lambda)\xi(s-h)] ds\right| \\
&\leq |\xi(0)| + k_1 \int_0^t [|\xi(s)| + |\xi(s-h)|] ds \\
&\leq |\xi(0)| + 2k_1 \int_{-h}^{t} |\xi(s)| ds.
\end{aligned}$$

$$\begin{bmatrix} \Omega(\lambda) & \Psi_1(\lambda) & h\Psi_2(\lambda) & hY & h\tilde{A}(\lambda)^T Z_1 \\ \Psi_1(\lambda)^T & W + W^T - Q & h\Psi_3(\lambda) & hW & h\tilde{A}_1(\lambda)^T Z_1 \\ h\Psi_2(\lambda)^T & h\Psi_3(\lambda)^T & -hZ_2 & h^2 S & 0 \\ hY^T & hW^T & h^2 S^T & -hZ_1 & 0 \\ hZ_1\tilde{A}(\lambda) & hZ_1\tilde{A}_1(\lambda) & 0 & 0 & -hZ_1 \end{bmatrix} < 0 \tag{3}$$

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} > 0 \tag{4}$$

Then, for any $0 \le t \le h$, we have

$$|\xi(t)| \le |\xi(0)| + 2k_1 \int_{-h}^{0} |\xi(s)|ds + 2k_1 \int_0^t |\xi(s)|ds \le (2k_1 h + 1)|\phi|_h + 2k_1 \int_0^t \sup_{0\le r\le s} |\xi(r)|ds.$$

Therefore, for any $0 \le t \le h$,

$$\sup_{0\le s\le t} |\xi(s)| \le (2k_1 h + 1)|\phi|_h + 2k_1 \int_0^t \sup_{0\le r\le s} |\xi(r)|ds.$$

Applying the Gronwall–Bellman Lemma to this inequality gives that for any $0 \le t \le h$

$$\sup_{0\le s\le t} |\xi(s)| \le (2k_1 h + 1)|\phi|_h \exp(2k_1 h). \tag{20}$$

Thus

$$\sup_{0\le s\le h} |\xi(s)|^2 \le (2k_1 h + 1)^2 |\phi|_h^2 \exp(4k_1 h). \tag{21}$$

Note that

$$\int_{-h}^{0}\int_{t+\beta}^{t} \dot{\xi}(s)^T Z_1 \dot{\xi}(s) ds d\beta \le h \int_{t-h}^{t} \dot{\xi}(s)^T Z_1 \dot{\xi}(s)\, ds \tag{22}$$

$$\int_{-h}^{0}\int_{t+\beta}^{t} \xi(s)^T Z_2 \xi(s) ds d\beta \le h \int_{t-h}^{t} \xi(s)^T Z_2 \xi(s)\, ds. \tag{23}$$

Then, by (14) and (21)–(23), we have

$$\begin{aligned} V(\xi_h) &\le k_2 \left[|\xi(h)|^2 + \left(\int_0^h \xi(s)ds\right)^2 + \int_0^h |\dot{\xi}(s)|^2 ds + 2\int_0^h |\xi(s)|^2 ds \right] \\ &\le k_2 (h^2 + 2h + 1) \sup_{0\le s\le h} |\xi(s)|^2 + k_2 \int_0^h |\dot{\xi}(s)|^2 ds \\ &\le k_2 (h+1)^2 \sup_{0\le s\le h} |\xi(s)|^2 + k_2 k_1^2 \int_0^h [|\xi(s)| + |\xi(s-h)|]^2 ds \\ &\le k_2 \left[(h+1)^2 + 2hk_1^2\right] \sup_{0\le s\le h} |\xi(s)|^2 + 2k_2 k_1^2 h |\phi|_h^2 \\ &\le k_3 |\phi|_h^2 \end{aligned} \tag{24}$$

where

$$\begin{aligned} k_2 &= \max(\lambda_{\max}(P), h\lambda_{\max}(Z_1), h\lambda_{\max}(Z_2), \lambda_{\max}(Q)) \\ k_3 &= k_2 \left[(h+1)^2 + 2hk_1^2\right] \times (2k_1 h + 1)^2 \exp(4k_1 h) + 2k_2 k_1^2 h \\ P &= \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}. \end{aligned}$$

Now, by (4), (14), and (19), it is easy to see that for any $t \ge h$

$$\lambda_{\min}(P)|\xi(t)|^2 \le V(\xi_t) \le V(\xi_h).$$

This together with (24) implies that for any $t \ge h$

$$|\xi(t)|^2 \le \frac{k_3}{\lambda_{\min}(P)} |\phi|_h^2. \tag{25}$$

Noting the relationship in (11) and the inequality in (25), we have that for any $t \ge h$,

$$|x(t)|^2 \le \frac{k_3}{\lambda_{\min}(P)} e^{-2\lambda t} |\varphi|_h^2 \tag{26}$$

Similarly, by (20), we have

$$\sup_{0\le s\le h} |x(s)| = \sup_{0\le s\le h} e^{-\lambda s} |\xi(s)| \le (2k_1 h + 1)|\varphi|_h \exp(2k_1 h). \tag{27}$$

Then, it follows from (26) and (27) that for any $t > 0$

$$|x(t)| \le \max\left((2k_1 h + 1) \times \exp(2k_1 h + \lambda h), \sqrt{\frac{k_3}{\lambda_{\min}(P)}}\right) e^{-\lambda t} |\varphi|_h.$$

Therefore, by Definition 1, we have that the time-delay system (Σ) is exponentially stable with a decay rate λ. This completes the proof. □

Remark 1: Theorem 1 provides a new exponential stability condition for time-delay system (Σ) in terms of LMIs. With this result, an upper bound of the decay rate can be calculated easily.

Remark 2: It is worth pointing out that the method in Theorem 1 can also be used to obtain exponential stability condition for neutral systems. To show this, we consider the following neutral system:

$$\dot{x}(t) + D\dot{x}(t-h) = Ax(t) + A_1 x(t-h). \tag{28}$$

By (11), it is easy to see that the neutral system in (28) is transformed to

$$\dot{\xi}(t) + \hat{D}(\lambda)\dot{\xi}(t-h) = \hat{A}(\lambda)\xi(t) + \hat{A}_1(\lambda)\xi(t-h) \tag{29}$$

where

$$\hat{D}(\lambda) = e^{\lambda h} D \quad \hat{A}(\lambda) = A + \lambda I \quad \hat{A}_1(\lambda) = e^{\lambda h}(A_1 + \lambda D).$$

Choose a Lyapunov functional candidate for (28) as follows:

$$V(\xi_t) = V_1(\xi_t) + V_2(\xi_t) + V_3(\xi_t) + V_4(\xi_t)$$

where $V_i(\xi_t), i = 1, 2, 3$, are given in (14), and

$$V_4(\xi_t) = \int_{t-h}^{t} \dot{\xi}(s)^T Z_3 \dot{\xi}(s) ds d\beta$$

TABLE I
COMPARISON OF THE DECAY RATES IN EXAMPLE 1

h	0.8	1	1.2	1.4	1.6	1.8	2.0
$\bar{\lambda}$ by Mondié and Kharitonov [9]	0.7344	0.6715	0.6145	0.5642	0.5202	0.4818	0.4481
$\bar{\lambda}$ by Liu *et al.* [8]	0.9367	0.5903	0.3400	0.1813	0.0752	0.0014	0
$\bar{\lambda}$ by Theorem 1	0.9366	0.9192	0.8991	0.8115	0.6990	0.6148	0.5494

with $Z_3 > 0$. Then, following the same line as in the derivation of Theorem 1, we can easily obtain a sufficient condition for exponential stability of the neutral system in (28).

By Theorem 1, it is easy to obtain the following delay-dependent asymptotic stability result for time-delay system (Σ).

Corollary 1: The time-delay system (Σ) is asymptotically stable if there exist matrices $P_1 > 0, P_3 > 0, Q > 0, Z_1 > 0, Z_2 > 0, Y, W, S$, and P_2 such that the following LMIs hold:

$$\begin{bmatrix} \hat{\Omega} & \hat{\Psi}_1 & h\hat{\Psi}_2 & hY & hA^TZ_1 \\ \hat{\Psi}_1^T & W+W^T-Q & h\hat{\Psi}_3 & hW & hA_1^TZ_1 \\ h\hat{\Psi}_2^T & h\hat{\Psi}_3^T & -hZ_2 & h^2S & 0 \\ hY^T & hW^T & h^2S^T & -hZ_1 & 0 \\ hZ_1A & hZ_1A_1 & 0 & 0 & -hZ_1 \end{bmatrix} < 0$$

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} > 0$$

where

$$\hat{\Omega} = P_1A + A^TP_1 + P_2 + P_2^T - Y - Y^T + Q + hZ_2 \quad (30)$$

$$\hat{\Psi}_1 = P_1A_1 - P_2 + Y - W^T \quad (31)$$

$$\hat{\Psi}_2 = A^TP_2 + P_3 - S^T \quad (32)$$

$$\hat{\Psi}_3 = A_1^TP_2 - P_3 + S^T. \quad (33)$$

Remark 3: It is worth mentioning that although the Lyapunov–Krasovskii functional in (14) was also used in [14] to investigate the asymptotic stability of time-delay systems, the slack variable S in Theorem 1 has not been introduced in [14] since in the derivation of asymptotic stability in [14] only single integrals were used while we use double integrals (see the proof of Theorem 1); the use of double integrals makes it possible to introduce the slack variable S in our case. It is now well known that it is helpful to reduce conservatism in stability results for delay systems by introducing slack variables. Thus, the introduction of the slack variable S in Corollary 1 may also reduce conservatism in the asymptotic stability condition in [14].

To show the reduced conservatism of the exponential stability condition in Theorem 1, we consider the time-delay system in [8] in the form of (1) with

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} \quad A_1 = \begin{bmatrix} -0.5 & 0.1 \\ 0.3 & 0 \end{bmatrix}.$$

The upper bounds of the decay rate $\bar{\lambda}$ calculated by the methods in [8], [9] and Theorem 1 are compared in Table I. It can be seen that the result in Theorem 1 is less conservative than those in [8] and [9] for this example.

Now, we consider a time-delay system with time-varying norm-bounded parameter uncertainties described by

$$(\hat{\Sigma}): \quad \dot{x}(t) = (A + \Delta A(t))x(t) + (A_1 + \Delta A_1(t))x(t-h) \quad (34)$$

$$x(t) = \phi(t) \qquad \forall t \in [-h, 0] \quad (35)$$

$$\begin{bmatrix} \Omega(\lambda)+\epsilon E^TE & \Psi_1(\lambda)+\epsilon E^TE_1 & h\Psi_2(\lambda) & hY & h\tilde{A}(\lambda)^TZ_1 & P_1D \\ \Psi_1(\lambda)^T+\epsilon E_1^TE & W+W^T-Q+\epsilon E_1^TE_1 & h\Psi_3(\lambda) & hW & h\tilde{A}_1(\lambda)^TZ_1 & 0 \\ h\Psi_2(\lambda)^T & h\Psi_3(\lambda)^T & -hZ_2 & h^2S & 0 & hP_2^TD \\ hY^T & hW^T & h^2S^T & -hZ_1 & 0 & 0 \\ hZ_1\tilde{A}(\lambda) & hZ_1\tilde{A}_1(\lambda) & 0 & 0 & -hZ_1 & hZ_1D \\ D^TP_1 & 0 & hD^TP_2 & 0 & hD^TZ_1 & -\epsilon I \end{bmatrix} < 0$$

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} > 0$$

$$\begin{bmatrix} \hat{\Omega}+\epsilon E^TE & \hat{\Psi}_1+\epsilon E^TE_1 & h\hat{\Psi}_2 & hY & hA^TZ_1 & P_1D \\ \hat{\Psi}_1^T+\epsilon E_1^TE & W+W^T-Q+\epsilon E_1^TE_1 & h\hat{\Psi}_3 & hW & hA_1^TZ_1 & 0 \\ h\hat{\Psi}_2^T & h\hat{\Psi}_3^T & -hZ_2 & h^2S & 0 & hP_2^TD \\ hY^T & hW^T & h^2S^T & -hZ_1 & 0 & 0 \\ hZ_1A & hZ_1A_1 & 0 & 0 & -hZ_1 & hZ_1D \\ D^TP_1 & 0 & hD^TP_2 & 0 & hD^TZ_1 & -\epsilon I \end{bmatrix} < 0$$

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} > 0$$

TABLE II
COMPARISON OF THE DECAY RATES IN EXAMPLE 2

h	0.3	0.5	0.7	0.9	1.1	1.3	1.5
$\bar{\lambda}$ by Mondié and Kharitonov [9]	0.6255	0.4760	0.3825	0.3191	0.2735	0.2392	0.2125
$\bar{\lambda}$ by Theorem 2	1.0108	0.8366	0.7103	0.6156	0.5425	0.4845	0.4375

where

$$[\,\Delta A(t) \quad \Delta A_h(t)\,] = DF(t)[\,E \quad E_1\,] \tag{36}$$

and $F(t) \in \mathbb{R}^{k\times l}$ is an unknown time-varying matrix function bounded by

$$F(t)^T F(t) \leq I \qquad \forall t. \tag{37}$$

By Theorem 1, it is easy to have the following result.

Theorem 2: For given scalars $\lambda > 0$ and $h > 0$, the uncertain time-delay system $(\hat{\Sigma})$ is robustly exponentially stable with a decay rate λ if there exist matrices $P_1 > 0, P_3 > 0, Q > 0, Z_1 > 0, Z_2 > 0, Y, W, S, P_2$ and a scalar $\epsilon > 0$ such that the first set of LMIs shown at the bottom of the previous page hold, where $\Omega(\lambda), \Psi_1(\lambda), \Psi_2(\lambda)$, and $\Psi_3(\lambda)$ are given in (7)–(10), respectively.

By Theorem 2, it is easy to have the following robust asymptotic stability results.

Corollary 2: The uncertain time-delay system $(\hat{\Sigma})$ is robustly asymptotically stable if there exist matrices $P_1 > 0, P_3 > 0, Q > 0, Z_1 > 0, Z_2 > 0, Y, W, S, P_2$ and a scalar $\epsilon > 0$ such that the second set of LMIs shown at the bottom of the previous page hold, where $\hat{\Omega}, \hat{\Psi}_1, \hat{\Psi}_2$, and $\hat{\Psi}_3$ are given in (30)–(33), respectively.

To compare the robust exponential stability result in Theorem 2 with that in [9], we consider an uncertain time-delay system in the form of (34)–(37) with

$$A = \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix} \quad A_1 = \begin{bmatrix} 0.1 & 0 \\ 4 & 0.1 \end{bmatrix}$$

and $D = 0.2I, E = E_1 = I$. Then, it is to see that this system can be rewritten in the form of that in [9, Ex. 2] with $\|\Delta A(t)\| \leq 0.2, \|\Delta A_1(t)\| \leq 0.2$. The comparison of the upper bound of the decay rates $\bar{\lambda}$ obtained by [9] and Theorem 2 is given in Table II, which shows that the condition in Theorem 2 is less conservative than that in [9] for this example.

III. CONCLUSION

This note has provided a new exponential stability condition for time-delay systems in terms of LMIs. Based on this, an upper bound of the decay rate can be calculated easily. When parameter uncertainties appear in a time-delay system, a new robust exponential stability condition has been proposed. Both the exponential stability and the robust exponential stability conditions proposed in this note are less conservative than some of those in the literature, which has been illustrated via two numerical examples.

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