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NEW FAMILIES OF SPECIAL NUMBERS FOR COMPUTING NEGATIVE ORDER EULER NUMBERS AND RELATED NUMBERS AND POLYNOMIALS

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The main purpose of this paper is to construct new families of special numbers with their generating functions. These numbers are related to many well-known numbers, which are Bernoulli numbers, Fibonacci numbers, Lucas numbers, Stirling numbers of the second kind and central factorial numbers. Our other inspiration of this paper is related to the Golombek's problem [**15**] "Aufgabe 1088. El. Math., 49 (1994), 126–127". Our first numbers are not only related to the Golombek's problem, but also computation of the negative order Euler numbers. We compute a few values of the numbers which are given by tables. We give some applications in probability and statistics. That is, special values of mathematical expectation of the binomial distribution and the Bernstein polynomials give us the value of our numbers. Taking derivative of our generating functions, we give partial differential equations and also functional equations. By using these equations, we derive recurrence relations and some formulas of our numbers. Moreover, we come up with a conjecture with two open questions related to our new numbers. We give two algorithms for computation of our numbers. We also give some combinatorial applications, further remarks on our new numbers and their generating functions.

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1. INTRODUCTION

In this section, we consider the following question:

What could be more basic tools to compute the negative order of the first and the second kind Euler numbers? One of motivations of this paper is associated with this question and its answer. Another motivation of this paper is related to the work of Golombek [**15**], which is entitled **Aufgabe 1088**.

Here, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$ be the sets of complex numbers, real numbers, rational numbers, integers, and positive integers, respectively and let $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\} = \mathbb{N} \cup \{0\}$.

Golombek gave the following novel combinatorial sum:

$$\sum_{j=0}^{k} \binom{k}{j} j^n = \frac{d^n}{dt^n} \left(e^t + 1\right)^k |_{t=0} , \tag{1}$$

where $n \in \mathbb{N}$. Golombek [**15**] also mentioned that this sum is related to the following sequence

$$n2^{n-1}, n(n+1)2^{n-2}, \dots$$

We introduce new families of special numbers, which are not only used in counting techniques and problems, but also computing negative order of the first and the second kind Euler numbers and other combinatorial sums. Here, our technique is related to the generating functions and their functional equations. In the historical development of mathematics, we can observe that the generating functions play a very important role in pure and applied mathematics. These function are powerful tools in solving counting problems and investigating properties of the special numbers and polynomials. In addition, the generating functions are also used in computer programming, in physics, and in other areas. Briefly, in Physics, generating functions, which arise in Hamiltonian mechanics, are quite different from generating functions in mathematics. The generating functions are functions whose partial derivatives generate the differential equations that determine a system's dynamics. These functions are also related to the partition function of statistical mechanics (cf. [**11**], [**21**], [**37**]). In mathematics, a generating function can be expanded as formal power series in one indeterminate whose coefficients encode information about a sequence of numbers and that is indexed by the natural numbers (cf. [**11**], [**12**], [**14**], [**13**], [**21**], [**37**], [**29**], [**40**]). As far as we know, the generating function is firstly discovered by Abraham de Moivre (26 May 1667 -27 November 1754, French mathematician) (cf. [**21**]). In order to solve the general linear recurrence problem, Moivre constructed the concept of the generating functions in 1730. In work of Doubilet et al. [**14**], we also see that Laplace (23 March 1749-5 March 1827, French mathematician, physicist and statistician) discovered the remarkable correspondence between set theoretic operations and operations on formal power series. Their method gives us great success to solve a variety of combinatorial problems. They developed new kinds of algebras of generating functions better suited to combinatorial and probabilistic problems. Their method depends on group algebra (or semigroup algebra) (see, for details, [**14**]). It is well-known that there are many different ways or approaches to generate a sequence of numbers and polynomials from the series or the generating functions. The purpose of this paper is to construct the generating functions for new families of numbers involving Golombek's identity in (1), Stirling numbers, central factorial numbers, Euler numbers of negative order, rook numbers and combinatorial sums. Our method and approach provides a way of constructing new special families of numbers and combinatorial sums. We show how several of these numbers and these combinatorial sums relate to each other. We pose a conjecture with two open questions associated with our new numbers and their generating functions.

We organize our paper as follows:

In Section 2, we briefly review some special numbers and polynomials, which are Bernoulli numbers, Euler numbers, Stirling numbers, central factorial numbers and array polynomials.

In Section 3, we give a generating function. By using this function, we define a family of new numbers $y_1(n,k;\lambda)$. We investigate many properties including recurrence relations of these numbers by using their generating functions. We compute a few values of the numbers $y_1(n,k;\lambda)$, which are given by tables. We give some remarks and comments related to the Golombek's identity and the numbers $y_1(n,k;1)$. Finally, we give a conjecture with two open questions.

In Section 4, we give a generating function for a new family of the other numbers $y_2(n,k;\lambda)$. By using this function, we investigate many properties with a recurrence relation of these numbers. We compute a few values of the numbers $y_2(n,k;\lambda)$, which are given by tables. We give relations between these numbers, Fibonacci numbers, Lucas numbers, and λ-Stirling numbers of the second kind. We also give some combinatorial sums.

In Section 5, we define λ-central factorial numbers $C(n,k;\lambda)$. By using their generating function, we derive some identities and relations including these numbers and the others.

In Section 6, we give some applications related to the special values of mathematical expectation for the binomial distribution, the Bernstein polynomials and the Bernoulli polynomials.

In Section 7, by using the numbers $y_1(n,k;\lambda)$, we compute the Euler numbers of negative order. In addition, we compute a few values of these numbers, which are given by tables.

In Section 8, we give two algorithms for our computations.

In Section 9, we give combinatorial applications, including a rook numbers and polynomials. We also give combinatorial interpretation for the numbers $y_1(n,k;1)$. Finally in the last section, we give further remarks with conclusion.

The principal value $\ln z$ is the logarithm whose imaginary part lies in the interval $(-\pi,\pi]$. Moreover we also use the following notational conventions:

$$0^n = \begin{cases} 1, & (n=0) \\ 0, & (n\in\mathbb{N}) \end{cases}$$

and

$$\begin{pmatrix} \lambda \\ 0 \end{pmatrix} = 1 \text{ and } \begin{pmatrix} \lambda \\ v \end{pmatrix} = \frac{\lambda(\lambda-1)\cdots(\lambda-v+1)}{v!} = \frac{(\lambda)_v}{v!} \; (n \in \mathbb{N},\, \lambda \in \mathbb{C})$$

(*cf.* [**4**], [**12**], [**42**]). For combinatorial example, we will use the notations of Bona [**5**], that is the set $\{1, 2, \ldots, n\}$ is an n-element set, that is, n distinct objects. Therefore, Bona introduced the shorter notation $[n]$ for this set. The number $n(n-1)(n-2)\cdots(n-k+1)$ of all k-element lists from $[n]$ without repetition occurs so often in combinatorics that there is a symbol for it, namely

$$(n)_k = n(n-1)(n-2)\cdots(n-k+1)$$

(cf. [**5**, pp. 11-13.]).

2. Background

In this section, we give a brief introduction about Bernoulli numbers, Euler numbers, the (λ-) Stirling numbers and array polynomials, which will be used in subsequent sections.

In [**2**]-[**45**], we see that there are many known properties and relations involving various kind of the special numbers and polynomials such as Bernoulli polynomials and numbers, Euler polynomials and numbers, Stirling numbers and also rook polynomials and numbers by making use of some standard techniques based upon generating functions and other known techniques.

Bernoulli polynomials are defined by means of the following generating function (*cf.* [**13**]-[**45**]):

$$\frac{t}{e^t-1}e^{tx} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}$$

($|t| < 2\pi$). One can observe that

$$B_n = B_n(0),$$

which denotes Bernoulli numbers (cf. [**13**]-[**45**]; see also the references cited in each of these earlier works).

The sum of powers of integers is related to the Bernoulli numbers and polynomials:

$$\sum_{k=0}^{n} k^r = \frac{1}{r+1}\left(B_{r+1}(n+1) - B_{r+1}\right), \tag{2}$$

(cf. [**13**], [**40**], [**42**]).

The first kind Apostol-Euler polynomials of order k, with $k \geq 0$, $E_n^{(k)}(x;\lambda)$ are defined by means of the following generating function:

$$F_{P1}(t,x;k,\lambda)=\left(\frac{2}{\lambda e^t+1}\right)^k e^{tx}=\sum_{n=0}^{\infty}E_n^{(k)}(x;\lambda)\frac{t^n}{n!} \tag{3}$$

($|t|<\pi$ when $\lambda=1$ and $|t|<|\ln(-\lambda)|$ when $\lambda\neq 1$), $\lambda\in\mathbb{C}$, $k\in\mathbb{N}$ with, of course,

$$E_n^{(k)}(\lambda)=E_n^{(k)}(0;\lambda),$$

which denote the first kind Apostol-Euler numbers of order k (cf. [**20**], [**13**], [**28**], [**26**], [**30**], [**40**], [**45**]). Substituting $k=\lambda=1$ into (3), we have the first kind Euler numbers $E_n = E_n^{(1)}(1)$, which are defined by means of the following generating function:

$$\frac{2}{e^t+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!},$$

($|t|<\pi$) (cf. [**13**]-[**45**]; see also the references cited in each of these earlier works).

The second kind Euler numbers E_n^* are defined by means of the following generating function:

$$\frac{2}{e^t+e^{-t}}=\sum_{n=0}^{\infty}E_n^*\frac{t^n}{n!}$$

($|t|<\frac{\pi}{2}$) (cf. [**8**], [**13**], [**24**], [**26**], [**30**], [**42**], [**45**]; see also the references cited in each of these earlier works).

Stirling numbers of the second kind are used in pure and applied mathematics. These numbers occur in combinatorics and in the theory of partitions. The Stirling numbers of the second kind, denoted by $S_2(n,v)$, the number of ways to partition a set of n objects into k groups ([**5**], [**7**], [**12**], [**37**], [**42**]).

Let $v\in\mathbb{N}_0$ and $\lambda\in\mathbb{C}$. The λ-Stirling numbers of the second kind $S_2(n,v;\lambda)$ are generalized of the Stirling number of the second kind. These numbers $S_2(n,v;\lambda)$ are defined by means of the following generating function:

$$F_S(t,v;\lambda)=\frac{(\lambda e^t-1)^v}{v!}=\sum_{n=0}^{\infty}S_2(n,v;\lambda)\frac{t^n}{n!}. \tag{4}$$

For further information about these numbers, the reader may be referred to [**25**] and ([**34**], [**33**], [**39**]; see also the references cited in each of these earlier works).

Observe that

$$S_2(n,v)=S_2(n,v;1),$$

which are computing by the following formulas:

$$x^n=\sum_{v=0}^{n}\binom{x}{v}v!S_2(n,v)$$

or

$$S_2(n,v)=\frac{1}{v!}\sum_{j=0}^{v}\left(\begin{array}{c} v \\ j \end{array}\right)(-1)^j\,(v-j)^n$$

(cf. [**13**]-[**45**]; see also the references cited in each of these earlier works). A recurrence relation for these numbers is given by

$$S_2(n,k)=S_2(n-1,k-1)+kS_2(n-1,k),$$

with

$$S_2(n,0)=0\ (n\in\mathbb{N});\ S_2(n,n)=1\ (n\in\mathbb{N});\ S_2(n,1)=1\ (n\in\mathbb{N})$$

and $S_2(n,k)=0$ ($n<k$ or $k<0$) (*cf.* [**13**]-[**45**]; see also the references cited in each of these earlier works).

Let $v\in\mathbb{N}_0$ and $\lambda\in\mathbb{C}$. In [**34**], we defined the λ-array polynomials $S_v^n(x;\lambda)$ by means of the following generating function:

$$F_A(t,x,v;\lambda)=\frac{(\lambda e^t-1)^v}{v!}e^{tx}=\sum_{n=0}^{\infty}S_v^n(x;\lambda)\frac{t^n}{n!} \tag{5}$$

(cf. [**4**], [**34**]).

The array polynomials $S_v^n(x)$ are defined by means of the following generating function:

$$F_A(t,x,v)=\frac{(e^t-1)^v}{v!}e^{tx}=\sum_{n=0}^{\infty}S_v^n(x)\frac{t^n}{n!}, \tag{6}$$

(cf. [**4**], [**10**], [**34**]; see also the references cited in each of these earlier works). By using the above generating function, we have

$$S_v^n(x)=\frac{1}{v!}\sum_{j=0}^{v}(-1)^{v-j}\left(\begin{array}{c} v \\ j \end{array}\right)(x+j)^n$$

with

$$S_0^0(x)=S_n^n(x)=1, S_0^n(x)=x^n$$

and for $v>n$,

$$S_v^n(x)=0$$

(cf. [**10**], [**34**], [**35**]; see also the references cited in each of these earlier works).

Recently, central factorial numbers $T(n,k)$ have been studied by many authors. These numbers are used in theory of numbers, combinatorics and probability. Central factorial numbers $T(n,k)$ (of the second kind) are defined by means of the following generating function:

$$F_T(t,k)=\frac{1}{(2k)!}\left(e^t+e^{-t}-2\right)^k=\sum_{n=0}^{\infty}T(n,k)\frac{t^{2n}}{(2n)!} \tag{7}$$

(cf. [**5**], [**7**], [**12**], [**41**], [**35**], [**37**]; see also the references cited in each of these earlier works).

These numbers have the following relations:

$$x^n = \sum_{k=0}^{n} T(n,k)x(x-1)(x-2^2)(x-3^2)\cdots(x-(k-1)^2).$$

Combining the above equation with (7), we also have

$$T(n,k) = T(n-1,k-1) + k^2 T(n-1,k),$$

where $n \geq 1$, $k \geq 1$, $(n,k) \neq (1,1)$. For n, $k \in \mathbb{N}$, $T(0,k) = T(n,0) = 0$ and $T(n,1) = 1$ (*cf.* [**5**], [**7**], [**12**], [**41**], [**35**], [**37**]).

3. A family of new numbers $y_1(n,k;\lambda)$

In this section, we give generating function for the numbers $y_1(n,k;\lambda)$. We give some functional equations and differential equations of this generating function. By using these equations, we derive various new identities and combinatorial relations involving these numbers. Some of our observations on these numbers can be briefly expressed as follows: the numbers $y_1(n,k;\lambda)$ are related to the λ-Stirling numbers of the second kind, the central factorial numbers, the Euler numbers of negative orders and the Golombek's identity.

Let $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$. We define these numbers, $y_1(n,k;\lambda)$ by the following generating function:

$$F_{y_1}(t,k;\lambda) = \frac{1}{k!}\left(\lambda e^t + 1\right)^k = \sum_{n=0}^{\infty} y_1(n,k;\lambda)\frac{t^n}{n!}. \tag{8}$$

The function $F_{y_1}(t,k;\lambda)$ is an analytic function.

By using (8), we get

$$\sum_{n=0}^{\infty} y_1(n,k;\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j} j^n\lambda^j\right)\frac{t^n}{n!}.$$

Comparing the coefficients of t^n on both sides of the above equation, we arrive at the the following theorem:

Theorem 1. *Let $n \in \mathbb{N}_0$. The following identity holds:*

$$y_1(n,k;\lambda) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j} j^n\lambda^j. \tag{9}$$

We assume that $\lambda \neq 0$. For $k = 0, 1, 2, 3, 4$ and $n = 0, 1, 2, 3, 4, 5$, we use (9) to compute a few values of the numbers $y_1(n, k; \lambda)$ as follows:

$n\backslash k$	0	1	2	3	4
0	1	$\lambda+1$	$\frac{1}{2}\lambda^2+\lambda+\frac{1}{2}$	$\frac{1}{6}\lambda^3+\frac{1}{2}\lambda^2+\frac{1}{2}\lambda+\frac{1}{6}$	$\frac{1}{24}\lambda^4+\frac{1}{6}\lambda^3+\frac{1}{4}\lambda^2+\frac{1}{6}\lambda+\frac{1}{24}$
1	0	λ	$\lambda^2+\lambda$	$\frac{1}{2}\lambda^3+\lambda^2+\frac{1}{2}\lambda$	$\frac{1}{6}\lambda^4+\frac{1}{2}\lambda^3+\frac{1}{2}\lambda^2+\frac{1}{6}\lambda$
2	0	λ	$2\lambda^2+\lambda$	$\frac{3}{2}\lambda^3+2\lambda^2+\frac{1}{2}\lambda$	$\frac{2}{3}\lambda^4+\frac{3}{2}\lambda^3+\lambda^2+\frac{1}{6}\lambda$
3	0	λ	$4\lambda^2+\lambda$	$\frac{9}{2}\lambda^3+4\lambda^2+\frac{1}{2}\lambda$	$\frac{8}{3}\lambda^4+\frac{9}{2}\lambda^3+2\lambda^2+\frac{1}{6}\lambda$
4	0	λ	$8\lambda^2+\lambda$	$\frac{27}{2}\lambda^3+8\lambda^2+\frac{1}{2}\lambda$	$\frac{32}{3}\lambda^4+\frac{27}{2}\lambda^3+4\lambda^2+\frac{1}{6}\lambda$
5	0	λ	$16\lambda^2+\lambda$	$\frac{81}{2}\lambda^3+16\lambda^2+\frac{1}{2}\lambda$	$\frac{128}{3}\lambda^4+\frac{81}{2}\lambda^3+8\lambda^2+\frac{1}{6}\lambda$

Table 1: Some numerical values of the numbers $y_1(n, k; \lambda)$.

For $k = 0, 1, 2, \dots, 9$ and $n = 0, 1, 2, \dots, 9$, we also use (9) to compute a few values of the numbers $y_1(n, k; 1)$ as follows:

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
0	1	2	2	$\frac{4}{3}$	$\frac{2}{3}$	$\frac{4}{15}$	$\frac{4}{45}$	$\frac{8}{315}$	$\frac{2}{315}$	$\frac{4}{2835}$
1	0	1	2	2	$\frac{4}{3}$	$\frac{2}{3}$	$\frac{4}{15}$	$\frac{4}{45}$	$\frac{8}{315}$	$\frac{2}{315}$
2	0	1	3	4	$\frac{10}{3}$	2	$\frac{14}{15}$	$\frac{16}{45}$	$\frac{4}{35}$	$\frac{2}{63}$
3	0	1	5	9	$\frac{28}{3}$	$\frac{20}{3}$	$\frac{18}{5}$	$\frac{14}{9}$	$\frac{176}{315}$	$\frac{6}{35}$
4	0	1	9	22	$\frac{85}{3}$	24	$\frac{224}{15}$	$\frac{328}{45}$	$\frac{102}{35}$	$\frac{62}{63}$
5	0	1	17	57	$\frac{274}{3}$	$\frac{275}{3}$	$\frac{328}{5}$	$\frac{1624}{45}$	$\frac{5048}{315}$	$\frac{208}{35}$
6	0	1	33	154	$\frac{925}{3}$	367	$\frac{4529}{15}$	$\frac{8416}{45}$	$\frac{3224}{35}$	$\frac{2360}{63}$
7	0	1	65	429	$\frac{3238}{3}$	$\frac{4580}{3}$	$\frac{7223}{5}$	$\frac{9065}{9}$	$\frac{173216}{315}$	$\frac{8576}{35}$
8	0	1	129	1222	$\frac{11665}{3}$	6554	$\frac{107114}{15}$	$\frac{252268}{45}$	$\frac{118717}{35}$	$\frac{104288}{63}$
9	0	1	257	3537	$\frac{42994}{3}$	$\frac{86645}{3}$	$\frac{181458}{5}$	$\frac{1444534}{45}$	$\frac{6781748}{315}$	$\frac{402723}{35}$

Table 2: Some numerical values of the numbers $y_1(n, k; 1)$.

Some special values of $y_1(n, k; \lambda)$ are given as follows:

$$y_1(0, k; \lambda) = \frac{1}{k!}(\lambda + 1)^k,$$

$$y_1(n, 0; \lambda) = 0, \quad (n \in \mathbb{N})$$

and

$$y_1(n, 1; \lambda) = \lambda, \quad (n \in \mathbb{N}).$$

By using (8), we derive the following functional equation

$$\lambda^k e^{kt} = \sum_{l=0}^{k} (-1)^{k-l} \left(\begin{array}{c} k \\ l \end{array} \right) l! F_{y_1}(t, l; \lambda).$$

Combining (8) with the above equation, we get

$$\lambda^k \sum_{n=0}^{\infty} \frac{(kt)^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} l! y_1(n,l;\lambda) \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the following theorem:

Theorem 2.

$$k^n \lambda^k = \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} l! y_1(n,l;\lambda).$$

We give a relationship between the numbers $y_1(n,k;\lambda)$ and the λ-Stirling numbers of the second kind by the following theorem:

Theorem 3.

$$S_2(n,k;\lambda^2) = \frac{k!}{2^n} \sum_{l=0}^{n} \binom{n}{l} S_2(l,k;\lambda) y_1(n-l,k;\lambda).$$

Proof. By using (4) and (8), we derive the following functional equation:

$$F_S(2t,k;\lambda^2) = k! F_S(t,k;\lambda) F_{y_1}(t,k;\lambda).$$

From the above equation, we have

$$\sum_{n=0}^{\infty} 2^n S_2\left(n,k;\lambda^2\right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} S_2\left(n,k;\lambda\right) \frac{t^n}{n!} \sum_{n=0}^{\infty} y_1(n,k;\lambda) \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} 2^n S_2\left(n,k;\lambda^2\right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(k! \sum_{l=0}^{n} \binom{n}{l} S_2(l,k;\lambda) y_1(n-l,k;\lambda) \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the desired result. $\square$

A relationship between the numbers $y_1(n,k;\lambda)$, $S_2(n,k;\lambda^3)$ and the array polynomials $S_k^n(x;\lambda)$ is given by the following theorem:

Theorem 4.

$$S_2(n,k;\lambda^3) = \sum_{l=0}^{n} \sum_{j=0}^{k} \binom{n}{l} \binom{k}{j} \frac{\lambda^{2k-2j} j!}{3^n} y_1(l,j;\lambda) S_k^{n-l}(2k-2j;\lambda).$$

Proof. Combining (4), (5) and (8), we get

$$F_S(3t,k;\lambda^3)=\sum_{j=0}^{k}\frac{k!}{(k-j)!}\lambda^{2k-2j}F_A(t,2k-2j,k;\lambda)F_{y_1}(t,j;\lambda).$$

By using the above functional equation, we obtain

$$\sum_{n=0}^{\infty}3^nS_2\left(n,k;\lambda^3\right)\frac{t^n}{n!}=\sum_{j=0}^{k}\frac{k!}{(k-j)!}\lambda^{2k-2j}\sum_{n=0}^{\infty}S_k^n\left(n,2k-2j;\lambda\right)\frac{t^n}{n!}\sum_{n=0}^{\infty}y_1(n,j;\lambda)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty}3^nS_2\left(n,k;\lambda^3\right)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{j=0}^{k}\left(\begin{array}{c}n\\l\end{array}\right)\left(\begin{array}{c}k\\j\end{array}\right)j!\lambda^{2k-2j}y_1(l,j;\lambda)S_k^{n-l}(2k-2j;\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. $\square$

There are many combinatorial applications for (9). That is, by substituting $\lambda=1$ into (9), we set

$$B(n,k)=k!y_1(n,k;1). \tag{10}$$

In [**15**], Golombek gave the following formula for (9):

$$B(n,k)=\frac{d^n}{dt^n}\left(e^t+1\right)^k|_{t=0}\ .$$

Remark 1. *If we substitute $\lambda=-1$ into* (9)*, then we get the Stirling numbers of the second kind (*cf. [**5**]-[**44**]*):*

$$S_2(n,k)=(-1)^ky_1(n,k;-1)=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\left(\begin{array}{c}k\\j\end{array}\right)j^n.$$

Remark 2. *We claim that the numbers $B(n,k)$ are related to the following numbers:*

$$a_k2^k$$

where the sequence a_k is a positive integer depend on k. Consequently, in the work of Spivey Identity 8*-Identity* 10 Spevy*, we see*

$$B(0,k)=2^k,$$

$$B(1,k)=k2^{k-1},$$

$$B(2,k)=k(k+1)2^{k-2},$$

see also [**5**, p. 56, *Exercise* 21] and [**11**, p. 117]*.*

Remark 3. *In* [**38**, *Identity* 12.], *Spivey also proved the following novel identity by the falling factorial method:*

$$B(m,n)=\sum_{j=0}^{m}\binom{n}{j}j!2^{n-j}S_2(m,j). \tag{11}$$

The numbers $B(0,k)$ are given by means of the following well-known generating function: Let $|x|<\frac{1}{2}$, we have

$$\sum_{k=0}^{\infty}B(0,k)x^k=\frac{1}{1-2x}.$$

The numbers $B(1,k)$ are given by means of the following well-known generating function: Let $|x|<\frac{1}{2}$, we have

$$\sum_{k=1}^{\infty}B(1,k)x^k=\frac{x}{(1-2x)^2}.$$

Remark 4. *In work of Boyadzhiev* [**6**, p.4, Eq-(7)], *we have*

$$\sum_{j=0}^{k}\binom{k}{j}j^n x^j=\sum_{j=0}^{n}\binom{k}{j}j!S_2(n,j)x^j(1+x)^{k-j}.$$

Substituting $x=1$ into the above equation, we arrive at (11).

Theorem 5. *Let $d\in\mathbb{N}$ and $m_0,m_1,m_2,...,m_d\in\mathbb{Q}$. Let $m_0\neq 0$. Thus we have*

$$\sum_{v=0}^{d-1}m_v B(d-v,k)=2^{k-d}\binom{k}{d}. \tag{12}$$

Proof. It is well-known that

$$(1+x)^k=\sum_{j=0}^{k}\binom{k}{j}x^j.$$

Taking the d^{th} derivative, with respect to x, we obtain

$$\binom{k}{d}(1+x)^{k-d}=\sum_{j=0}^{k}\binom{k}{j}\binom{j}{d}x^{j-d}. \tag{13}$$

Substituting $x=1$ into the above equation, we get

$$2^{k-d}\binom{k}{d}=\sum_{j=0}^{k}\binom{k}{j}\binom{j}{d}. \tag{14}$$

In [**36**], we have

$$\binom{j}{d} = m_0 j^d + m_1 j^{d-1} + \cdots + m_{d-1} j,$$

where $m_0, m_1, \ldots, m_{d-1} \in \mathbb{Q}$. Therefore

$$2^{k-d}\binom{k}{d} = \sum_{j=0}^{k} \binom{k}{j} \left(m_0 j^d + m_1 j^{d-1} + \cdots + m_{d-1} j\right).$$

Thus we get

$$2^{k-d}\binom{k}{d} = \sum_{v=0}^{d-1} m_v \sum_{j=0}^{k} \binom{k}{j} j^{d-v}.$$

Combining (10) with the above equation, we have

$$2^{k-d}\binom{k}{d} = \sum_{v=0}^{d-1} m_v B(d-v,k).$$

This completes the proof. □

There are many combinatorial arguments of (13). That is, if we substitute $d = 3$ and 4 into (13), then we compute $B(3,k)$ and $B(4,k)$, respectively, as follows:

$$B(3,k) = k^2(k+3)2^{k-3}$$

and

$$B(4,k) = k(k^3 + 6k^2 + 3k - 2)2^{k-4}.$$

By using (12), we derive the following result:

$$B(d,k) = \frac{2^{k-d}}{m_0}\binom{k}{d} - \sum_{v=1}^{d-1} \frac{m_v}{m_0} B(d-v,k).$$

Therefore, we *conjecture* that

$$B(d,k) = (k^d + x_1 k^{d-1} + x_2 k^{d-2} + \cdots + +x_{d-2} k^2 + x_{d-1} k)2^{k-d},$$

where $x_1, x_2, \ldots, x_{d-1}$ and d are positive integers. Consequently, we arrive at the following open questions:

1-How can we compute the coefficients $x_1, x_2, \ldots, x_{d-1}$?

2-We assume that for $|x| < r$

$$\sum_{k=1}^{\infty} B(d,k) x^k = f_d(x).$$

Is it possible to find $f_d(x)$ function?

3.1. Recurrence relation and some identities for the numbers $y_1(n,k;\lambda)$

Here, by applying derivative operator to the generating functions (8), we give a recurrence relation and other formulas for the numbers $y_1(n,k;\lambda)$.

Theorem 6. *Let $k \in \mathbb{N}$. The following identity holds:*

$$y_1(n+1,k;\lambda) = k y_1(n,k;\lambda) - y_1(n,k-1;\lambda).$$

Proof. Taking derivative of (8), with respect to t, we obtain the following partial differential equation:

$$\frac{\partial}{\partial t} F_{y_1}(t,k;\lambda) = k F_{y_1}(t,k;\lambda) - F_{y_1}(t,k-1;\lambda).$$

Combining (8) with the above equation, we get

$$\sum_{n=1}^{\infty} y_1(n,k;\lambda)\frac{t^{n-1}}{(n-1)!} = k\sum_{n=0}^{\infty} y_1(n,k;\lambda)\frac{t^n}{n!} - \sum_{n=0}^{\infty} y_1(n,k-1;\lambda)\frac{t^n}{n!}.$$

After some elementary calculations, comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the desired result. $\square$

Theorem 7. *Let $k \in \mathbb{N}$. The following identity holds:*

$$\frac{\partial}{\partial \lambda} y_1(n,k;\lambda) = \sum_{j=0}^{n} \binom{n}{j} y_1(j,k-1;\lambda).$$

Proof. Taking derivative of (8), with respect to λ, we obtain the following partial differential equation:

$$\frac{\partial}{\partial \lambda} F_{y_1}(t,k;\lambda) = e^t F_{y_1}(t,k-1;\lambda). \tag{15}$$

Combining (8) with the above equation, we get

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial \lambda} y_1(n,k;\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{n} \binom{n}{j} y_1(j,k-1;\lambda)\frac{t^n}{n!}.$$

After some elementary calculations, comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the desired result. $\square$

Theorem 8. *Let $k \in \mathbb{N}$. The following identity holds:*

$$\lambda \frac{\partial}{\partial \lambda} y_1(n,k;\lambda) = k y_1(n,k;\lambda) - y_1(n,k-1;\lambda).$$

Proof. By using (15), we obtain the following partial differential equation:

$$\lambda\frac{\partial}{\partial\lambda}F_{y_1}(t,k;\lambda)=kF_{y_1}(t,k;\lambda)-F_{y_1}(t,k-1;\lambda).$$

Combining (8) with the above equation, we get

$$\sum_{n=0}^{\infty}\lambda\frac{\partial}{\partial\lambda}y_1(n,k;\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}ky_1(n,k;\lambda)\frac{t^n}{n!}-\sum_{n=0}^{\infty}y_1(n,k-1;\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the desired result. $\square$

4. A family of new numbers $y_2(n,k;\lambda)$

In this section, we define a family of new numbers $y_2(n,k;\lambda)$ by means of the following generating function:

$$F_{y_2}(t,k;\lambda)=\frac{1}{(2k)!}\left(\lambda e^t+\lambda^{-1}e^{-t}+2\right)^k=\sum_{n=0}^{\infty}y_2(n,k;\lambda)\frac{t^n}{n!}, \tag{16}$$

where $k\in\mathbb{N}_0$ and $\lambda\in\mathbb{C}$.

By using (16) with their functional equation, we derive various identities and relations including our new numbers, the Fibonacci numbers, the Lucas numbers, the Stirling numbers and the central factorial numbers.

We get the following explicit formula for the numbers $y_2(n,k;\lambda)$:

Theorem 9. *Let $n,k\in\mathbb{N}$. The following identity holds:*

$$y_2(n,k;\lambda)=\frac{1}{(2k)!}\sum_{j=0}^{k}\binom{k}{j}2^{k-j}\sum_{l=0}^{j}\binom{j}{l}(2l-j)^n\lambda^{2l-j}. \tag{17}$$

Proof. By (16), we have

$$\sum_{n=0}^{\infty}y_2(n,k;\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(\frac{1}{(2k)!}\sum_{j=0}^{k}\binom{k}{j}2^{k-j}\sum_{l=0}^{j}\binom{j}{l}(2l-j)^n\lambda^{2l-j}\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the desired result. $\square$

For $k=0,1,2,3$ and $n=0,1,2,3,4,5$,we use (17) to compute a few values of the numbers $y_2(n,k;\lambda)$ as follows:

$n\backslash k$	0	1	2	3
0	1	$\frac{1}{2\lambda}+\frac{\lambda}{2}+1$	$\frac{\lambda^2+4\lambda}{24}+\frac{4\lambda+1}{24\lambda^2}+\frac{1}{4}$	$\frac{\lambda^3+6\lambda^2}{720}+\frac{\lambda}{48}+\frac{1}{48\lambda}+\frac{6\lambda+1}{720\lambda^3}+\frac{1}{36}$
1	0	$\frac{\lambda}{2}-\frac{1}{2\lambda}$	$\frac{\lambda^2+2\lambda}{12}-\frac{2\lambda+1}{6\lambda^2}$	$\frac{\lambda^3+4\lambda^2}{240}+\frac{\lambda}{48}-\frac{1}{48\lambda}-\frac{4\lambda+1}{240\lambda^3}$
2	0	$\frac{\lambda}{2}+\frac{1}{2\lambda}$	$\frac{\lambda^2+\lambda}{6}+\frac{\lambda+1}{6\lambda^2}$	$\frac{\lambda^3}{80}+\frac{\lambda^2}{30}+\frac{\lambda}{48}+\frac{1}{48\lambda}+\frac{1}{30\lambda^2}+\frac{1}{80\lambda^3}$
3	0	$\frac{\lambda}{2}-\frac{1}{2\lambda}$	$\frac{2\lambda^2+\lambda}{6}-\frac{\lambda+2}{6\lambda^2}$	$\frac{3\lambda^3}{80}+\frac{\lambda^2}{15}+\frac{\lambda}{48}-\frac{1}{48\lambda}-\frac{1}{15\lambda^2}-\frac{3}{80\lambda^3}$
4	0	$\frac{\lambda}{2}+\frac{1}{2\lambda}$	$\frac{2\lambda^2+\lambda}{3}+\frac{\lambda+4}{6\lambda^2}$	$\frac{9\lambda^3}{80}+\frac{2\lambda^2}{15}+\frac{\lambda}{48}+\frac{1}{48\lambda}+\frac{2}{15\lambda^2}+\frac{9}{80\lambda^3}$
5	0	$\frac{\lambda}{2}-\frac{1}{2\lambda}$	$\frac{8\lambda^2+\lambda}{6}-\frac{\lambda+8}{6\lambda^2}$	$\frac{27\lambda^3}{80}+\frac{4\lambda^2}{15}+\frac{\lambda}{48}-\frac{1}{48\lambda}-\frac{4}{15\lambda^2}-\frac{27}{80\lambda^3}$

Table 3: Some numerical values of the numbers $y_2(n,k;\lambda)$.

By using (8) and (16), we get the following functional equation:

$$F_{y_2}(t,k;\lambda)=\frac{k!}{(2k)!}\sum_{j=0}^{k}F_{y_1}(t,j;\lambda)F_{y_1}\left(-t,k-j;\lambda^{-1}\right).$$

By combining (8) and (16) with the above equation, we obtain

$$\sum_{n=0}^{\infty}y_2(n,k;\lambda)\frac{t^n}{n!}=\frac{k!}{(2k)!}\sum_{j=0}^{k}\left(\sum_{n=0}^{\infty}y_1(n,j;\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}(-1)^n y_1(n,k-j;\lambda^{-1})\frac{t^n}{n!}\right).$$

Therefore

$$\sum_{n=0}^{\infty}y_2(n,k;\lambda)\frac{t^n}{n!}=\frac{k!}{(2k)!}\sum_{n=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{n}(-1)^{n-l}\left(\begin{array}{c}n\\l\end{array}\right)y_1(l,j;\lambda)y_1(n-l,k-j;\lambda^{-1})\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, the numbers $y_2(n,k;\lambda)$ is given in terms of the numbers $y_1(n,k;\lambda)$ by the following theorem:

Theorem 10. *The following identity holds:*

$$y_2(n,k;\lambda)=\frac{k!}{(2k)!}\sum_{j=0}^{k}\sum_{l=0}^{n}(-1)^{n-l}\left(\begin{array}{c}n\\l\end{array}\right)y_1(l,j;\lambda)y_1(n-l,k-j;\lambda^{-1}). \tag{18}$$

Theorem 11. *The following identity holds:*

$$y_1(n,2k;\lambda)=\lambda^k\sum_{j=0}^{n}\left(\begin{array}{c}n\\j\end{array}\right)k^{n-j}y_2(j,k;\lambda).$$

Proof. By using (8) and (16), we get the following functional equation:

$$\lambda^k e^{kt}F_{y_2}(t,k;\lambda)=F_{y_1}(t,2k;\lambda).$$

From the above functional equation, we obtain

$$\sum_{n=0}^{\infty} y_1(n,2k;\lambda)\frac{t^n}{n!} = \lambda^k \sum_{n=0}^{\infty} \frac{(kt)^n}{n!} \sum_{n=0}^{\infty} y_2(n,k;\lambda)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} y_1(n,2k;\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\lambda^k \sum_{j=0}^{n} \binom{n}{j} k^{n-j} y_2(j,k;\lambda) \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the desired result. $\square$

By substituting $\lambda = 1$ into (16), we have

$$F_{y_2}(t,k) = \frac{1}{(2k)!}\left(e^t + e^{-t} + 2\right)^k.$$

The function $F_{y_2}(t,k)$ is an even function. Consequently, we get the following result:

$$y_2(2n+1,k;1) = 0.$$

Thus, we get

$$F_{y_2}(t,k) = F_{y_2}(t,k;1) = \sum_{n=0}^{\infty} y_2(n,k)\frac{t^{2n}}{(2n)!}. \tag{19}$$

By using (19), we give the following explicit formula for the numbers ($y_2(n,k) = y_2(n,k;1)$):

Corollary 1. *The following identity holds:*

$$y_2(n,k) = \frac{1}{(2k)!}\sum_{j=0}^{k} \binom{k}{j} 2^{k-j} \sum_{l=0}^{j} \binom{j}{l} (2l-j)^n. \tag{20}$$

From the equation (20), we see that

$$y_2(0,0) = 1.$$

For $k = 0,1,2,\ldots,9$ and $n \in \mathbb{N}$, we use (20) to compute a few values of the numbers $y_2(n,k)$ as follows:

$$y_2(n,0) = 0,$$

$$y_2(n,1) = \frac{(-1)^n + 1}{2},$$

$$y_2(n,2)=\frac{(-1)^n+1}{6}+\frac{2^{n-3}-(-2)^{n-3}}{3},$$

$$y_2(n,3)=\frac{(-1)^n+1}{48}+\frac{2^{n-3}-(-2)^{n-3}}{15}+\frac{3^{n-2}+(-3)^{n-2}}{80},$$

$$\begin{aligned}y_2(n,4)=\ &\frac{(-1)^n+1}{720}+\frac{2^{n-4}+(-2)^{n-4}}{105}+\frac{2^{n-5}-(-2)^{n-5}}{315}+\frac{3^{n-2}+(-3)^{n-2}}{560}\\&+\frac{4^{n-3}-(-4)^{n-3}}{630},\end{aligned}$$

$$\begin{aligned}y_2(n,5)=\ &\frac{(-1)^n+1}{17280}+\frac{2^{n-4}+(-2)^{n-4}+2^{n-5}-(-2)^{n-5}}{2835}+\frac{3^{n-2}+(-3)^{n-2}}{8960}\\&+\frac{2\left(4^{n-4}+(-4)^{n-4}\right)}{2835}+\frac{5^{n-2}+(-5)^{n-2}}{145152},\end{aligned}$$

$$\begin{aligned}y_2(n,6)=\ &\frac{(-1)^n+1}{604800}+\frac{2^{n-5}-(-2)^{n-5}+2^{n-10}+(-2)^{n-10}}{31185}+\frac{3^{n-4}+(-3)^{n-4}}{98560}\\&+\frac{3^{n-5}-(-3)^{n-5}}{12320}+\frac{4^{n-4}+(-4)^{n-4}}{31185}+\frac{2\left(4^{n-5}-(-4)^{n-5}\right)}{155925}\\&+\frac{5^{n-2}+(-5)^{n-2}}{1596672}+\frac{6^{n-5}-(-6)^{n-5}}{61600},\end{aligned}$$

$$\begin{aligned}y_2(n,7)=\ &\frac{(-1)^n+1}{29030400}+\frac{2^{n-3}-(-2)^{n-3}}{6081075}+\frac{2^{n-10}+(-2)^{n-10}}{405405}+\frac{3^{n-3}-(-3)^{n-3}}{7321600}\\&+\frac{3^{n-5}-(-3)^{n-5}}{640640}+\frac{4^{n-4}+(-4)^{n-4}}{1216215}+\frac{2\left(4^{n-5}-(-4)^{n-5}\right)}{2027025}\\&+\frac{5^{n-2}+(-5)^{n-2}}{38320128}+\frac{6^{n-5}-(-6)^{n-5}}{800800}+\frac{7^{n-2}+(-7)^{n-2}}{1779148800},\end{aligned}$$

$$\begin{aligned}y_2(n,8)=\ &\frac{(-1)^n+1}{1828915200}+\frac{7^{n-2}+(-7)^{n-2}}{26687232000}+\frac{5^{n-2}+(-5)^{n-2}}{1494484992}+\frac{3^{n-4}+(-3)^{n-4}}{64064000}\\&+\frac{4^{n-5}-(-4)^{n-5}}{18243225}+\frac{8^{n-5}-(-8)^{n-5}}{638512875}+\frac{4^{n-5}-(-4)^{n-5}}{30405375}+\frac{3^{n-5}-(-3)^{n-5}}{256256000}\\&+\frac{4^{n-6}+(-4)^{n-6}}{182432250}+\frac{3\left(6^{n-6}+(-6)^{n-6}\right)}{11211200}+\frac{2^{n-8}+(-2)^{n-8}}{18243225}\\&+\frac{2^{n-11}-(-2)^{n-11}}{6081075}+\frac{2^{n-7}-(-2)^{n-7}+2^{n-12}+(-2)^{n-12}}{91216125},\end{aligned}$$

$$
\begin{aligned}
y_2(n,9) = \;& \frac{(-1)^n+1}{146313216000} + \frac{7^{n-2}+(-7)^{n-2}}{853991424000} + \frac{5^{n-2}+(-5)^{n-2}}{83691159552} + \frac{9^{n-4}+(-9)^{n-4}}{975822848000} \\
&+ \frac{8^{n-5}-(-8)^{n-5}}{10854718875} + \frac{3^{n-5}-(-3)^{n-5}}{8712704000} + \frac{2\left(4^{n-5}-(-4)^{n-5}\right)}{1550674125} \\
&+ \frac{6^{n-6}+(-6)^{n-6}}{952952000} + \frac{3^{n-6}+(-3)^{n-6}}{871270400} + \frac{6^{n-7}-(-6)^{n-7}}{34034000} \\
&+ \frac{2\left(4^{n-7}-(-4)^{n-7}\right)}{1550674125} + \frac{3^{n-7}-(-3)^{n-7}}{536166400} + \frac{2^{n-7}-(-2)^{n-7}}{10854718875} \\
&+ \frac{2^{n-8}+(-2)^{n-8}}{1550674125} + \frac{2^{n-11}-(-2)^{n-11}}{310134825} + \frac{2^{n-12}+(-2)^{n-12}}{1550674125}.
\end{aligned}
$$

For $n=0$, we have

$$
y_2(0,0)=1, \quad y_2(0,1)=2, \quad y_2(0,2)=\frac{2}{3}, \quad y_2(0,3)=\frac{4}{45}, \quad y_2(0,4)=\frac{2}{315},
$$

and for $k=0,1,2,\ldots,9$ and $n=1,2,\ldots,9$, we use (20) to compute a few values of the numbers $y_2(n,k)$, as follows:

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
1	0	0	0	0	0	0	0	0	0	0
2	0	1	$\frac{2}{3}$	$\frac{2}{15}$	$\frac{4}{315}$	$\frac{2}{2835}$	$\frac{4}{155925}$	$\frac{4}{6081075}$	$\frac{8}{638512875}$	$\frac{2}{10854718875}$
3	0	0	0	0	0	0	0	0	0	0
4	0	1	$\frac{5}{3}$	$\frac{8}{15}$	$\frac{22}{315}$	$\frac{2}{405}$	$\frac{34}{155925}$	$\frac{8}{1216215}$	$\frac{92}{638512875}$	$\frac{2}{834978375}$
5	0	0	0	0	0	0	0	0	0	0
6	0	1	$\frac{17}{3}$	$\frac{47}{15}$	$\frac{184}{315}$	$\frac{152}{2835}$	$\frac{454}{155925}$	$\frac{634}{6081075}$	$\frac{1688}{638512875}$	$\frac{542}{10854718875}$
7	0	0	0	0	0	0	0	0	0	0
8	0	1	$\frac{65}{3}$	$\frac{338}{15}$	$\frac{1957}{315}$	$\frac{2144}{2835}$	$\frac{7984}{155925}$	$\frac{2672}{1216215}$	$\frac{41462}{638512875}$	$\frac{15206}{10854718875}$
9	0	0	0	0	0	0	0	0	0	0

Table 4: Some numerical values of the numbers $y_2(n,k)$.

This function is related to the $\cosh t$. That is,

$$
F_{y_2}(t,k) = \frac{2^k}{(2k)!}\left(\cosh t + 1\right)^k.
$$

By using this function, we get the following combinatorial sums:

Theorem 12. *Each of the following identities holds true:*

$$
y_2(n,k;1) = \frac{1}{(2k)!}\sum_{j=0}^{k}\binom{k}{j}2^{k-j}\sum_{l=0}^{j}\binom{j}{l}(2l-j)^{2n}.
$$

Also

$$\sum_{j=0}^{k}\left(\begin{array}{c}k\\j\end{array}\right)2^{k-j}\sum_{l=0}^{j}\left(\begin{array}{c}j\\l\end{array}\right)(2l-j)^{2n+1}=0.$$

Proof. By using (19), we have

$$\sum_{n=0}^{\infty}y_2(n,k)\frac{t^{2n}}{(2n)!}=\sum_{n=0}^{\infty}\left(\frac{1}{(2k)!}\sum_{j=0}^{k}\left(\begin{array}{c}k\\j\end{array}\right)2^{k-j}\sum_{l=0}^{j}\left(\begin{array}{c}j\\l\end{array}\right)(2l-j)^{2n}\right)\frac{t^n}{n!}.$$

Comparing the coefficients of t^{2n} on both sides of the above equation, we obtain the desired result. □

By using (19), we obtain

$$F_{y_1}(t,2k;1)e^{-kt}=\frac{k!}{(2k)!}\sum_{v=0}^{k}F_{y_1}(t,v;1)F_{y_1}(-t,k-v;1).$$

By using the above functional equation, we obtain the following theorem:

Theorem 13. *The following identity holds:*

$$\sum_{j=0}^{n}\left(\begin{array}{c}n\\j\end{array}\right)(-k)^{n-j}y_1(j,2k;1)=\frac{k!}{(2k)!}\sum_{j=0}^{n}(-1)^{n-j}\left(\begin{array}{c}n\\j\end{array}\right)\sum_{v=0}^{k}y_1(j,v;1)y_1(n-j,k-v;1).$$

Recall that the following identity has very important applications in theory of double series and its applications ([**29**, Lemma 11, Eq-(7)]):

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(n,k)=\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]}A(n,n-2k), \tag{21}$$

where $[x]$ denotes the greatest integer function.

Theorem 14. *The following identity holds:*

$$y_1(n,2k;1)=\sum_{j=0}^{\left[\frac{n}{2}\right]}\left(\begin{array}{c}n\\2j\end{array}\right)k^{n-2j}y_2(j,k;1).$$

Proof. By using (19), we obtain the following functional equation:

$$F_{y_1}(t,2k;1)=F_{y_2}(t,k)e^{kt}.$$

Combining this equation with (8), we get

$$\sum_{n=0}^{\infty}y_1(n,2k;1)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\frac{(kt)^n}{n!}\sum_{n=0}^{\infty}y_2(n,k;1)\frac{t^{2n}}{(2n)!}.$$

By using (21) in the above equation, we obtain

$$\sum_{n=0}^{\infty} y_1(n,2k;1)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{\left[\frac{n}{2}\right]} \frac{k^{n-2j}}{(2j)!(n-2j)!} y_2(j,k;1)\right) t^n.$$

Comparing the coefficients of t^n on both sides of the above equation, we obtain the desired result. $\square$

We now present a relation between the Lucas numbers L_n and the numbers $y_2(n,k;1)$ by the following theorem:

Theorem 15. *Let $a+b=1$, $ab=-1$ and $\frac{a-b}{2}=c=\frac{\sqrt{5}}{2}$. Then*

$$L_n^{(k)} = \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array}\right) (2j)!(-2)^{k-j} \sum_{m=0}^{\left[\frac{n}{2}\right]} \left(\begin{array}{c} n \\ 2m \end{array}\right) c^{2m} y_2(m,j;1)\left(\frac{k}{2}\right)^{n-2m},$$

where $L_n^{(k)}$ denote Lucas numbers of order k.

Proof. In [**22**, pp. 232-233] and [**8**], the Lucas numbers L_n are defined by means of the following generating function:

$$e^{at}+e^{bt} = \sum_{n=0}^{\infty} L_n \frac{t^n}{n!}.$$

From the above, we have

$$F_L(t,k;a,b) = \left(e^{at}+e^{bt}\right)^k = \sum_{n=0}^{\infty} L_n^{(k)} \frac{t^n}{n!}. \tag{22}$$

where

$$L_n^{(k)} = \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array}\right) L_n^{(m)} L_n^{(k-m)}.$$

By combining (22) with (16), we obtain the following functional equation

$$F_L(t,k;a,b) = e^{\frac{tk}{2}} \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array}\right) (-2)^{k-j}(2j)! F_{y_2}(ct,j;1).$$

Since $F_{y_2}(ct,j;1)$ is an even function, we have

$$\sum_{n=0}^{\infty} L_n^{(k)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{k}{2}\right)^n \frac{t^n}{n!} \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array}\right) (-2)^{k-j}(2j)! \sum_{m=0}^{\infty} y_2(m,j;1) c^m \frac{t^{2m}}{(2m)!}.$$

Using (21) in the above equation, we get

$$\sum_{n=0}^{\infty} L_n^{(k)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} (2j)!(-2)^{k-j} \sum_{m=0}^{\left[\frac{n}{2}\right]} \binom{n}{2m} c^{2m} y_2(m,j;1) \left(\frac{k}{2}\right)^{n-2m} \frac{t^n}{n!}.$$

Comparing the coefficients of t^n on both sides of the above equation, we obtain the desired result. □

We also present an identity including the Fibonacci numbers f_n, the Lucas numbers L_n and the numbers $y_1(n,k;1)$ by the following theorem:

Theorem 16. *Let $a+b=1$, $ab=-1$ and $\frac{a-b}{2}=c=\frac{\sqrt{5}}{2}$. Then*

$$L_n^{(k)} = k! \sum_{j=0}^{n} \binom{n}{j} (2c)^{n-j} y_1(n-j,k;1) \left(\left(a-2ck^j\right) f_j + f_{j-1} \right).$$

Proof. We set

$$F_f(t,a,b) = \frac{e^{at}-e^{bt}}{a-b} = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!}$$

(*cf.* [**22**, p. 232], [**8**]). By combining (22) and (8) with the above equation, we obtain the following functional equation

$$F_L(t,k;a,b) = k! F_{y_1}(2ct,k;1) \left(e^{akt} - 2cF_f(kt,a,b)\right).$$

Therefore

$$\begin{aligned} \sum_{n=0}^{\infty} L_n^{(k)} \frac{t^n}{n!} &= k! \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} a^j (2c)^{n-j} y_1(n-j,k;1) \frac{t^n}{n!} \\ &\quad - k! \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} (2c)^{n-j+1} y_1(n-j,k;1) k^j f_j \frac{t^n}{n!}. \end{aligned}$$

After some elementary calculations and comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the desired result. □

4.1. Recurrence relation for the numbers $y_2(n,k;\lambda)$

Here, taking derivative of (16), with respect to t, we give a recurrence relation for the numbers $y_2(n,k;\lambda)$.

Theorem 17. *Let $k \in \mathbb{N}$. Then*

$$y_2(n+1,k;\lambda) = ky_2(n,k;\lambda) - y_2(n,k-1;\lambda) - \lambda^{-1} \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} y_2(j,k-1;\lambda).$$

Proof. Taking derivative of (16), with respect to t, we obtain the following partial differential equation:

$$\frac{\partial}{\partial t}F_{y_2}(t,k;\lambda)=kF_{y_2}(t,k;\lambda)-F_{y_2}(t,k-1;\lambda)-\lambda^{-1}e^{-t}F_{y_2}(t,k-1;\lambda).$$

Combining (16) with the above equation, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}y_2(n,k;\lambda)\frac{t^{n-1}}{(n-1)!} &= k\sum_{n=0}^{\infty}y_2(n,k;\lambda)\frac{t^n}{n!}-\sum_{n=0}^{\infty}y_2(n,k-1;\lambda)\frac{t^n}{n!}\\ &\quad -\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}y_2(j,k-1;\lambda)\frac{t^n}{n!}.\end{aligned}$$

After some elementary calculation, comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the desired result. □

Theorem 18. *Let $n\in\mathbb{N}_0$ and $k\in\mathbb{N}$. Then*

$$2\lambda^2\frac{\partial}{\partial\lambda}y_2(n,k;\lambda)=\lambda^2\sum_{j=0}^{n}\binom{n}{j}y_2(j,k-1;\lambda)-\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}y_2(j,k-1;\lambda).$$

Proof. Taking derivative of (16), with respect to λ, we obtain the following partial differential equation:

$$2\lambda^2\frac{\partial}{\partial\lambda}F_{y_2}(t,k;\lambda)=F_{y_2}(t,k-1;\lambda)\left(e^t-\frac{1}{\lambda^2}e^{-t}\right).$$

Combining (16) with the above equation, we get

$$\begin{aligned}2\lambda^2\sum_{n=0}^{\infty}\frac{\partial}{\partial\lambda}y_2(n,k;\lambda)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}y_2(j,k-1;\lambda)\frac{t^n}{n!}\\ &\quad -\frac{1}{\lambda^2}\sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}y_2(j,k-1;\lambda)\frac{t^n}{n!}.\end{aligned}$$

After some elementary calculation, comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the desired result. □

5. λ-central factorial numbers $C(n,k;\lambda)$

In this section, we define λ-central factorial numbers $C(n,k;\lambda)$ by means of the following generating function: Let $k\in\mathbb{N}_0$ and $\lambda\in\mathbb{C}$. Then

(23) $$F_C(t,k;\lambda)=\frac{1}{(2k)!}\left(\lambda e^t+\lambda^{-1}e^{-t}-2\right)^k=\sum_{n=0}^{\infty}C(n,k;\lambda)\frac{t^n}{n!}.$$

For $\lambda = 1$, we have the central factorial numbers

$$T(n,k) = C(n,k;1)$$

(cf. [**2**], [**7**], [**18**], [**35**], [**41**]).

Theorem 19. *The following identity holds:*

$$C\left(n,k;\lambda^2\right) = 2^{-n}(2k)!\sum_{j=0}^{n}\binom{n}{j}C(j,k;\lambda)y_2(n-j,k;\lambda).$$

Proof. By using (16) and (23), we get the following functional equation:

$$F_C\left(2t,k;\lambda^2\right) = (2k)!F_C(t,k;\lambda)F_{y_2}(t,k;\lambda).$$

From this equation, we get

$$\sum_{n=0}^{\infty}C\left(n,k;\lambda^2\right)\frac{(2t)^n}{n!} = (2k)!\sum_{n=0}^{\infty}C(n,k;\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}y_2(n,k;\lambda)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty}C(n,k;\lambda^2)\frac{2^nt^n}{n!} = \sum_{n=0}^{\infty}(2k)!\sum_{j=0}^{n}\binom{n}{j}C(j,k;\lambda)y_2(n-j,k;\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the desired result. $\square$

By using (7) and (19), we obtain the following functional equation:

$$F_T(t,k;1)F_{y_2}(t,k;1) = \frac{1}{(2k)!}F_T(2t,k;1).$$

Combining the above equation with (7) and (19), we get

$$\frac{1}{(2k)!}\sum_{n=0}^{\infty}T(n,k)\frac{4^nt^{2n}}{(2n)!} = \sum_{n=0}^{\infty}y_2(n,k)\frac{t^n}{n!}\sum_{n=0}^{\infty}T(n,k)\frac{t^{2n}}{(2n)!}.$$

Therefore

$$\frac{1}{(2k)!}\sum_{n=0}^{\infty}4^nT(n,k)\frac{t^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{\left[\frac{n}{2}\right]}\binom{n}{2j}y_2(n-2j,k)T(j,k)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^{2n}}{(2n)!}$ on both sides of the above equation, we obtain a relationship between the central factorial numbers $T(n,k)$ and the numbers $y_2(j,k)$ by the following theorem:

Theorem 20. *If n is even an integer, then we have*

$$T(n,k) = 4^{-n}(2k)! \sum_{j=0}^{\left[\frac{n}{2}\right]} \binom{n}{2j} y_2(n-2j,k)T(j,k).$$

If n is an odd integer, then we have

$$\sum_{j=0}^{\left[\frac{n}{2}\right]} \binom{n}{2j} y_2(n-2j,k)T(j,k) = 0.$$

Remark 5. *In* [**3**]*, Alayont et al. have studied the rook polynomials, which count the number of ways of placing non-attacking rooks on a chess board. By using generalization of these polynomials, they gave the rook number interpretations of generalized central factorial and the Genocchi numbers.*

In [**2**]*, Alayont and Krzywonos gave the following result for the classical central factorial numbers:*

The number of ways to place k rooks on a size m triangle board in three dimensions is equal to $T(m+1, m+1-k)$, where $0 \leq k \leq m$.

6. Application: in the binomial distribution and in the Bernstein polynomials

Let n be a nonnegative integer. For every function $f : [0,1] \to \mathbb{R}$ and the n^{th} Bernstein polynomial of f is defined by

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{n}{k}\right) B_k^n(x),$$

where $B_k^n(x)$ denotes the Bernstein basis functions:

$$B_k^n(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

and $x \in [0,1]$. Let $(U_k)_{k \geq 1}$ be a sequence of independent distributed random variable having the uniform distribution on $[0,1]$ and defined by Adell *et al.* [**1**]:

$$S_n(x) = \sum_{k=1}^{n} 1_{[0,x]}(U_k).$$

In [**1**], it is well-know that, $S_n(x)$ is a binomial random variable. That is the theory of Probability and Statistics, the binomial distribution is very useful. This distribution, with parameters n and probability x, is the discrete probability distribution. This distribution is defined as follows:

$$P(S_n(x) = k) = \binom{n}{k} x^k (1-x)^{n-k},$$

where $k = 0, 1, 2, \cdots, n$. Let E denote mathematical expectation. Than

$$Ef\left(\frac{S_n(x)}{n}\right) = B_n(f, x)$$

(cf. [**1**]). For any $x \in (0, 1)$, $n \geq 2$, and $r > 1$, Adel *et al.* [**1**] defined

$$E\left(S_n(x)\right)^r = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) k^r x^k (1-x)^{n-k}. \tag{24}$$

Substituting $x = \frac{1}{2}$ into (24), we get

$$E\left(S_n\left(\frac{1}{2}\right)\right)^r = \frac{1}{2^n} \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) k^r. \tag{25}$$

By combining (9) with (25), we arrive at the following theorem:

Theorem 21. *Let $n \in \mathbb{N} \backslash \{1\}$. Let $r \in \mathbb{N}$. Then*

$$y_1(r, n) = \frac{2^n}{n!} E\left(S_n\left(\frac{1}{2}\right)\right)^r.$$

Integrating (24) from 0 to 1, we get

$$\int_0^1 E\left(S_n(x)\right)^r dx = \frac{1}{n+1} \sum_{k=0}^{n} k^r.$$

By substituting (2) into the above equation, after some elementary calculations, we get the following theorem:

Theorem 22. *The following identity holds:*

$$\int_0^1 E\left(S_n(x)\right)^r dx = \frac{1}{(n+1)(r+1)} \sum_{j=0}^{r} \sum_{l=0}^{r+1-j} \left(\begin{array}{c} n+1 \\ j \end{array}\right) \left(\begin{array}{c} r+1-j \\ l \end{array}\right) n^l B_l,$$

where B_l denotes the Bernoulli numbers.

7. Computation of the Euler numbers of negative order

In this section, we not only give elementary properties of the first and second kind Euler polynomials and numbers, but also compute the first kind of Apostol type Euler numbers associated with the numbers $y_1(n, k; \lambda)$ and $y_2(n, k; \lambda)$.

The second kind Apostol type Euler polynomials of order k, $E_n^{*(k)}(x;\lambda)$ are given by means of the following generating function:

$$F_P(t,x;k,\lambda)=\left(\frac{2}{\lambda e^t+\lambda^{-1}e^{-t}}\right)^k e^{tx}=\sum_{n=0}^{\infty}E_n^{*(k)}(x;\lambda)\frac{t^n}{n!}.$$

Substituting $x=0$ into the above equation, we get the second kind Apostol type Euler numbers of order k, with $k\geq 0$, $E_n^{*(k)}(\lambda)$ by means of the following generating function:

$$F_N(t;k,\lambda)=\left(\frac{2}{\lambda e^t+\lambda^{-1}e^{-t}}\right)^k=\sum_{n=0}^{\infty}E_n^{*(k)}(\lambda)\frac{t^n}{n!}.$$

If we substitute $k=\lambda=1$ into the above generating function, then we have

$$E_n^*=E_n^{*(1)}(1).$$

Substituting $x=0$ into the equation (3) with $-k$, we get the first kind Apostol-Euler numbers of order $-k$, $E_n^{(-k)}(\lambda)$ are given by means of the following generating function:

$$G_E(t,-k;\lambda)=\left(\frac{\lambda e^t+1}{2}\right)^k=\sum_{n=0}^{\infty}E_n^{(-k)}(\lambda)\frac{t^n}{n!}. \tag{26}$$

The second kind Apostol type Euler numbers of order $-k$ are defined by means of the following generating function:

$$F_N(t;-k,\lambda)=\left(\frac{\lambda e^t+\lambda^{-1}e^{-t}}{2}\right)^k=\sum_{n=0}^{\infty}E_n^{*(-k)}(\lambda)\frac{t^n}{n!}. \tag{27}$$

The numbers $E_n^{*(-k)}(\lambda)$ are related to the numbers $E_n^{(-k)}(\lambda)$ and the Apostol Bernoulli numbers $B_n^{(-k)}(\lambda)$ of the negative order. By using (27), we get the following functional equation:

$$F_N(t;-k,\lambda)=\sum_{j=0}^{k}\binom{k}{j}2^{j-k}t^{k-j}G_E(t,-j;\lambda)H_B(-t,-k+j;\lambda^{-1}),$$

where

$$H_B(t,-k;\lambda)=\left(\frac{\lambda e^t-1}{t}\right)^k=\sum_{n=0}^{\infty}B_n^{(-k)}(\lambda)\frac{t^n}{n!}$$

(cf. [**25**], [**27**], [**39**]). By using this equation, we get

$$\sum_{n=0}^{\infty}E_n^{*(-k)}(\lambda)\frac{t^n}{n!}=\sum_{j=0}^{k}\binom{k}{j}2^{j-k}t^{k-j}\sum_{n=0}^{\infty}E_n^{(-j)}(\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}B_n^{(-k+j)}(\lambda^{-1})\frac{(-t)^n}{n!}.$$

Therefore

$$\begin{aligned}\sum_{n=0}^{\infty} E_n^{*(-k)}(\lambda)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\sum_{l=0}^{n-k+j}(-1)^{n+j-k-l}\binom{n-k+j}{l} \\ &\times 2^{j-k}(n)_{k-j}E_l^{(-j)}(\lambda)B_{n+j-k-l}^{(-k+j)}(\lambda^{-1})\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem.

Theorem 23. *The following identity holds:*

$$\begin{aligned}E_n^{*(-k)}(\lambda) &= \sum_{j=0}^{k}\binom{k}{j}\sum_{l=0}^{n-k+j}(-1)^{n+j-k-l}\binom{n-k+j}{l}2^{j-k}(n)_{k-j} \\ &\times E_l^{(-j)}(\lambda)B_{n+j-k-l}^{(-k+j)}(\lambda^{-1}).\end{aligned}$$

We observe that the second kind Euler numbers of negative order $E_n^{*(-k)}$ have been computed by Liu [**24**].

By using the numbers $y_1(n,k;\lambda)$, we are ready to compute the first kind Euler numbers of negative order.

Combining (8) and (26), we get

$$k!2^k\sum_{n=0}^{\infty} y_1(n,k;\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty} E_n^{(-k)}(\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem.

Theorem 24. *Let* $k\in\mathbb{N}_0$*. Then*

$$E_n^{(-k)}(\lambda) = k!2^{-k}y_1(n,k;\lambda). \tag{28}$$

Remark 6. *Substituting* $\lambda = 1$ *into* (28)*, we obtain the following explicit formula for the first kind Euler numbers of order* $-k$*:*

$$E_n^{(-k)} = 2^{-k}B(n,k). \tag{29}$$

From the equation (29), we see that

$$E_0^{(0)} = 1.$$

For $k = 0,1,2,\ldots,7$ and $n\in\mathbb{N}$, we use (29) to compute a few values of the numbers

$E_n^{(-k)}$ as follows:

$$
\begin{aligned}
E_n^{(0)} &= 0, \\
E_n^{(-1)} &= \frac{1}{2}, \\
E_n^{(-2)} &= 2^{n-2} + \frac{1}{2}, \\
E_n^{(-3)} &= \frac{3^n}{8} + 3.2^{n-3} + \frac{3}{8}, \\
E_n^{(-4)} &= \frac{3^n}{4} + 4^{n-2} + 3.2^{n-3} + \frac{1}{4}, \\
E_n^{(-5)} &= \frac{5^n}{32} + \frac{5.3^n}{16} + \frac{5.4^{n-2}}{2} + 5.2^{n-4} + \frac{5}{32}, \\
E_n^{(-6)} &= \frac{6^n}{64} + \frac{3.5^n}{32} + \frac{5.3^n}{16} + 15.4^{n-3} + 15.2^{n-6} + \frac{3}{32}, \\
E_n^{(-7)} &= \frac{7^n}{128} + \frac{7.6^n}{128} + \frac{21.5^n}{128} + \frac{35.3^n}{128} + \frac{35.4^{n-3}}{2} + 21.2^{n-7} + \frac{7}{128}, \\
&\vdots
\end{aligned}
$$

That is for $n = 0, 1, 2, \ldots, 9$ and $k = 0, 1, 2, \ldots, 9$, we compute a few values of the numbers $E_n^{(-k)}$, given by the above relations, as follows:

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1	1
1	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$
2	0	$\frac{1}{2}$	$\frac{3}{2}$	3	5	$\frac{15}{2}$	$\frac{21}{2}$	14	18	$\frac{45}{2}$
3	0	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{27}{4}$	14	25	$\frac{81}{2}$	$\frac{245}{4}$	88	$\frac{243}{2}$
4	0	$\frac{1}{2}$	$\frac{9}{2}$	$\frac{33}{2}$	$\frac{85}{2}$	90	168	287	459	$\frac{1395}{2}$
5	0	$\frac{1}{2}$	$\frac{17}{2}$	$\frac{171}{4}$	137	$\frac{1375}{4}$	738	1421	2524	4212
6	0	$\frac{1}{2}$	$\frac{33}{2}$	$\frac{231}{2}$	$\frac{925}{2}$	$\frac{5505}{4}$	$\frac{13587}{4}$	7364	14508	26550
7	0	$\frac{1}{2}$	$\frac{65}{2}$	$\frac{1287}{4}$	1619	5725	$\frac{65007}{4}$	$\frac{317275}{8}$	86608	173664
8	0	$\frac{1}{2}$	$\frac{129}{2}$	$\frac{1833}{2}$	$\frac{11665}{2}$	$\frac{49155}{2}$	$\frac{160671}{2}$	$\frac{441469}{2}$	$\frac{1068453}{2}$	1173240
9	0	$\frac{1}{2}$	$\frac{257}{2}$	$\frac{10611}{4}$	21497	$\frac{433225}{4}$	$\frac{816561}{2}$	$\frac{5055869}{4}$	3390874	$\frac{32620563}{4}$

Table 5: Some numerical values of the numbers $E_n^{(-k)}$.

Theorem 25. *Let $k \in \mathbb{N}_0$. Then*

$$
y_2(n, k; \lambda) = \frac{2^k}{(2k)!} \sum_{l=0}^{k} \binom{k}{l} E_n^{*(-l)}(\lambda).
$$

Proof. By using (16) and (27), we get the following functional equation:

$$F_{y_2}(t,k;\lambda)=\frac{2^k}{(2k)!}\sum_{l=0}^{k}\binom{k}{l}F_N(t;-l,\lambda).$$

From this equation, we obtain

$$\sum_{n=0}^{\infty}y_2(n,k;\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(\frac{1}{(2k)!}\sum_{l=0}^{k}\binom{k}{l}2^kE_n^{*(-l)}(\lambda)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the desired result. □

Theorem 26. *Let $k\in\mathbb{N}_0$. Then*

$$y_2(n,k;\lambda)=\frac{2^k}{(2k)!}\sum_{l=0}^{k}\binom{k}{l}\lambda^{-l}E_n^{(-k)}\left(-l;\lambda\right).$$

Proof. By using (16) and (3), we get the following functional equation:

$$F_{y_2}(t,k;\lambda)=\frac{2^k}{(2k)!}\sum_{l=0}^{k}\binom{k}{l}\lambda^{-l}F_{P1}\left(t,-l;-k,\lambda\right).$$

From this equation, we obtain

$$\sum_{n=0}^{\infty}y_2(n,k;\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(\frac{2^k}{(2k)!}\sum_{l=0}^{k}\binom{k}{l}\lambda^{-l}E_n^{(-k)}\left(-l;\lambda\right)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the desired result. □

By applying derivative operator to the generating function in (8), we give a relationship between the numbers $y_1(n,k;\lambda)$ and $E_n^{(-1)}(\lambda)$ as in the following theorem:

Theorem 27. *Let $n,k\in\mathbb{N}_0$. Then*

$$\begin{aligned}(30)\qquad y_1(n+2,k;\lambda) &= k^2y_1(n,k;\lambda)\\ &\quad+\frac{k\left(2k-3\right)}{2}\sum_{l=0}^{n}\binom{n}{l}y_1(n-l,k;\lambda)E_l\left(\lambda\right)\\ &\quad+\frac{k(k-1)}{4}\sum_{l=0}^{n}\binom{n}{l}E_l^{(2)}\left(\lambda\right)y_1(n-l,k;\lambda).\end{aligned}$$

Let $n\in\mathbb{N}\backslash\left\{1\right\}$. Then

$$(31)\qquad y_1(n+2,k;\lambda)=k^2y_1(n,k;\lambda)+y_1(n,k-2;\lambda)-(2k-1)y_1(n,k-1;\lambda).$$

Proof of (30). By applying derivative operator to (8) with respect to t, we obtain the following partial differential equation:

$$\begin{aligned}\frac{\partial^2}{\partial t^2}F_{y_1}(t,k;\lambda) &= k^2F_{y_1}(t,k;\lambda)+\frac{k\,(2k-3)}{2}F_{P1}(t,0;1,\lambda)F_{y_1}(t,k;\lambda)\\ &\quad+\frac{k\,(k-1)}{4}F_{P1}(t,0;2,\lambda)F_{y_1}(t,k;\lambda).\end{aligned}$$

Combining (8) and (3) with the above equation, we get

$$\begin{aligned}&\sum_{n=2}^{\infty}y_1(n,k;\lambda)\frac{t^{n-2}}{(n-2)!}\\ =\;& k^2\sum_{n=0}^{\infty}y_1(n,k;\lambda)\frac{t^n}{n!}+\frac{k\,(2k-3)}{2}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\begin{pmatrix} n\\ l\end{pmatrix}y_1(n-l,k;\lambda)E_l\left(\lambda\right)\frac{t^n}{n!}\\ &+\frac{k\,(k-1)}{4}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\begin{pmatrix} n\\ l\end{pmatrix}E_l^{(2)}\left(\lambda\right)y_1(n-l,k;\lambda)\frac{t^n}{n!}.\end{aligned}$$

We make a suitable arrangement of the series and then compare the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, and we obtain the first assertion (30). $\square$

Proof of (31). Similarly, by applying derivative operator to (8) with respect to t, we obtain the following partial differential equation:

$$\frac{\partial^2}{\partial t^2}F_{y_1}(t,k;\lambda)=k^2F_{y_1}(t,k;\lambda)-\left(2k-1\right)F_{y_1}(t,k-1;\lambda)+F_{y_1}(t,k-2;\lambda).$$

Combining (8) with the above equation, we get

$$\begin{aligned}&\sum_{n=2}^{\infty}y_1(n,k;\lambda)\frac{t^{n-2}}{(n-2)!}\\ =\;&\sum_{n=0}^{\infty}\left(k^2y_1(n,k;\lambda)-\left(2k-1\right)y_1(n,k-1;\lambda)+y_1(n,k-2;\lambda)\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the second assertion (31). $\square$

8. Algorithms and Computation

In computer science and applied mathematics, one investigate information and computation and also their theoretical foundations. In these areas practical techniques are very important. Therefore algorithmic processes play a very important role in both areas. Thus, in this section, we give two algorithms for the computation of the numbers $y_1(n,k;\lambda)$ and $y_2(n,k;\lambda)$.

Algorithm 1 Let n be a positive integer and $\lambda \neq 0$. This algorithm will return value of the numbers $y_1(n,k;\lambda)$ given by equation (9).

```
procedure y_1(n, k, λ)
    Begin
    Inputs:
    Y_1 ← 0
    Outputs:
    y_1(n, k, λ) ← Y_1
    if n = 0 and k = 0 then
        Y_1 = 1
    else
        for all j in {0, 1, 2, ..., k} do
            Y_1 ← Y_1 + Binomial_Coef (k, j) * Power (j, n) * Power (λ, j)
        end for
    end if
    Y_1 ← (1/k!) * Y_1
    return Y_1
end procedure
```

Algorithm 2 Let n be a positive integer and $\lambda \neq 0$. This algorithm will return value of the numbers $y_2(n,k;\lambda)$ given by equation (18) obtained by the y_1 numbers in equation (9).

```
procedure y_2(n, k, λ)
    Begin
    Inputs:
    Y_2 ← 0
    Outputs:
    y_2(n, k, λ) ← Y_2
    if n = 0 and k = 0 then
        Y_2 = 1
    else
        for all j in {0, 1, 2, ..., k} do
            for all l in {0, 1, 2, ..., n} do
                Y_2 ← Y_2 + Power (−1, n − l) * Binomial_Coef (n, l) * y_1(l, j, λ)
                * y_1(n − l, k − j, Power (λ, −1))
            end for
        end for
    end if
    Y_2 ← (k!/(2k)!) * Y_2
    return Y_2
end procedure
```

9. Combinatorial applications and further remarks

In this section, we discuss some combinatorial interpretations of these numbers, as well as the generalization of the central factorial numbers given in Section 3-5. These interpretations include the rook numbers and polynomials and combinatorial interpretation for the numbers $y_1(n,k)$. We see that our numbers are associated with known counting problems. By using counting techniques and generating function techniques, Bona [**5**] rederived several known properties and novel relations involving enumerative combinatorics and related areas. A very interesting further special case of the numbers $B(n,k)$ is worthy of note by the work of Bona [**5**]. That is, in [**5**, P. 46, Exercise 3-4], Bona gave the following two exercises which are associated with the numbers $B(n,k)$:

Exercise 3. Find the number of ways to place n rooks on an $n \times n$ chess board so that no two of them attack each other.

Exercise 4. How many ways are there to place some rooks on an $n \times n$ chess board so that no two of them attack each other?

Remark 7. *Our numbers occur in combinatorics applications. In* [**5**, p. 46, Exercise 3-4]*, Bona gave detailed and descriptive solution of these two exercises, which are related to the numbers $B(n,k)$, respectively, as follows:*

There has to be one rook in each column. The first rook can be anywhere in its column (n possibilities). The second rook can be anywhere in its column except in the same row where the first rook is, which leaves $n-1$ possibilities. The third rook can be anywhere in its column, except in the rows taken by the first and second rook, which leaves $n-2$ possibilities, and so on, leading to $n(n-1)\cdots 2.1 = n!$ possibilities.

Exercise 4. If we place k rooks, then we first need to choose the k columns in which these rooks will be placed. We can do that in $\binom{n}{k}$ ways. Continuing the line of thought of the solution of the previous exercise, we can then place our k rooks into the chosen columns in $(n)_k$ ways. Therefore, the total number of possibilities is

$$\sum_{k=1}^{n} \binom{n}{k} (n)_k.$$

Remark 8. *In* (14)*, for $j < d$, it is well-known that*

$$\binom{j}{d} = 0.$$

Therefore, we arrive at solutions of Exercise 16 (a) in [**5**, p. 55, Exercise 16(a)] *and also Exercise* 10 [**11**, p. 126] *as follows:*

$$2^{n-k} \binom{k}{d} = \sum_{j=d}^{k} \binom{k}{j} \binom{j}{d}.$$

10. Conclusions

In this paper, we have constructed some new families of special numbers with their generating functions. We give many properties of these numbers. These numbers are related to many well-known numbers, which are Bernoulli numbers, Euler numbers, Stirling numbers of the second kind, central factorial numbers and also related to the Golombek's problem [**15**] "Aufgabe 1088". We have discussed some combinatorial interpretations of these numbers. Besides, we give some applications about not only rook polynomials and numbers, but also combinatorial sum.

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