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New families of Copulas based on periodic functions

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Abstract

Although there exists a large variety of copula functions, only a few are practically manageable, and often the choice in dependence modeling falls on the Gaussian copula. Further, most copulas are exchangeable, thus implying symmetric dependence. We introduce a way to construct copulas based on periodic functions. We study the two-dimensional case based on one dependence parameter and then provide a way to extend the construction to the n-dimensional framework. We can thus construct families of copulas in dimension n and parameterized by $n-1$ parameters, implying possibly asymmetric relations. Such "periodic" copulas can be simulated easily.

Key words: Dependence Modeling, Copula Functions, Gaussian Copula, Archimedean Copula, Periodic Copula, Simulation

1 Introduction and Motivation

Consider a random vector $X = (X_1, ..., X_n)$, and suppose that we wish to analyze the dependence between its components. The whole information on the distribution of the vector is given by the joint cumulative distribution function of X. If $\mathbb{P}$ denotes the probability measure in our setting, such function in the point $(x_1, \ldots, x_n)$ is given by $\mathbb{P}(X_1 \leq x_1, ..., X_n \leq x_n)$. However, this function mixes information on the dependence between the different components of the vector with information on the distribution of the single components themselves. Copula functions have been introduced in order to allow a separation between the marginal cumulative distribution functions (cdf for short) and the dependence structure. The former concerns single components, taken one at the time, and is given by the cdf's $F_i(x) := \mathbb{P}(X_i \leq x)$, $i = 1, \ldots, n$, which we assume to be continuous. The latter is entirely represented by the copula function we introduce now. It is well known that $U_1 = F_1(X_1), ..., U_n = F_n(X_n)$ are uniformly distributed random variables on $[0, 1]$. The joint cumulative distribution function of $(U_1, ..., U_n)$, that we denote by

$$C(u_1, ..., u_n) = \mathbb{P}(U_1 \leq u_1, ..., U_n \leq u_n),$$

is called the copula function of $(X_1, ..., X_n)$ and has the following link with the multivariate cdf:

$$\mathbb{P}(X_1 \leq x_1, ..., X_n \leq x_n) = C(\mathbb{P}(X_1 \leq x_1), ..., \mathbb{P}(X_n \leq x_n)). \tag{1}$$

One can easily check that a copula has the following properties:

1. $C(u_1, .., u_{i-1}, 0, u_{i+1}, .., u_n) = 0$
2. $C(1, .., 1, u_k, 1, .., 1) = u_k$
3. $\partial_{u_1 \ldots u_n} C$ is a positive measure in the sense of Schwartz distributions. This means concretely that for any hypercube $H = [a_1, b_1] \times ... \times [a_n, b_n] \subset [0, 1]^n$,

   $$\mathbb{P}[(U_1, .., U_n) \in H] \geq 0.$$

   When $n = 2$, this can be written as
   $C(b_1, b_2) - C(a_1, b_2) - C(b_1, a_2) + C(a_1, a_2) \geq 0.$

Conversely, one can show that any function that satisfies these three conditions can be viewed as the joint cdf of a vector of uniform variables on $[0, 1]$ and is thus a copula. This is known as Sklar's theorem, see for example Joe (1997) or Nelsen (1999).

In the following, the expression "simulating a copula C" will denote the simulation of a random vector of uniform variables $(U_1, .., U_n)$ on $[0, 1]$ whose joint cdf is C.

Among the different ways to define specific copula functions, there are following two. The first one consists in seeking functions C satisfying the three above properties. Archimedean copulas are an example of this approach. Indeed, Archimedean copulas come from the remark that if φ is a convex decreasing function such that $\varphi(1) = 0$, then

$$C(u_1, .., u_n) = \mathbf{1}_{\{\varphi(u_1)+..+\varphi(u_n)\leq\varphi(0)\}}\varphi^{-1}(\varphi(u_1) + .. + \varphi(u_n))$$

has the above three properties and is thus a copula. Therefore, by specifying families of decreasing convex functions that vanish in 1 we specify families of copulas (e.g. Gumbel, Joe, Frank...), see Bouyé et al. (2000), Nelsen (1999) and Joe (1997).

The second method consists in working directly with joint cdf's $F(x_1, ..., x_n)$ and the related marginal cdf's F_i. The associated copula is then defined as $F(F_1^{-1}(u_1), ..., F_n^{-1}(u_n))$. Even if this method does not always lead to analytically tractable copulas, it can provide us copulas that are easy to simulate. Indeed, the main example of this kind of construction is the well known fundamental family of Gaussian copulas. A Gaussian copula is defined as the copula of a joint Gaussian random vector X with standard Gaussian marginals and correlation matrix ρ, and is thus given by $N_\rho(N^{-1}(u_1), ..., N^{-1}(u_n))$ where N is the cdf of a standard normal variable and N_ρ is the joint cdf of X. This copula cannot be computed explicitly. The simulation is however straightforward: it is sufficient to consider $(N(X_1), ..., N(X_n))$ where $X = (X_1, ..., X_n)$ is a Gaussian vector with correlation ρ that can be easily simulated by resorting to a standard Gaussian simulator and to a Cholesky decomposition of ρ. A similar approach leads to Student's copula (see Bouyé et al. (2000). and Genz and Bretz (2002)

A possible major drawback of Archimedean and Gaussian copulas is that they are n-exchangeable (n-symmetric): if σ is a permutation on $\{1, ..., n\}$, we have $C(u_1, .., u_n) = C(u_{\sigma(1)}, .., u_{\sigma(n)})$. Let us extend this notion by the following definition.

Definition 1.1. (k-exchangeability). *Let us consider a copula C that is the cdf of the random vector $(U_1, .., U_n)$. We will say that the copula C is k-exchangeable $(2 \leq k \leq n)$ if, for any $1 \leq i_1 < i_2 < .. < i_k \leq n$ and any permutation σ on $\{1, ..., k\}$, $(U_{i_1}, .., U_{i_k})$ and $(U_{i_{\sigma(1)}}, .., U_{i_{\sigma(k)}})$ have the same law.*

It is clear, with this definition, that a k'-exchangeable copula is also a k-exchangeable copula whenever $k' > k$. In the two-dimensional case we resort directly to the term "exchangeable" rather than "2-exchangeable". When dealing with defaultable bonds as in Jouanin and al. (2001), n-exchangeable copulas cannot model situations where the dependence is asymmetric and based on the assets themselves. With 2-echangeable copulas (such as for example Gaussian or Archimedean copulas), we cannot model asymmetric relations featuring a first entity that influences a second one more than the latter influences the former.

In recent years, copula functions have received a great deal of attention, see for example the papers of Genz and Bretz (2002), Hürlimann (2002, 2003), Juri and Wüthrich (2002), Nelsen et al. (2001), Wei and Hu (2002), and the books of Joe (1997) and Nelsen (1999). For financial and insurance applications, recent applications on copulas include for example Bouyé et al. (2000), Cherubini et al. (2002), Embrechts et al. (2001), Jouanin et al. (2001), Klugman and Parsa (1999), Prampolini (2003), and Schönbucher and Schubert (2001).

In this paper, we will build new families of copulas based on the first approach, using periodic functions following Alfonsi (2002). We first begin to work in the two-dimensional case, obtaining a one-parameter copula, and then give a way to extend the result to the n-dimensional case with $n > 2$, getting a family with $n - 1$ parameters. Finally, we explain how such copulas can be simulated.

2 Construction of copulas based on periodic functions

2.1 The construction of families in two dimensions

We begin by defining our new copula functions for bivariate dependence, i.e. for possible dependence structures between two random variables. It is helpful to first recall three particular "limit" copulas. The "middle" one, which is typically denoted by $C^{\perp}$, is the copula obtained when U_1 and U_2 are independent uniform variables on $[0,1]$, that is:

$$C^{\perp}(u_1,u_2) := u_1 u_2.$$

The two other "limit" copulas, denoted by C_F^- and C_F^+ respectively, are the two Frechet bounds of the convex subset of copulas:

$$C_F^+(u_1,u_2) = \min(u_1,u_2), \quad C_F^-(u_1,u_2) = (u_1+u_2-1)^+,$$

where $x^+ = \max(x,0)$ denotes the positive part operator. Naming U a uniform random variable on $[0,1]$, C_F^+ can be obtained as the copula of (U,U) and corresponds obviously to perfect positive dependence, whereas C_F^- is obtained as the copula of $(U, 1-U)$ and describes total negative dependence. Moreover, for any copula C, we have

$$C_F^-(u_1,u_2) \le C(u_1,u_2) \le C_F^+(u_1,u_2), \ \forall (u_1,u_2) \in [0,1]^2.$$

Recall the above characterization of a copula function for the bivariate case: the function C defined on $[0,1]^2$ is a copula if and only if i) $C(u_1,0)=0$ and $C(0,u_2)=u_2$, ii) $C(u_1,1)=u_1$ and $C(1,u_2)=u_2$, and iii) $\frac{\partial^2 C}{\partial u_1 \partial u_2}$ is a positive measure in the sense of Schwartz distributions.

We will say in the following that a copula admits a *density* when $\frac{\partial^2 C}{\partial u_1 \partial u_2} = c(u_1,u_2)$ exists in the ordinary sense. In this paper we propose copulas that have a density that can be written in the form

$$c(u_1,u_2) = \tilde{c}(u_1+u_2) \quad (\text{resp. } c(u_1,u_2) = \tilde{c}(u_1-u_2))$$

for a function $\tilde{c} : \mathbb{R} \to \mathbb{R}$. To satisfy properties i), ii) and iii), $\tilde{c}$ must be nonnegative and verify:

$$\int_0^{u_1}\int_0^1 \tilde{c}(x_1 \pm x_2) dx_1 dx_2 = u_1, \ \forall u_1 \in [0,1],$$
$$\int_0^1\int_0^{u_2} \tilde{c}(x_1 \pm x_2) dx_1 dx_2 = u_2, \ \forall u_2 \in [0,1].$$

Differentiating with respect to u_1 and u_2 respectively, we obtain

$$\int_0^1 \tilde{c}(u_1 \pm x_2) dx_2 = 1, \quad \forall u_1 \in [0,1],$$
$$\int_0^1 \tilde{c}(x_1 \pm u_2) dx_1 = 1, \quad \forall u_2 \in [0,1].$$

Differentiating further the first relation, since $\int_0^1 \tilde{c}(u_1+x_2)dx_2 = \int_{u_1}^{u_1+1} \tilde{c}(x_2)dx_2$ (resp. $\int_0^1 \tilde{c}(u_1-x_2)dx_2 = \int_{u_1-1}^{u_1} \tilde{c}(x_2)dx_2$) , we obtain:

$$\tilde{c}(u_1+1) = \tilde{c}(u_1)\ \forall u_1 \in [0,1], \quad (\text{resp. } \tilde{c}(u_1-1) = \tilde{c}(u_1)\ \forall u_1 \in [0,1]).$$

Thus, a consequence of requiring $c(u_1,u_2) := \tilde{c}(u_1 \pm u_2)$ to be the density of a copula is that $\tilde{c}$ has to be 1-periodic (at least on $[-1,2]$, but its value outside this interval is irrelevant) and that $\int_0^1 \tilde{c}(u)du = 1$. Conversely, it is easy to see that if $\tilde{c}$ is a nonnegative 1-periodic function such that $\int_0^1 \tilde{c}(u)du = 1$, then

$$\widetilde{C}^-(u_1,u_2) := \int_0^{u_1}\int_0^{u_2} \tilde{c}(x_1+x_2)dx_1dx_2 \quad (\text{resp. } \widetilde{C}^+(u_1,u_2) := \int_0^{u_1}\int_0^{u_2} \tilde{c}(x_1-x_2)dx_1dx_2)$$

satisfies conditions i), ii) and iii), and so is a copula function that we call, with a slight abuse of language, *periodic copula*. We note here that copulas obtained with these densities form a convex set since a convex combination of 1-periodic nonnegative functions satisfying $\int_0^1 \tilde{c}(u)du = 1$ is also a 1-periodic nonnegative function with integral 1 on a period. Notice further that the use of the "-" and "+" signs appears to be counterintuitive (one would exchange the above signs), but there is a reason for this that will be clarified later on.

At times, rather than characterizing copulas through their densities, it is preferable to have a direct characterization of the copula itself. To characterize periodic copulas without explicitly referring to their densities, denote by φ the primitive of $\tilde{c}$ that vanishes at 0, and set $\Phi(x) := \int_0^x \varphi(u)du$, so that Φ is a double primitive of $\tilde{c}$. We can then rewrite the above periodic copula as follows:

$$\begin{aligned}
\widetilde{C}^-(u_1,u_2) &= \int_0^{u_1}\int_0^{u_2} \tilde{c}(x_1+x_2)dx_1dx_2 &= \Phi(u_1+u_2) - \Phi(u_1) - \Phi(u_2), \\
\widetilde{C}^+(u_1,u_2) &= \int_0^{u_1}\int_0^{u_2} \tilde{c}(x_1-x_2)dx_1dx_2 &= \Phi(u_1) + \Phi(-u_2) - \Phi(u_1-u_2)
\end{aligned} \tag{2}$$

and we see that the first copula is always exchangeable (symmetric), in that $\widetilde{C}^-(u_1,u_2) = \widetilde{C}^-(u_2,u_1)$, whereas the second one can be non symmetric if Φ is not par, i.e. if $\Phi(-x) \neq \Phi(x)$ for some x. We have thus characterized our periodic copulas in terms of double primitives Φ of periodic functions $\tilde{c}$.

A first example of such a function which arises naturally is $\tilde{c}(x) = 1+\sin(2\pi x+\varphi)$ where φ is a parameter that we can take in $[0,\ 2\pi)$. It gives respectively the following families of copulas:

- $\widetilde{C}^{-}(u_1,u_2) = u_1u_2 + (\sin(2\pi u_1+\varphi) - \sin(\varphi) - \sin(2\pi(u_1+u_2)+\varphi) + \sin(2\pi u_2 + \varphi))/(2\pi)^2$
- $\widetilde{C}^{+}(u_1,u_2) = u_1u_2 + (\sin(\varphi) - \sin(2\pi u_1+\varphi) + \sin(2\pi(u_1-u_2)+\varphi) - \sin(-2\pi u_2 + \varphi))/(2\pi)^2$.

These copulas, however, cannot model strong positive or negative dependence, since these copulas cannot approach neither C_F^- nor C_F^+. *On the contrary, it might be interesting to have a family of copulas which attains the copulas C_F^+, $C^\perp$ and C_F^- as limit cases in order to be able to describe a large range of dependence structures.* By expressing copulas by means of $\frac{\partial^2 C}{\partial u_1 \partial u_2}$, attaining C_F^+, C_F^- and $C^\perp$ amounts to attaining

$$\mu^+ = \delta_{x_1}(dx_2) \otimes dx_1, \quad \mu^- = \delta_{1-x_1}(dx_2) \otimes dx_1, \quad c^\perp = 1.$$

with the copula density. Since $\frac{\partial^2 C^\perp}{\partial u_1 \partial u_2}$ exists in the ordinary sense of differentiation, it has a density $c^\perp = 1$ that corresponds to the Lebesgue measure on the square $[0,1]^2$, $\mu^\perp = dx_1 \otimes dx_2$. Instead, μ^+ and μ^- are to be interpreted in the generalized sense. Moreover, μ^+ and μ^- charge only the diagonals of the square $[0,1]^2$, i.e. $\Delta^+ = \{(x,x), x \in [0,1]\}$ and $\Delta^- = \{(x,1-x), x \in [0,1]\}$ respectively. The idea is then to find a family of periodic functions $\tilde{c}_\gamma$ indexed by a parameter γ and such that the density $\tilde{c}_\gamma(x_1 - x_2)$ (resp. $\tilde{c}_\gamma(x_1+x_2)$) concentrates on Δ^+ (resp. Δ^-) for some values of γ. Thus, if we define the piecewise 1-periodic function $\tilde{c}_\gamma$ for $0 < \gamma \leq \frac{1}{2}$ by

$$\tilde{c}_\gamma(x) := \frac{1}{2\gamma}\left(\mathbf{1}_{[0,\ \gamma]}(x) + \mathbf{1}_{(1-\gamma,\ 1)}(x)\right) \quad \text{for} \quad x \in [0,1), \tag{3}$$

we see that the family of densities defined as $c_\gamma^+(x_1,x_2) := \tilde{c}_\gamma(x_1 - x_2)$ verifies:

$$c_{1/2}^+ = c^\perp, \quad c_\gamma^+(x_1,x_2)dx_1dx_2 \xrightarrow[\gamma \to 0]{\mathcal{D}} \mu^+$$

$\xrightarrow{\mathcal{D}}$ denoting convergence in distribution. The corresponding convergence in law for random variables is denoted by $\mathcal{L}$. To calculate the associated copula $C_\gamma^+ := \widetilde{C}^+$, it is best to try a drawing and see that its value in (u_1,u_2) is the area in the intersection of the rectangle delimited by $(0,0)$ and (u_1,u_2) with $\{(x_1,x_2) \in [0,1]^2, -\gamma \leq x_1 - x_2 \leq \gamma$ or $x_1 - x_2 \leq \gamma - 1$ or $x_1 - x_2 \geq 1 - \gamma\}$. We obtain, for $u_1 \leq u_2$ which is not restrictive since $C_\gamma^+(u_1,u_2) = C_\gamma^+(u_2,u_1)$,

$$\begin{aligned} C_\gamma^+(u_1,u_2) = \ & \tfrac{1}{2\gamma}[u_1u_2 + \tfrac{1}{2}[-((u_2-u_1-\gamma)^+ + (u_2-\gamma)^+)(\min(u_1,u_2-\gamma))^+ - ((u_1-\gamma)^+)^2 \\ & +((u_2-1+\gamma)^+ + (u_2-1+\gamma-u_1)^+)\cdot(\min(u_1,u_2-1+\gamma))^+ + ((u_1-1+\gamma)^+)^2]] \end{aligned} \tag{4}$$

In order to obtain a family that reaches C_F^- instead, we use the other family, precisely $c_\gamma^-(x_1, x_2) = \tilde{c}_\gamma(x_1 + x_2)$. We obtain

$$c_{1/2}^- = c^\perp, \quad c_\gamma^-(x_1, x_2) dx_1 dx_2 \xrightarrow[\gamma \to 0]{\mathcal{D}} \mu^-.$$

Fortunately, a simple geometric remark links the related copula C_γ^- to C_γ^+, thus avoiding a new calculation:

$$C_\gamma^-(u_1, u_2) = u_2 - C_\gamma^+(1 - u_1, u_2) \tag{5}$$

Thus, with this method, we have obtained a family of copula quite "exhaustive" going from C_F^- to C_F^+ and taking the in-between value $C^\perp$. Incidentally, we see now why we chose to name $\widetilde{C}^+$ the copula coming from $\tilde{c}(x_1 - x_2)$ and $\widetilde{C}^-$ the copula coming from $\tilde{c}(x_1 + x_2)$: this is done because in our case the former attains C_F^+ and the latter C_F^-.

At times it can be handy to have a single number measuring some stylized aspects of a given copula. The Spearman's rho is such a number and is a well known measure of concordance, see for example Embrechts et al. (2001). When defined in terms of copula functions, it is given by the following integral in the copula density c: $\rho := 12 \int_0^1 \int_0^1 u_1 u_2 c(u_1, u_2) du_1 du_2 - 3$. We obtain for the C_γ^+ and C_γ^- copulas respectively,

$$\rho_\gamma^+ = (2\gamma - 1)(\gamma - 1), \quad \rho_\gamma^- = (1 - 2\gamma)(\gamma - 1).$$

An interesting remark concerns the construction of non-exchangeable (non-symmetric) copulas ($C(u_1, u_2) \neq C(u_2, u_1)$) through this method. This can be relevant for example in credit risk when modelling the default dependence between two firms with asymmetric relations. One may have a first firm depending more on a second one than the latter depends on the former. This could be the case of a little firm that provides goods to a large one. A default of the large company could induce a dramatic effect on the smaller one, whereas a default of the small firm could have little relevance to the large one. Notice that, in this respect, Archimedean and Gaussian copulas only provide symmetric relations between the two firms defaults.

In order to provide an example of non-exchangeable copula obtained from our family, we see from (2) that our only chance is to select a periodic function $\tilde{c}$ whose double primitive Φ is not par and then take the related $\widetilde{C}^+$. The simplest such function is $\tilde{c} := \bar{c}_\gamma$ defined, for $\gamma \in [0, 1]$, as

$$\bar{c}_\gamma(x) := (1/\gamma) \mathbf{1}_{[0,\gamma]}(x), \quad \text{for} \quad x \in [0, 1). \tag{6}$$

We have then $\bar{c}_\gamma(x) = \tilde{c}_{\frac{\gamma}{2}}(x - \frac{\gamma}{2})$. Thus, we obtain the following copula $\bar{C}^+_\gamma := \widetilde{C}^+$ associated with the density $\tilde{c}(x_1 - x_2) := \bar{c}_\gamma(x_1 - x_2)$:

$$\begin{aligned}\bar{C}^+_\gamma(u_1, u_2) &= \int_0^{u_1}\int_0^{u_2} \bar{c}_\gamma(x_1 - x_2)dx_1dx_2 = \int_0^{u_1}\int_0^{u_2} \tilde{c}_{\gamma/2}(x_1 - x_2 - \gamma/2)dx_1dx_2 \\ &= \int_0^{u_1}\int_{\frac{\gamma}{2}}^{u_2+\gamma/2} \tilde{c}_{\gamma/2}(x_1 - x_2)dx_1dx_2 \\ &= C^+_{\gamma/2}(u_1, \min(u_2 + \gamma/2, 1)) - C^+_{\gamma/2}(u_1, \gamma/2) + C^+_{\gamma/2}(u_1, (u_2 + \gamma/2 - 1)^+)\end{aligned}$$

With this asymmetric periodic copula we still have good asymptotic properties, in that $\bar{C}^+_1 = C^\perp$ and $\bar{C}^+_\gamma \to C^+_F$ when $\gamma \to 0$. Calculating the Spearman's rho, we find again $\rho(\bar{C}^+_\gamma) = (2\gamma - 1)(\gamma - 1)$ for $\gamma \in [0, 1]$. Somehow surprisingly, $\rho(\bar{C}^+_\gamma)$ takes negative value between $\gamma = 1/2$ and $\gamma = 1$, and vanishes at $1/2$ for a copula different from $C^\perp$. To obtain a (symmetric) family that reaches C^-_F we need consider $\bar{C}^-_\gamma(u_1, u_2) := \int_0^{u_1}\int_0^{u_2} \bar{c}_\gamma(x_1 + x_2)dx_1dx_2$. We get

$$\bar{C}^-_\gamma(u_1, u_2) = C^-_{\gamma/2}(u_1, (u_2 - \gamma/2)^+) + C^-_{\gamma/2}(u_1, 1 - (\gamma/2 - u_2)^+) - C^-_{\gamma/2}(u_1, 1 - \gamma/2)$$

and can show, with a trivial change of variable, that $\rho(\bar{C}^-_\gamma) = -\rho(\bar{C}^+_\gamma) = -(2\gamma - 1)(\gamma - 1)$. Thus, if we wish to describe a negative asymmetric dependence, it is best to use C^+_γ with $1/2 < \gamma < 1$. However, we point out that we cannot describe negative dependence with an asymmetric copula attaining C^-_F.

Another interesting synthetic quantity concerning copulas is the upper-tail dependence. This indicator is defined as

$$\lambda := \lim_{u\to 0}\frac{1}{u}\int_{1-u}^1\int_{1-u}^1 c(x_1, x_2)dx_1dx_2$$

when the copula has a density c. Let us consider the general periodic case, where as before $\tilde{c}$ is a nonnegative 1-periodic function such that $\int_0^1 \tilde{c}(u)du = 1$, and φ is its primitive that vanishes at 0. Using the periodicity, we have $\frac{1}{u}\int_{1-u}^1\int_{1-u}^1 c(x_1 \pm x_2)dx_1dx_2 = \frac{1}{u}\int_{-u}^0\int_{-u}^0 c(x_1 \pm x_2)dx_1dx_2 = \frac{1}{u}\int_{-u}^0 \pm(\varphi(x_1) - \varphi(x_1 \mp u))dx_1 \underset{u\to 0}{\to} 0$ since $\lim_{u\to 0}\frac{1}{u}\int_{-u}^0 \varphi(x_1)dx_1 = \lim_{u\to 0}\frac{1}{u}\int_0^u \varphi(x_1)dx_1 = \varphi(0) = 0$. Thus, periodic copulas have no upper-tail dependence. However, if one wishes to obtain a copula with an upper-tail dependence equal to $\lambda > 0$, it is still possible to consider the convex combination $(1 - \lambda)C + \lambda C^+_F$ where C is a preferred periodic copula. This convex combination can be simulated easily when one knows how to simulate the basic C, as we do for the periodic copulas (with invertible φ, i.e. with a strictly positive periodic function $\tilde{c}$) we introduced here.

2.2 A smooth family that reaches C_F^-, $C^\perp$ and C_F^+

A drawback of the families C_γ^+ and C_γ^- is that these copulas are constant on some intervals and comes from the 0-1 nature of the density, and more precisely from the existence of a domain (with positive measure) where the density vanishes. This causes problems, especially when in need of simulating the copula. In order to avoid this drawback, the idea is then to replace γ with a random variable Γ and then take the expectation of $\tilde{c}_\Gamma$, using the convexity of the subset of the periodic copulas. Indeed, if $\Gamma \sim p$ where p is the density of a probability measure on $[0, 1/2]$ such that $p(\gamma) > 0\ \forall\gamma$, then $\widehat{c}_p(x) := E[\tilde{c}_\Gamma(x)] = \int_0^{\frac{1}{2}} \tilde{c}_\gamma(x)p(\gamma)d\gamma$ is a positive 1-periodic function. Thus, if we have a family of random variables $(\Gamma_\alpha)_{\alpha\geq 0}$ concentrated on $[0, 1/2]$ with densities $\{p_\alpha, \alpha \in]0, +\infty[\}$ on $[0, 1/2]$ and such that $\Gamma_\alpha \xrightarrow[\alpha\to 0]{\mathcal{L}} 1/2$ and $\Gamma_\alpha \xrightarrow[\alpha\to\infty]{\mathcal{L}} 0$, we can define

$$\widehat{C}_\alpha^+ := E[C_{\Gamma_\alpha}^+] = \int_0^{\frac{1}{2}} C_\gamma^+ p_\alpha(\gamma)d\gamma,\quad \widehat{C}_\alpha^- := E[C_{\Gamma_\alpha}^-] = \int_0^{\frac{1}{2}} C_\gamma^- p_\alpha(\gamma)d\gamma \qquad (7)$$

(that correspond respectively to the periodic densities $\tilde{c}(x_1 - x_2) = \widehat{c}_{p_\alpha}(x_1 - x_2)$ and $\tilde{c}(x_1+x_2) = \widehat{c}_{p_\alpha}(x_1+x_2)$). We have obtained a family of copulas with good asymptotic properties, in that

$$\widehat{C}_\alpha^+ \underset{\alpha\to 0}{\to} C^\perp,\quad \widehat{C}_\alpha^+ \underset{\alpha\to+\infty}{\to} C_F^+,\quad \widehat{C}_\alpha^- \underset{\alpha\to 0}{\to} C^\perp,\quad \text{and}\quad \widehat{C}_\alpha^- \underset{\alpha\to+\infty}{\to} C_F^-.$$

We can build easily a random variable with suitable density on $[0, \frac{1}{2}]$ by transforming a uniform variable U on [0,1] according to a homeomorphism. Indeed, consider $\Gamma_\alpha := \frac{1}{2}U^\alpha$, $\alpha \in]0, +\infty[$, so that we get a family of densities p_α on $[0, \frac{1}{2}]$ that feature the desired asymptotic properties in 0 and $+\infty$ and are immediately computed:

$$p_\alpha(u) = (2^{\frac{1}{\alpha}}/\alpha)u^{\frac{1-\alpha}{\alpha}}.$$

The calculation of C_α^+ does not present difficulties either. We first calculate the periodic function $\widehat{c}_\alpha := \widehat{c}_{p_\alpha}$, obtaining

$$\begin{aligned} \widehat{c}_\alpha(x) &= \frac{1}{1-\alpha}[1 - (2x)^{\frac{1-\alpha}{\alpha}}],\ \alpha \neq 1 \\ \widehat{c}_1(x) &= -\ln(2x) \end{aligned}$$

for $0 \leq x \leq 1/2$, and $\widehat{c}_\alpha(x) = \widehat{c}_\alpha(1 - x)$ for $1/2 \leq x \leq 1$, since the same property holds for the basic $\tilde{c}_\gamma$'s. Let us compute the primitive ψ^α of $\widehat{c}_\alpha$ that vanishes at $x = 0$. We obtain, for $0 \leq x \leq 1/2$:

$$\psi_\alpha(x) = \frac{1}{2(1-\alpha)}[2x - \alpha(2x)^{\frac{1}{\alpha}}],\ \alpha \neq 1, \qquad \psi_1(x) = x - x\ln(2x).$$

Using the symmetry property $\widehat{c}_\alpha(x) = \widehat{c}_\alpha(1-x)$ we obtain, for $1/2 \leq x \leq 1$, $\psi_\alpha(x) = \psi_\alpha(1/2) + (\psi_\alpha(1/2) - \psi_\alpha(1-x)) = 1 - \psi_\alpha(1-x)$, since $\psi_\alpha(1/2) = 1/2$. Instead, for $x \in [-1, 0]$ we use the periodicity of $\widehat{c}$ to get $\psi_\alpha(x) = -\psi_\alpha(-x)$. To proceed further, we need to know also the primitive Ψ_α of ψ_α, i.e. the double primitive of $\widehat{c}_\alpha$. We find, for $x \in [0, \frac{1}{2}]$:

$$\Psi_\alpha(x) = \frac{1}{2(1-\alpha)}[x^2 - \frac{\alpha^2}{\alpha+1} 2^{\frac{1}{\alpha}} x^{\frac{1+\alpha}{\alpha}}],\ \alpha \neq 1, \qquad \Psi_1(x) = \frac{3}{4}x^2 - \frac{x^2}{2}\ln(2x),$$

and, for $x \in [\frac{1}{2}, 1]$:

$$\Psi_\alpha(x) = x - \frac{1}{2} + \Psi_\alpha(1-x),$$

and finally $\Psi_\alpha(x) = \Psi_\alpha(-x)$ for $x \in [-1, 0]$, since ψ_α is an odd function. We are now able to calculate $\widehat{C}_\alpha^+(u_1, u_2) = \int_0^{u_2}\int_0^{u_1} \widehat{c}_\alpha(x_1 - x_2)dx_1dx_2 = \int_0^{u_2}(\psi_\alpha(u_1 - x_2) - \psi_\alpha(-x_2))dx_2 = \int_0^{u_2}(\psi_\alpha(u_1 - x_2) + \psi_\alpha(x_2))dx_2$ and so we get, in agreement with our earlier general result (2):

$$\widehat{C}_\alpha^+(u_1, u_2) = \Psi_\alpha(u_1) + \Psi_\alpha(u_2) - \Psi_\alpha(u_1 - u_2). \tag{8}$$

The copula $\widehat{C}_\alpha^-$ can then be calculated easily, since $C_\gamma^-(u_1, u_2) = u_2 - C_\gamma^+(1-u_1, u_2)$ and therefore $\widehat{C}_\alpha^-(u_1, u_2) = \int_0^{\frac{1}{2}}(u_2 - C_\gamma^-(1-u_1, u_2))p_\alpha(\gamma)d\gamma = u_2 - \widehat{C}_\alpha^+(1-u_1, u_2)$. We can also easily calculate the Spearman's rho of $\widehat{C}_\alpha^+$, since $\rho(\widehat{C}_\alpha^+) = \int_0^{1/2}(2\gamma - 1)(\gamma - 1)2^{1/\alpha}\gamma^{(1-\alpha)/\alpha}/\alpha d\gamma$ so that

$$\rho(\widehat{C}_\alpha^+) = 1 - \frac{3}{2(1+\alpha)} + \frac{1}{2(1+2\alpha)}$$

and we have $\rho(\widehat{C}_\alpha^-) = -\rho(\widehat{C}_\alpha^+)$ (this is a general relation between the rho of the periodic copulas with density $\tilde{c}(x_1 + x_2)$ and $\tilde{c}(x_1 - x_2)$). We can sum up in Table 1 the values for which a limit copula is reached. The families built previously are

α	C_F^-	$C^\perp$	C_F^+
C_α^+	/	$\frac{1}{2}$	0
C_α^-	0	$\frac{1}{2}$	/
$\widehat{C}_\alpha^+$	/	0	$+\infty$
$\widehat{C}_\alpha^-$	$+\infty$	0	/

Table 1: Limit copulas for the parameterization C_α and $\widehat{C}_\alpha$

exchangeable, since the related $\tilde{c}$ are expectations of functions leading to par double primitives and therefore lead themselves to par double primitives, so that (2) yields symmetry.

However, we can also construct a smooth family of non symmetric copulas by defining $\tilde{c}$ as the expectation of the previous non symmetric function (6) with a random γ.

Indeed, if $Z \sim q$ where q is a positive density of a probability measure on $[0,1]$, then $\bar{c}_q := E[\bar{c}_Z] = \int_0^1 \bar{c}_\gamma(x)q(\gamma)d\gamma$ is a positive 1-periodic function. Thus, as before, if we have a family of random variables $(Z_\alpha)_{\alpha\geq 0}$ concentrated on $[0,1]$ with densities $\{q_\alpha, \alpha \in]0,+\infty[\}$ on $[0,1]$ and such that $Z_\alpha \xrightarrow[\alpha\to 0]{\mathcal{L}} 1$ and $Z_\alpha \xrightarrow[\alpha\to\infty]{\mathcal{L}} 0$, we can define

$$\widehat{\widetilde{C}}^+_\alpha := E[C^+_{Z_\alpha}] = \int_0^1 C^+_\gamma q_\alpha(\gamma)d\gamma, \quad \widehat{\widetilde{C}}^-_\alpha := E[C^-_{Z_\alpha}] = \int_0^1 C^-_\gamma q_\alpha(\gamma)d\gamma \tag{9}$$

(that correspond respectively to the periodic densities $\tilde{c}(x_1 - x_2) = \bar{c}_{q_\alpha}(x_1 - x_2)$ and $\tilde{c}(x_1 + x_2) = \bar{c}_{q_\alpha}(x_1 + x_2) = E[\bar{c}_{Z_\alpha}(x_1 + x_2)]$).

We take $Z_\alpha := U^\alpha$, where U is a uniform random variable on $[0,1]$. Its density is

$$q_\alpha(u) = \frac{1}{\alpha}u^{\frac{1-\alpha}{\alpha}},$$

so that the associated periodic function $\tilde{c} = \bar{c}_\alpha$ is given in $[0,1]$ by

$$\widehat{\tilde{c}}_\alpha(x) := \bar{c}_{q_\alpha} = \frac{1}{1-\alpha}[1 - x^{\frac{1-\alpha}{\alpha}}], \ \alpha \neq 1, \quad \widehat{\tilde{c}}_1(x) := \bar{c}_{q_1} = -\ln(x).$$

In order to find an expression for the copula, we compute the primitive g_α of $\widehat{\tilde{c}}_\alpha$ that vanishes at $x = 0$. We have, for $x \in [0,1]$:

$$g_\alpha(x) = \frac{1}{1-\alpha}[x - \alpha x^{\frac{1}{\alpha}}], \ \alpha \neq 1, \quad g_1(x) = x - x\ln(x),$$

and for $x \in [-1,0]$ we have $g_\alpha(x) = g_\alpha(1+x) - 1$. Denote by G_α the primitive of g_α vanishing at 0. For $x \in [0,1]$ G_α is given by

$$G_\alpha(x) = \frac{1}{1-\alpha}\left[\frac{1}{2}x^2 - \frac{\alpha^2}{1+\alpha}x^{\frac{1+\alpha}{\alpha}}\right], \ \alpha \neq 1, \quad G_1(x) = \frac{3}{4}x^2 - \frac{1}{2}x^2\ln(x),$$

whereas for $x \in [-1,0]$ we have

$$G_\alpha(x) = G_\alpha(1+x) - x - G_\alpha(1).$$

By Fubini's theorem, the copulas defined by (9) are the same as the copulas associated with the periodic functions $\tilde{c} = \widehat{\tilde{c}}_\alpha$ and defined by (2). We thus have, by (2) and using periodicity of $\widehat{\tilde{c}}$ (thus replacing $\widehat{\tilde{c}}(\cdot)$ by $\widehat{\tilde{c}}(\cdot - 1)$, which is helpful in some computational respects):

$$\begin{aligned}\widehat{\bar{C}}^+_\alpha(u_1, u_2) &= G_\alpha(u_1) + G_\alpha(-u_2) - G_\alpha(u_1 - u_2) \\ \widehat{\bar{C}}^-_\alpha(u_1, u_2) &= G_\alpha(u_1 + u_2 - 1) - G_\alpha(u_2 - 1) - G_\alpha(u_1 - 1) + G_\alpha(-1).\end{aligned}$$

Calculating the related Spearman rho we find:

$$\rho(\widehat{\bar{C}}^+_\alpha) = 1 - \frac{3}{1+\alpha} + \frac{2}{1+2\alpha}$$

and $\rho(\widehat{\bar{C}}^-_\alpha) = -\rho(\widehat{\bar{C}}^+_\alpha)$. As we know from (2), $\widehat{\bar{C}}^-_\alpha$, contrary to $\widehat{\bar{C}}^+_\alpha$, is a family of symmetric copulas, but this family is however interesting because it completes "naturally" the family $\widehat{\bar{C}}^+_\alpha$. In Table 2 we sum up the values of the parameter α for which the $\widehat{\bar{C}}$ copulas reach the limit copulas.

	C_F^-	$C^\perp$	C_F^+
$\bar{C}_\alpha^+$	/	1	0
$\bar{C}_\alpha^-$	0	1	/
$\widehat{\bar{C}}^+_\alpha$	/	0	$+\infty$
$\widehat{\bar{C}}^-_\alpha$	$+\infty$	0	/

Table 2: Limit copulas for the parameterization $\bar{C}_\alpha$ and $\widehat{\bar{C}}_\alpha$

2.3 Beyond the bivariate case

There are several ways to extend the previous construction to build a copula in dimension $n > 2$. In dimension $n > 2$, we see that if $\tilde{c}$ is a 1-periodic function such that $\int_0^1 \tilde{c}(x)dx = 1$, then $\tilde{c}(\sum_{i=1}^n \varepsilon_i x_i)$ are densities of copulas when $\varepsilon_i \in \{-1, 1\}$. However, these copulas cannot be obtained directly in terms of bivariate copulas and therefore require cumbersome calculations. In order to keep the analysis simple, we work instead with the densities already defined for the bivariate case. Consider the following proposition.

Proposition 2.1. *Assume that $c_1, ..., c_{n-1}$ are densities of two dimensional copulas built through periodic densities $\tilde{c}_1, \dots, \tilde{c}_{n-1}$, i.e. $c_j(x, y) = \tilde{c}_j(x + \varepsilon_j y)$ with $\varepsilon_j \in$*

$\{-1,1\}$ and $\tilde{c}_j$ a nonnegative 1-periodic function with unit integral on a period. Set $\check{c} := (c_1, ..., c_{n-1})$. Then

$$\begin{aligned}\check{C}_1(u_1,..,u_n) &:= \int_0^{u_1} .. \int_0^{u_n} c_1(x_1,x_2)c_2(x_2,x_3)..c_{n-1}(x_{n-1},x_n)dx_1...dx_n \\ \check{C}_2(u_1,..,u_n) &:= \int_0^{u_1} .. \int_0^{u_n} c_1(x_1,x_2)c_2(x_1,x_3)..c_{n-1}(x_1,x_n)dx_1...dx_n\end{aligned}$$

are copulas.

The proof is quite immediate. Properties 1 and 3 in Section 1 are satisfied by construction. It remains to observe that $\check{C}(1,..,1,u_k,1,..,1) = \int_0^{u_k} dx_k = u_k$, by using Fubini's theorem, integrating first with respect to the x_i's with $i \neq k$, and using then the property $\int_x^{1+x} \tilde{c}(u)du = 1$.

The first copula $\check{C}_1$ is convenient if we wish to express the n-dependence in terms of dependences of two consecutive variables, whereas the second one $\check{C}_2$ allows us to express the n-dependence in terms of the dependence of a preferred variable (the first in our formulation) with all other variables. The second method could be referred to as a "preferred-" or "main-factor" approach.

3 The simulation of periodic copulas

Let us begin by recalling how to simulate a copula that admits a density $p(x_1, ..., x_n)$. We need simulate a vector of uniform variables $(U_1, ..., U_n)$ that has the following joint cdf:

$$C(u_1,...,u_n) = \int_0^{u_1} .. \int_0^{u_n} p(x_1,...,x_n)dx_1...dx_n.$$

This can be done according to the following steps.

- To simulate the first variable U_1, it suffices to sample from a uniform random variable $\tilde{U}_1$ in $[0,1]$. This can be easily done on a PC. Let us call u_1 the simulated sample.

- To obtain a sample u_2 from U_2 consistently with the earlier sampled u_1, we need to know the law of U_2 conditional on $U_1 = u_1$. Let us name $F_2(.|u_1)$ the cdf of this law,

$$\begin{aligned}F_2(u_2|u_1) &= \mathbb{P}(U_2 \le u_2|U_1 = u_1) = \partial_{u_1} C(u_1,u_2,1,..,1)/\partial_{u_1} C(u_1,1,1,..,1) \\ &= \partial_{u_1} C(u_1,u_2,1,..,1) = \int_0^{u_2}\int_0^1 \dots \int_0^1 p(u_1,x_2,..x_m)dx_2..dx_m.\end{aligned}$$

We take $u_2 = F_2^{-1}(\tilde{U}_2|u_1)$ where $\tilde{U}_2$ is a new uniform-$[0,1]$ sample independent of $\tilde{U}_1$.

- to simulate U_k consistently with the earlier sampled $u_1, \ldots, u_{k-1}$, we need the law of U_k conditional on $U_i = u_i$ for $i < k$. Denoting as usual by $F_k(\cdot|u_1, .., u_{k-1})$ the cdf of this law,

$$
\begin{aligned}
F_k(u_k|u_1, \ldots, u_{k-1}) &= \mathbb{P}(U_k \leq u_k | U_1 = u_1, .., U_{k-1} = u_{k-1}) \\
&= \frac{\partial_{u_1,..,u_{k-1}} C(u_1, .., u_k, 1, .., 1)}{\partial_{u_1,..,u_{k-1}} C(u_1, .., u_{k-1}, 1, .., 1)} \\
&= \frac{\int_0^{u_k} \int_0^1 .. \int_0^1 p(u_1, .., u_{k-1}, x_k, .., x_n) dx_k dx_{k+1}..dx_n}{\int_0^1 \int_0^1 .. \int_0^1 p(u_1, .., u_{k-1}, x_k, .., x_n) dx_k dx_{k+1}..dx_n}
\end{aligned}
$$

  we can take $U_k = F_k^{-1}(\tilde{U}_k|u_1, .., u_{k-1})$ where $\tilde{U}_k$ is a uniform-$[0,1]$ variable independent of $(\tilde{U}_1, ..., \tilde{U}_{k-1})$.

In the case of the periodic copulas $\check{C}_{1,2}$, maintaining the notation of Proposition 2.1, we have respectively $F_k^1(u_k|u_1, .., u_{k-1}) = \int_0^{u_k} c_{k-1}(u_{k-1}, x_k) dx_k$ and $F_k^2(u_k|u_1, .., u_{k-1}) = \int_0^{u_k} c_{k-1}(u_1, x_k) dx_k$, where the upper index refers to the copula we are considering. Taking the smooth families of the previous section, these F functions can be expressed in terms of ψ_α and g_α (for example $\int_0^{u_k} \widehat{c}_\alpha(u \pm x) dx = \pm(\psi_\alpha(u \pm u_k) - \psi_\alpha(u))$. Moreover they are strictly increasing, and can therefore be inverted easily numerically. We note here that if we choose the "non smooth" copulas $C^\pm$ and $\overline{C}^\pm$, this inversion is not feasible since the densities vanish on some intervals. Thus we have obtained families of n-dimensional copulas essentially characterized by $n-1$ parameters α_i plus the flags sgn_i, sym_i, for $i = 1, \ldots, n-1$, where sym_i is set according to whether we take a symmetric family or not (symbolized here by the $\overline{\text{bar}}$), and where sgn_i is taken from the set $\{-, +\}$.

4 Conclusions

The new family of "periodic" copulas introduced in this paper is an attempt at obtaining practically manageable and possibly asymmetric copulas. We have studied the two-dimensional case, based on a single dependence parameter, and then provided a means to construct an n-dimensional copula building on the two-dimensional case. We obtained families of copulas in dimension n and parameterized by $n-1$

parameters, implying possibly asymmetric relations. We explain how such copulas can be simulated.

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