Asadi et al. *Journal of Inequalities and Applications* 2014, 2014:18
http://www.journalofinequalitiesandapplications.com/content/2014/1/18

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

New extension of p-metric spaces with some fixed-point results on M-metric spaces

Mehdi Asadi[1*], Erdal Karapınar[2] and Peyman Salimi[3]

*Correspondence:
masadi.azu@gmail.com
[1]Department of Mathematics, Zanjan Branch, Islamic Azad University, Zanjan, 45156 58145, Iran
Full list of author information is available at the end of the article

Abstract

In this paper, we extend the p-metric space to an M-metric space, and we shall show that the definition we give is a real generalization of the p-metric by presenting some examples. In the sequel we prove some of the main theorems by generalized contractions for getting fixed points and common fixed points for mappings.

Keywords: fixed point; partial metric space

1 Introduction and preliminaries

In 1994, in [1] Matthews introduced the notion of a partial metric space and proved the contraction principle of Banach in this new framework. Next, many fixed-point theorems in partial metric spaces have been given by several mathematicians. Recently Haghi *et al.* published [2] a paper which stated that we should 'be careful on partial metric fixed point results' along with giving some results. They showed that fixed-point generalizations to partial metric spaces can be obtained from the corresponding results in metric spaces.

In this paper, we extend the p-metric space to an M-metric space, and we shall show that our definition is a real generalization of the p-metric by presenting some examples. In the sequel we prove some of the main theorems by generalized contractions for getting fixed points and common fixed points for mappings.

Definition 1.1 ([1], [3, Definition 1.1]) A partial metric on a nonempty set X is a function $p: X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(p1) $p(x,x) = p(y,y) = p(x,y) \iff x = y$,

(p2) $p(x,x) \le p(x,y)$,

(p3) $p(x,y) = p(y,x)$,

(p4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

Notation The following notation is useful in the sequel.

1. $m_{xy} := \min\{m(x,x), m(y,y)\}$,
2. $M_{xy} := \max\{m(x,x), m(y,y)\}$.

Now we want to extend Definition 1.1 as follows.

Definition 1.2 Let X be a nonempty set. A function $m : X \times X \to \mathbb{R}^+$ is called an m-metric if the following conditions are satisfied:

(m1) $m(x,x) = m(y,y) = m(x,y) \iff x = y$,
(m2) $m_{xy} \le m(x,y)$,
(m3) $m(x,y) = m(y,x)$,
(m4) $(m(x,y) - m_{xy}) \le (m(x,z) - m_{xz}) + (m(z,y) - m_{zy})$.

Then the pair (X,m) is called an M-metric space.

According to the above definition the condition (p1) in the definition of [1] changes to (m1), and (p2) is expressed for $p(x,x)$ where $p(y,y) = 0$ may become $p(y,y) \neq 0$. Thus we improve that condition by replacing it by $\min\{p(x,x), p(y,y)\} \le p(x,y)$, and also we improve the condition (p4) extending it to the form of (m4). In the sequel we present an example that holds for the m-metric but not for the p-metric.

Remark 1.1 For every $x, y \in X$

1. $0 \le M_{xy} + m_{xy} = m(x,x) + m(y,y)$,
2. $0 \le M_{xy} - m_{xy} = |m(x,x) - m(y,y)|$,
3. $M_{xy} - m_{xy} \le (M_{xz} - m_{xz}) + (M_{zy} - m_{zy})$.

The next examples show that m^s and m^w are ordinary metrics.

Example 1.1 Let $X := [0,\infty)$. Then $m(x,y) = \frac{x+y}{2}$ on X is an m-metric.

Example 1.2 Let m be an m-metric. Put

1. $m^w(x,y) = m(x,y) - 2m_{xy} + M_{xy}$,
2. $m^s(x,y) = m(x,y) - m_{xy}$ when $x \neq y$ and $m^s(x,y) = 0$ if $x = y$.

Then m^w and m^s are ordinary metrics.

Proof If $m^w(x,y) = 0$, then

$$m(x,y) = 2m_{xy} - M_{xy}. \tag{1}$$

But from equation (1) and $m_{xy} \le m(x,y)$ we get $m_{xy} = M_{xy} = m(x,x) = m(y,y)$, so by equation (1) we obtain $m(x,y) = m(x,x) = m(y,y)$ and therefore $x = y$. For the triangle inequality it is enough that we consider Remark 1.1 and (m4). □

Remark 1.2 For every $x, y \in X$

1. $m(x,y) - M_{xy} \le m^w(x,y) \le m(x,y) + M_{xy}$,
2. $(m(x,y) - M_{xy}) \le m^s(x,y) \le m(x,y)$.

In other words

$$\left|m^w(x,y) - m(x,y)\right| \le M_{xy}, \qquad \left|m^s(x,y) - m(x,y)\right| \le M_{xy}.$$

In the following example we present an example of an m-metric which is not a p-metric.

Example 1.3 Let $X = \{1, 2, 3\}$; define

$$m(1,1) = 1, \qquad m(2,2) = 9, \qquad m(3,3) = 5,$$
$$m(1,2) = m(2,1) = 10, \qquad m(1,3) = m(3,1) = 7, \qquad m(3,2) = m(2,3) = 7.$$

So m is an m-metric, but it is not p-metric.

Example 1.4 Let (X, d) be a metric space. Let $\phi : [0,\infty) \to [\phi(0),\infty)$ be a one to one and nondecreasing or strictly increasing mapping, with $\phi(0)$ defined such that

$$\phi(x + y) \le \phi(x) + \phi(y) - \phi(0), \quad \forall x, y \ge 0.$$

Then $m(x, y) = \phi(d(x, y))$ is an m-metric.

Proof (m1), (m2), and (m3) are clear. For (m4) we have

$$\begin{aligned}
&\phi\big(d(x,y)\big) \le \phi\big(d(x,z) + d(z,y)\big) \\
&\qquad\qquad \le \phi\big(d(x,z)\big) + \phi\big(d(z,y)\big) - \phi(0), \\
&\big(\phi\big(d(x,y)\big) - \phi(0)\big) \le \big(\phi\big(d(x,z)\big) - \phi(0)\big) + \big(\phi\big(d(z,y)\big) - \phi(0)\big), \\
&\big(m(x,y) - m_{xy}\big) \le \big(m(x,z) - m_{xz}\big) + \big(m(z,y) - m_{zy}\big).
\end{aligned}$$ □

Example 1.5 Let (X, d) be a metric space. Then $m(x, y) = ad(x, y) + b$ where $a, b > 0$ is an m-metric, because we can put $\phi(t) = at + b$.

Remark 1.3 According to Example 1.5, by the Banach contraction

$$\exists k \in [0,1), \quad m(Tx, Ty) \le km(x, y), \quad \text{for all } x, y \in X,$$

we have

$$m(Tx, Ty) = ad(Tx, Ty) + b \le kad(x, y) + kb \quad \Rightarrow \quad d(Tx, Ty) \le kd(x, y) + \frac{b(k-1)}{a},$$

which does not imply the ordinary Banach contraction

$$\exists k \in [0,1), \quad d(Tx, Ty) \le kd(x, y), \quad \text{for all } x, y \in X,$$

for all self-maps T on X. Thus, this states that even if the m-metric m and the ordinary metric d have the same topology, the Banach contraction of the m-metric does not imply the Banach contraction of the ordinary metric d.

Lemma 1.1 *Every p-metric is an m-metric.*

Proof Let m be a p-metric. It is enough that we consider the following cases:

1. $m(x, x) = m(y, y) = m(z, z)$,
2. $m(x, x) < m(y, y) < m(z, z)$,

3. $m(x,x) = m(y,y) < m(z,z)$,
4. $m(x,x) = m(y,y) > m(z,z)$,
5. $m(x,x) < m(y,y) = m(z,z)$,
6. $m(x,x) > m(y,y) = m(z,z)$.

For example, to prove (2), we have

$$m(x,y) \leq m(x,z) + m(z,y) - m(z,z),$$

$$m(x,y) \leq m(x,z) + m(z,y) - m(y,y),$$

$$m(x,y) - m(x,x) \leq m(x,z) - m(x,x) + m(z,y) - m(y,y),$$

$$m(x,y) - m_{x,y} \leq m(x,z) - m_{x,z} + m(z,y) - m_{z,y}.$$

□

2 Topology for *M*-metric space

It is clear that each m-metric p on X generates a T_0 topology τ_m on X. The set

$$\{B_m(x,\varepsilon) : x \in X, \varepsilon > 0\},$$

where

$$B_m(x,\varepsilon) = \{y \in X : m(x,y) < m_{x,y} + \varepsilon\},$$

for all $x \in X$ and $\varepsilon > 0$, forms a base of τ_m.

Definition 2.1 Let (X,m) be a m-metric space. Then:

1. A sequence $\{x_n\}$ in a M-metric space (X,m) converges to a point $x \in X$ if and only if

$$\lim_{n\to\infty} \bigl(m(x_n,x) - m_{x_n,x}\bigr) = 0. \tag{2}$$

2. A sequence $\{x_n\}$ in a M-metric space (X,m) is called an m-Cauchy sequence if

$$\lim_{n,m\to\infty} \bigl(m(x_n,x_m) - m_{x_n,x_m}\bigr), \qquad \lim_{n,m\to\infty} (M_{x_n,x_m} - m_{x_n,x_m}) \tag{3}$$

exist (and are finite).

3. An M-metric space (X,m) is said to be complete if every m-Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_m, to a point $x \in X$ such that

$$\Bigl(\lim_{n\to\infty} \bigl(m(x_n,x) - m_{x_n,x}\bigr) = 0 \ \& \ \lim_{n\to\infty} (M_{x_n,x} - m_{x_n,x}) = 0\Bigr).$$

Lemma 2.1 *Let (X,m) be a m-metric space. Then:*

1. *$\{x_n\}$ is an m-Cauchy sequence in (X,m) if and only if it is a Cauchy sequence in the metric space (X,m^w).*
2. *An M-metric space (X,m) is complete if and only if the metric space (X,m^w) is complete. Furthermore,*

$$\lim_{n\to\infty} m^w(x_n,x) = 0 \quad \Longleftrightarrow \quad \Bigl(\lim_{n\to\infty} \bigl(m(x_n,x) - m_{x_n,x}\bigr) = 0, \ \lim_{n\to\infty} (M_{x_n,x} - m_{x_n,x}) = 0\Bigr).$$

Likewise the above definition holds also for m^s.

Lemma 2.2 *Assume that $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in an M-metric space (X, m). Then*

$$\lim_{n\to\infty} \bigl(m(x_n, y_n) - m_{x_n,y_n}\bigr) = m(x, y) - m_{xy}.$$

Proof We have

$$\bigl|\bigl(m(x_n, y_n) - m_{x_n,y_n}\bigr) - \bigl(m(x, y) - m_{x,y}\bigr)\bigr| \le \bigl(m(x_n, x) - m_{x_n,x}\bigr) + \bigl(m(y, y_n) - m_{y,y_n}\bigr). \qquad \square$$

From Lemma 2.2 we deduce the following lemma.

Lemma 2.3 *Assume that $x_n \to x$ as $n \to \infty$ in an M-metric space (X, m). Then*

$$\lim_{n\to\infty} \bigl(m(x_n, y) - m_{x_n,y}\bigr) = m(x, y) - m_{x,y},$$

for all $y \in X$.

Lemma 2.4 *Assume that $x_n \to x$ and $x_n \to y$ as $n \to \infty$ in an M-metric space (X, m). Then $m(x, y) = m_{xy}$. Furthermore, if $m(x, x) = m(y, y)$, then $x = y$.*

Proof By Lemma 2.2 we have

$$0 = \lim_{n\to\infty} \bigl(m(x_n, x_n) - m_{x_n,x_n}\bigr) = m(x, y) - m_{xy}. \qquad \square$$

Lemma 2.5 *Let $\{x_n\}$ be a sequence in an m-metric space (X, m), such that*

$$\exists r \in [0, 1), \quad m(x_{n+1}, x_n) \le rm(x_n, x_{n-1}), \quad \forall n \in \mathbb{N}. \tag{4}$$

Then

(A) $\lim_{n\to\infty} m(x_n, x_{n-1}) = 0$,
(B) $\lim_{n\to\infty} m(x_n, x_n) = 0$,
(C) $\lim_{m,n\to\infty} m_{x_m x_n} = 0$,
(D) *$\{x_n\}$ is an m-Cauchy sequence.*

Proof From equation (4) we have

$$m(x_n, x_{n-1}) \le rm(x_{n-1}, x_{n-2}) \le r^2 m(x_{n-2}, x_{n-3}) \le \cdots \le r^n m(x_0, x_1),$$

thus,

$$\lim_{n\to\infty} m(x_n, x_{n-1}) = 0,$$

which implies that (A) holds.
From (m2) and (A) we have

$$\lim_{n\to\infty} \min\bigl\{m(x_n, x_n), m(x_{n-1}, x_{n-1})\bigr\} = \lim_{n\to\infty} m_{x_n x_{n-1}} \le \lim_{n\to\infty} m(x_n, x_{n-1}) = 0.$$

That is, (B) holds.

Clearly, (C) holds, since $\lim_{n\to\infty} m(x_n, x_n) = 0$. □

Theorem 2.1 *The topology τ_m is not Hausdorff.*

Proof Let $x, y, z \in X$ be such that

$$a := m(x,x) < m(z,z) = \frac{a+b}{2} < b := m(y,y)$$

with

$$\frac{b}{2} < \frac{k}{2} < m(x,y) < M_{x,y} = b, \qquad r := 2m(x,y) - a - b > 0$$

and

$$\max\{m(x,z), m(z,y)\} \leq \bigl(2m(x,y) - k\bigr)\frac{\varepsilon}{r};$$

without loss of generality we assume that for each $\varepsilon > 0$ we have $\varepsilon < r$. We want to show that the intersection of the following neighborhoods is not empty:

$$U_x = \{z \in X : m(x,z) - m_{xz} < \varepsilon\}, \qquad V_y = \{z \in X : m(y,z) - m_{yz} < \varepsilon\}.$$

To prove $z \in U_x$, we have

$$\begin{aligned} m(x,z) &< \bigl(2m(x,y) - k\bigr)\frac{\varepsilon}{r}, \\ m(x,z) - m_{xz} &< \bigl(2m(x,y) - k\bigr)\frac{\varepsilon}{r} - a \\ &< \bigl(2m(x,y) - k - a\bigr)\frac{\varepsilon}{r} \\ &< \bigl(2m(x,y) - a - b\bigr)\frac{\varepsilon}{r} = \varepsilon \end{aligned}$$

and for $z \in V_y$

$$\begin{aligned} m(y,z) &< \bigl(2m(x,y) - k\bigr)\frac{\varepsilon}{r}, \\ m(x,z) - m_{yz} &< \bigl(2m(x,y) - k\bigr)\frac{\varepsilon}{r} - \frac{a+b}{2} \\ &< \bigl(2m(x,y) - k\bigr)\frac{\varepsilon}{r} - \frac{a+b}{2}\frac{\varepsilon}{r} \\ &< \left(2m(x,y) - k - \frac{a+b}{2}\right)\frac{\varepsilon}{r} \\ &< \bigl(2m(x,y) - a - b\bigr)\frac{\varepsilon}{r} = \varepsilon, \end{aligned}$$

so we can find $x, y \in X$ such that for all nonempty neighborhoods U_x of x and V_y of y we have $U_x \cap V_y \neq \emptyset$. □

3 Fixed point results on M-metric space

Theorem 3.1 *Let (X,m) be a complete M-metric space and let $T:X\to X$ be a mapping satisfying the following condition*:

$$\exists k\in[0,1)\ \textit{such that}\ m(Tx,Ty)\le km(x,y)\quad \textit{for all}\ x,y\in X. \tag{5}$$

Then T has a unique fixed point.

Proof Let $x_0\in X$ and $x_n:=Tx_{n-1}$, so we have

$$m(x_n,x_{n-1})=m(Tx_{n-1},Tx_{n-2})\le km(x_{n-1},x_{n-2}) \tag{6}$$

and so (A), (B), (C), and (D) of Lemma 2.5 hold. By completeness of X we get $x_n\to x$ for some $x\in X$. Thus by equation (5) $m(Tx_n,Tx)\le km(x_n,x)\to 0$. Hence by (m2) $m_{Tx_n,Tx}\le m(Tx_n,Tx)\to 0$ so by equation (2) $Tx_n\to Tx$.

Contraction (5) implies that $m(x_n,Tx_n)\to 0$ and $m(Tx,Tx)<m(x,x)$. Since $m_{x_n,Tx_n}\to 0$, by Lemma 2.2, we get $m(x,Tx)=m_{x,Tx}=m(Tx,Tx)$.

On the other hand, by Lemma 2.2 and $x_n=Tx_{n-1}\to x$,

$$0=\lim_{n\to\infty}\bigl(m(x_n,Tx_n)-m_{x_n,Tx_n}\bigr)=\lim_{n\to\infty}\bigl(m(x_n,x_{n-1})-m_{x_n,Tx_n}\bigr)=m(x,x)-m_{x,Tx},$$

thus $m(x,x)=m(x,Tx)$. Since $m(x,Tx)=m_{x,Tx}=m(Tx,Tx)$ now by (m1) $x=Tx$. Uniqueness by the contraction (5) is clear. □

Theorem 3.2 *Let (X,m) be a complete M-metric space and let $T:X\to X$ be a mapping satisfying the following condition*:

$$\exists k\in\left[0,\frac{1}{2}\right)\ \textit{such that}\ m(Tx,Ty)\le k\bigl(m(x,Tx)+m(y,Ty)\bigr)\quad \textit{for all}\ x,y\in X. \tag{7}$$

Then T has an unique fixed point.

Proof Let $x_0\in X$ and $x_n:=Tx_{n-1}$, so we have

$$\begin{aligned} m(x_n,x_{n-1}) &= m(Tx_{n-1},Tx_{n-2}) \\ &\le k\bigl(m(x_{n-1},x_n)+m(x_{n-2},x_{n-1})\bigr). \end{aligned}$$

So

$$m(x_n,x_{n-1})\le rm(x_{n-2},x_{n-1}),$$

where $0\le r=\frac{k}{1-k}<1$.

By Lemma 2.5 and completeness of X, $x_n\to x$ for some $x\in X$. So

$$m(x_n,x)-m_{x_n,x}\to 0,\qquad M_{x_n,x}-m_{x_n,x}\to 0,$$

and since $m_{x_n,x} \to 0$, we have $m(x_n, x) \to 0$ and $M_{x_n,x} \to 0$. Therefore by Remark 1.1, $m(x,x) = 0 = m_{x,Tx}$;

$$m(x_{n+1}, Tx) = m(Tx_n, Tx) \leq k\big(m(x_n, x_{n+1}) + m(x, Tx)\big),$$

hence by $m(x_n, x_{n+1}) \to 0$

$$\limsup_{n\to\infty} m(x_{n+1}, Tx) = \limsup_{n\to\infty} m(Tx_n, Tx) \leq km(x, Tx).$$

On the other hand

$$m(x, Tx) - m_{x,Tx} \leq m(x, x_n) + m(x_n, Tx)$$

implies that

$$m(x, Tx) \leq \limsup_{n\to\infty}\big(m(x, x_n) + m(x_n, Tx)\big) \leq km(x, Tx),$$

because $m_{x,Tx} = 0$ and $m(x_n, x) \to 0$. So $m(x, Tx) = 0$. Now by contraction (7) we have $m(Tx, Tx) \leq 2km(x, Tx) = 0$, so $m(Tx, Tx) = 0 = m(x, x) = m(x, Tx)$, thus $x = Tx$ by (m1). □

The next theorem is still open.

Theorem 3.3 *Let (X, m) be a complete M-metric space and let $T : X \to X$ be a mapping satisfying the following condition*:

$$\exists k \in \left[0, \frac{1}{2}\right) \textit{ such that } m(Tx, Ty) \leq k\big(m(x, Ty) + m(Tx, y)\big) \quad \textit{for all } x, y \in X. \tag{8}$$

Then T has a unique fixed point.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have read and approved the final manuscript.

Author details
[1]Department of Mathematics, Zanjan Branch, Islamic Azad University, Zanjan, 45156 58145, Iran. [2]Department of Mathematics, Atilim University, İncek, Ankara 06836, Turkey. [3]Young Researchers and Elite Club, Rasht Branch, Islamic Azad University, Rasht, Iran.

Acknowledgements
The authors express their deep gratitude to the referee for his/her valuable comments and suggestions. This paper has been supported by the I.A.U., Zanjan Branch, Zanjan, Iran. The first author would like to thank for this support. The authors would like to thank Professors William A (Art) Kirk and Billy E. Rhoades for helpful advise which led them to present this paper.

Received: 1 October 2013 Accepted: 17 December 2013 Published: 14 Jan 2014

References

1. Matthews, S: Partial metric topology. Ann. N.Y. Acad. Sci. **728**, 183-197 (1994)
2. Haghi, RH, Rezapour, Sh, Shahzad, N: Be careful on partial metric fixed point results. Topol. Appl. **160**(3), 450-454 (2013)
3. Shatanawi, W, Postolache, M: Coincidence and fixed point results for generalized weak contractions in the sense of Berinde on partial metric spaces. Fixed Point Theory Appl. **2013**, Article ID 54 (2013)

10.1186/1029-242X-2014-18
Cite this article as: Asadi et al.: New extension of *p*-metric spaces with some fixed-point results on *M*-metric spaces. *Journal of Inequalities and Applications* 2014, 2014:18