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New Finite Difference Methods for Singularly Perturbed Convection-diffusion Equations

Xuefei He and Kun Wang*

Abstract. In this paper, a family of new finite difference (NFD) methods for solving the convection-diffusion equation with singularly perturbed parameters are considered. By taking account of infinite terms in the Taylor's expansions and using the triangle function theorem, we construct a series of NFD schemes for the one-dimensional problems firstly and derive the error estimates as well. Then, applying the ADI technique, the idea is extended to two dimensional equations. Besides no numerical oscillation, there are mainly three advantages for the proposed methods: one is that the schemes can achieve the predicted convergence orders on uniform mesh regardless of the perturbed parameter for 1D equations; Secondly, no matter which convergence order the scheme is, the generated linear systems have diagonal structures; Thirdly, the methods are easily expanded to the special mesh technique such as Shishkin mesh. Some numerical experiments are shown to verify the prediction.

1. Introduction

We consider the following convection-diffusion equation

$$-\varepsilon\Delta u + \alpha u_x + \beta u_y = f, \quad (x,y) \in \Omega = (0,1)\times(0,1), \tag{1.1}$$

where $\varepsilon > 0$ is the diffusion coefficient, the convection coefficients α, β and the source term f are assumed to be sufficiently smooth functions. As is known to all, this problem plays an important role in the computational fluid dynamics. When ε is small enough, the equation (1.1) becomes a singularly perturbed problem, which is very difficult to simulate due to the well-known nonphysical oscillation [21].

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*Corresponding author.

To overcome this difficulty, lots of methods have been put forward. One of the popular methods is the up-wind scheme, which is considered in [10, 14, 18]. Unfortunately, simple up-wind scheme can not capture the features of the boundary layer exactly because of the pollution effects. Due to the existence of the boundary layer where the solution of this problem is the most troublesome, lots of researchers also focus on constructing some special meshes to divide the computational domain (see [1, 8, 9]). One of the most famous meshes is the Shishkin mesh proposed in 1990s and has been continued in [15, 16, 18]. Many kinds of robust numerical methods based on layer-adapted meshes were introduced in [20], which can improve the accuracy and stabilities of the numerical scheme. Based on the Richardson extrapolation technique, Sun [23] proposed an operator interpolation scheme that can improve the computational accuracy. Beside special adaptively graded and patched meshes, another technique named defect-correction is also used in [1, 6, 7]. On the other hand, as one of the famous numerical methods, high-order finite difference schemes have also been developed for solving the singularly perturbed problem in the past decades. To enhance the convective stability of the convection-diffusion equation, Chiu and Sheu [3] constructed a dispersion-relation-preserving dual-compact upwind scheme. Chu and Fan [4, 5] developed a general combined compact difference scheme which has sixth-order accuracy. Another high-order finite difference method is the exponential finite difference method. Pillai [19] developed a fourth-order exponential finite difference method for the convection-diffusion problem. Similarly research is also done by Tian and Dai [24], in which fourth order schemes for the convection-diffusion equation with both constant and variable coefficients are obtained. Other kinds of methods, such as finite element method and finite volume method, for solving the convection-diffusion equation are investigated in [2, 11–13, 17, 29].

It is well-known that only finite terms are calculated when deducing the traditional finite difference methods based on Taylor's expansions. And more terms are taken into account in constructing the scheme, higher convergence order and accurate results will be got. However, the solution of convection-diffusion equation with singularly perturbed coefficient satisfies $|u^{(k)}| \leq 1/\varepsilon^k$. Thus, the local truncation error will be huge when ε is very small by applying the classical finite difference method. In this paper, by extending the idea in [27, 28] to the singularly perturbed problem, we investigate a kind of new finite difference methods by calculating infinite terms in Taylor's expansions. The proposed schemes can achieve the predicted convergence order on uniform mesh regardless of the singularly perturbed coefficient, and reach higher computational accuracy compared with other well-known ones.

Furthermore, these new finite difference schemes are easily expanded to special mesh techniques, such as Shishkin mesh, and much better approximation results can be got than the ones in the literatures. But different from the well-known Shishkin mesh method which strongly depend on the location of the boundary layer and the asymptotic behavior of the analytical solution, the proposed new methods can be easily extend to the multiple turning point problems and the nonlinear problems in which the oscillation and asymptotic behavior are much complicated.

The remainder paper is organized as follows. A kind of new finite difference methods are proposed for one dimensional convection-diffusion equations with constant and variable coefficients in Section 2. The convergence order is also derived here. Then, in Section 3, the idea is extended to the problems in two dimension. Some numerical experiments are shown to verify the efficiency of the algorithms in Section 4 and conclusions are stated in Section 5.

2. New scheme in 1D

In this section, we will begin with a series of NFD schemes for the one dimensional (1D) equation with Dirichlet boundary condition which can be reduced from (1.1) directly

$$-\varepsilon\frac{\partial^2 u}{\partial x^2}+\alpha(x)\frac{\partial u}{\partial x}=f(x),\quad x\in\Omega=(0,1), \tag{2.1}$$

where $\alpha(x)\neq 0$ for all $x\in\Omega$ is the convection coefficient. Before proceeding the deduction of the algorithms, we recall the stability results for the equation (2.1). Since we are particularly interested in the problem with small singularly perturbed coefficient ε, all of the deduction in the following will be based on this assumption.

Lemma 2.1. [21] *Assume that $\alpha(x)>\alpha_0>0$ with α_0 being a constant and $\alpha(x)$, $f(x)$ are sufficiently smooth. Then the solution of the problem* (2.1) *with homogeneous Direchlet boundary conditions satisfies*

$$|u^{(n)}(x)|\leq C\left[1+\varepsilon^{-n}\exp\left(-\alpha_0\frac{1-x}{\varepsilon}\right)\right],\quad n\in Z^+,\ x\in(0,1). \tag{2.2}$$

It is well-known that finite difference schemes are based on the following Taylor's expansions

$$u_{i+1}-u_i=hu_i^{(1)}+\frac{h^2}{2!}u_i^{(2)}+\cdots+\frac{h^n}{n!}u_i^{(n)}+\cdots, \tag{2.3}$$

$$u_{i-1}-u_i=(-h)u_i^{(1)}+\frac{(-h)^2}{2!}u_i^{(2)}+\cdots+\frac{(-h)^n}{n!}u_i^{(n)}+\cdots, \tag{2.4}$$

where $0<h<1$ is an uniform mesh size satisfying $h=1/N$, and $x_i=ih$, $u_i=u(x_i)$, $f_i=f(x_i)$, $(i=0,1,2,\ldots,N)$.

For example, the first two terms are contained in the central finite difference scheme, the first four terms are included in the compact fourth order finite difference scheme and so on. In all of these cases, a local truncation error in the form of $\frac{h^n}{n!}u_i^{(n)}$ will be generated. Due to Lemma 2.1, it holds that $\left|\frac{h^n}{n!}u_i^{(n)}\right| = O((h/\varepsilon)^n)$, which suggests that the predicted convergence order of the classical methods can't be achieved when ε is sufficient small.

Remark 2.2. For general domain (a, b) and inhomogeneous boundary condition, we can first transform them to $(0, 1)$ and the homogeneous one respectively, then the similar conclusion as Lemma 2.1 can be obtained as well. Moreover, when $\alpha(x) < 0$, we just need to replace $1 - x$ with x in (2.2).

2.1. Constant coefficient

In this subsection, we will first develop the new finite difference (NFD) schemes for (2.1) under the assumption that $\alpha > 0$ is a constant function.

In fact, according to the original equation (2.1), it yields that, for $n \geq 2$

$$u^{(n)} = \left(\frac{\alpha}{\varepsilon}\right)^{n-1} u^{(1)} - \frac{1}{\varepsilon}\sum_{k=0}^{n-2}\left(\frac{\alpha}{\varepsilon}\right)^{n-2-k} f^{(k)}. \tag{2.5}$$

Substituting (2.5) into the right-hand side of (2.3), we obtain

$$\begin{aligned} u_{i+1} - u_i &= hu_i^{(1)} + \sum_{n=2}^{+\infty}\frac{h^n}{n!}\left[\left(\frac{\alpha}{\varepsilon}\right)^{n-1} u_i^{(1)} - \frac{1}{\varepsilon}\sum_{k=0}^{n-2}\left(\frac{\alpha}{\varepsilon}\right)^{n-2-k} f_i^{(k)}\right] \\ &= \sum_{n=1}^{+\infty}\frac{h^n}{n!}\left(\frac{\alpha}{\varepsilon}\right)^{n-1} u_i^{(1)} - \frac{1}{\varepsilon}\sum_{n=2}^{+\infty}\frac{h^n}{n!}\sum_{k=0}^{n-2}\left(\frac{\alpha}{\varepsilon}\right)^{n-2-k} f_i^{(k)} \\ &= \frac{\varepsilon}{\alpha}(e^r - 1)u_i^{(1)} - F^+. \end{aligned} \tag{2.6}$$

Similarly, (2.4) could also be rewritten in the same way

$$u_{i-1} - u_i = \frac{\varepsilon}{\alpha}(e^{-r} - 1)u_i^{(1)} - F^-, \tag{2.7}$$

where

$$\begin{aligned} r &= \frac{\alpha h}{\varepsilon}, \\ F^+ &= \frac{1}{\varepsilon}\sum_{n=2}^{+\infty}\frac{h^n}{n!}\sum_{k=0}^{n-2}\left(\frac{\alpha}{\varepsilon}\right)^{n-2-k} f_i^{(k)} = \sum_{n=0}^{+\infty}\frac{\varepsilon^{n+1}}{\alpha^{n+2}}\left[e^r - \sum_{l=0}^{n+1}\frac{r^l}{l!}\right] f_i^{(n)}, \\ F^- &= \frac{1}{\varepsilon}\sum_{n=2}^{+\infty}\frac{(-h)^n}{n!}\sum_{k=0}^{n-2}\left(\frac{\alpha}{\varepsilon}\right)^{n-2-k} f_i^{(k)} = \sum_{n=0}^{+\infty}\frac{\varepsilon^{n+1}}{\alpha^{n+2}}\left[e^{-r} - \sum_{l=0}^{n+1}\frac{(-r)^l}{l!}\right] f_i^{(n)}. \end{aligned}$$

Multiplying (2.6) and (2.7) by e^{-r} and eliminating $u_i^{(1)}$, we have

$$\begin{aligned} &(e^{-2r} - e^{-r})u_{i-1} + (e^{-r} - e^{-3r})u_i + (e^{-3r} - e^{-2r})u_{i+1} \\ &= (e^{-r} - e^{-2r})F^- - (e^{-3r} - e^{-2r})F^+. \end{aligned}$$

Multiplying $1/(e^{-2r} - e^{-r})$ in the above formula, we get

$$u_{i-1} - (e^{-r}+1)u_i + e^{-r}u_{i+1} = \sum_{m=0}^{+\infty} Y_m, \tag{2.8}$$

where

$$Y_m = \frac{\varepsilon^{m+1}}{\alpha^{m+2}} \left[\sum_{l=1}^{m+1} \frac{r^l}{l!} \left[e^{-r} + (-1)^l \right] \right] f_i^{(m)}.$$

Let U_i denote the approximation of u_i, taking the first n terms on the right-hand side of (2.8), we will arrive at the NFD scheme for the 1D convection-diffusion equation (2.1) at the interior grid point as follows

$$U_{i-1} - (e^{-r}+1)U_i + e^{-r}U_{i+1} = \sum_{m=0}^{n} Y_m. \tag{2.9}$$

And a family of NFD schemes will be got when we take different n into calculation.

2.2. Variable coefficient

This subsection is devoted to deriving the new finite difference (NFD) schemes for the convection-diffusion equation when $\alpha(x)$ is a function with respect to x. We also assume that $\alpha(x) > 0$ for all $x \in \Omega$. Thanks to (2.1), it holds, for $n \geq 2$, that

$$\begin{aligned} u^{(n)} &= \frac{1}{\varepsilon}[\alpha(x)u^{(1)} - f]^{(n-2)} \\ &= \frac{1}{\varepsilon}\left[C_{n-2}^{n-2}\alpha u^{(n-1)} + \cdots + C_{n-2}^{k}\alpha^{(n-2-k)}u^{(k+1)} + \cdots + C_{n-2}^{0}\alpha^{(n-2)}u^{(1)} - f^{(n-2)} \right], \end{aligned}$$

where $\alpha^{(k)}$, $u^{(k)}$ denote the k-th order derivative of $\alpha(x)$ and $u(x)$ respectively.

Then, using (2.1) recursively, we can also rewrite $u^{(n)}$ as an analogous form to (2.5) which only contains $u^{(1)}$ and the derivatives of f. The relationship is shown in Table 2.1.

	$u^{(1)}$	f	$f^{(1)}$	$f^{(2)}$	$f^{(3)}$	$\cdots$	$f^{(n-2)}$	$\cdots$
$u^{(2)}$	1	1						
$u^{(3)}$	$2,1$	2	1					
$u^{(4)}$	$3,2,1$	$3,2$	2	1				
$u^{(5)}$	$4,3,2,1$	$4,3,2$	$3,2$	2	1			
$\vdots$								
$u^{(n)}$	$n-1,\ldots,1$	$n-1,\ldots,2$	$n-2,\ldots,2$	$n-3,\ldots,2$	$n-4,\ldots,2$	$\cdots$	1	
$\vdots$								

Table 2.1: The power of $1/\varepsilon$ in the coefficients of $u^{(1)}$ and $f^{(k)}$ $(k \geq 0)$.

Obviously, $u^{(n)}$ $(n \geq 3)$ has the following form which is similar to (2.5)

$$u^{(n)} = \sum_{k=1}^{n-1} P^k u^{(1)} + \sum_{j=0}^{n-3} \left[\sum_{k=2}^{n-j-1} Q_j^k \right] f^{(j)} + \frac{1}{\varepsilon} f^{(n-2)}, \tag{2.10}$$

where both P^k and Q_j^k are the coefficients with respect to $1/\varepsilon^k$.

Although there are infinite terms in $u^{(n)}$ when n tends to infinity according to Table 2.1, these terms can be arranged according to the power of $1/\varepsilon$. By collecting the contribution of $1/\varepsilon$ with different order, we get different finite-term-approximations to $u^{(n)}$ for all n. Then substituting these approximation terms into (2.3) and (2.4), and using the triangle function formula, we have different new finite difference (NFD) schemes. Three of them are shown in the following.

First, we collect the contribution of the terms which with respect to $1/\varepsilon^{n-1}$ in $u^{(n)}$ $(n \geq 2)$ which are included in the coefficients of $u^{(1)}$, f only according to Table 2.1. For each $u^{(n)}$ $(n \geq 2)$, after simply calculating, we have

$$u^{(n)} \approx \frac{\alpha^{n-1}}{\varepsilon^{n-1}} u^{(1)} - \frac{\alpha^{n-2}}{\varepsilon^{n-1}} f. \tag{2.11}$$

For the interior point x_i, substituting (2.11) into (2.3) and (2.4), and multiplying them by e^{-r_i}, it follows

$$e^{-r_i}(u_{i+1} - u_i) \approx \widehat{P}_i^1 u_i^{(1)} + \widehat{Q}_{0,i}^1 f_i, \tag{2.12}$$

$$e^{-r_i}(u_{i-1} - u_i) \approx \overline{P}_i^1 u_i^{(1)} + \overline{Q}_{0,i}^1 f_i, \tag{2.13}$$

where

$$\begin{aligned} r_i &= \frac{h\alpha_i}{\varepsilon}, \\ \widehat{P}_i^1 &= \frac{\varepsilon}{\alpha_i}(1 - e^{-r_i}), & \widehat{Q}_{0,i}^1 &= -\frac{\varepsilon}{\alpha_i^2}(1 - e^{-r_i} - r_i e^{-r_i}), \\ \overline{P}_i^1 &= \frac{\varepsilon}{\alpha_i}(e^{-2r_i} - e^{-r_i}), & \overline{Q}_{0,i}^1 &= -\frac{\varepsilon}{\alpha_i^2}(e^{-2r_i} - e^{-r_i} + r_i e^{-r_i}). \end{aligned}$$

Combing with (2.12) and (2.13), we have

$$A_1 U_{i-1} + A_2 U_i + A_3 U_{i+1} = Y_0 f_i,$$

where

$$\begin{aligned} A_1 &= -e^{-r_i} \widehat{P}_i^1, & A_2 &= e^{-r_i}(\widehat{P}_i^1 - \overline{P}_i^1), \\ A_3 &= e^{-r_i} \overline{P}_i^1, & Y_0 &= \overline{P}_i^1 \widehat{Q}_{0,i}^1 - \widehat{P}_i^1 \overline{Q}_{0,i}^1. \end{aligned}$$

To determine the second NFD scheme, all terms which are with respect to $1/\varepsilon^{n-1}$, $1/\varepsilon^{n-2}$ on the right-hand side of (2.10) are taken into account. The finite-term-approximations of $u^{(n)}$ for all n, after simple calculation, is

$$u^{(n)} \approx \frac{\alpha^{n-1}}{\varepsilon^{n-1}}u^{(1)} - \frac{\alpha^{n-2}}{\varepsilon^{n-1}}f + \frac{(n-1)(n-2)}{2}\frac{\alpha^{n-3}\alpha^{(1)}}{\varepsilon^{n-2}}u^{(1)} - \frac{n(n-3)}{2}\frac{\alpha^{n-4}\alpha^{(1)}}{\varepsilon^{n-2}}f - \frac{\alpha^{n-3}}{\varepsilon^{n-2}}f^{(1)}. \tag{2.14}$$

Similarly, replacing $u^{(n)}$ $(n \geq 3)$ in (2.3) and (2.4) with (2.14) and multiplying by e^{-r_i} respectively, we have

$$e^{-r_i}(u_{i+1} - u_i) \approx (\widehat{P}_i^1 + \widehat{P}_i^2)u_i^{(1)} + (\widehat{Q}_{0,i}^1 + \widehat{Q}_{0,i}^2)f_i + \widehat{Q}_{1,i}^2 f_i^{(1)},$$
$$e^{-r_i}(u_{i-1} - u_i) \approx (\overline{P}_i^1 + \overline{P}_i^2)u_i^{(1)} + (\overline{Q}_{0,i}^1 + \overline{Q}_{0,i}^2)f_i + \overline{Q}_{1,i}^2 f_i^{(1)}.$$

After simply calculating, we can obtain the specific expression of $\widehat{P}_i^2$, $\overline{P}_i^2$, $\widehat{Q}_{0,i}^2$, $\overline{Q}_{0,i}^2$ and $\widehat{Q}_{1,i}^2$, $\overline{Q}_{1,i}^2$ as follows ($\widehat{P}_i^1$, $\overline{P}_i^1$ and $\widehat{Q}_{0,i}^1$, $\overline{Q}_{0,i}^1$ have been got before)

$$\begin{aligned}
\widehat{P}_i^2 &= \frac{\alpha_i^{(1)}\varepsilon^2}{2\alpha_i^3}\left[(r_i^2 - 2r_i + 2) - 2e^{-r_i}\right], \\
\overline{P}_i^2 &= \frac{\alpha_i^{(1)}\varepsilon^2}{2\alpha_i^3}\left[e^{-2r_i}(r_i^2 + 2r_i + 2) - 2e^{-r_i}\right], \\
\widehat{Q}_{0,i}^2 &= -\frac{\alpha_i^{(1)}\varepsilon^2}{2\alpha_i^4}\left[(1 - e^{-r_i} - r_i e^{-r_i})(r_i^2 - 2r_i) + r_i^3 e^{-r_i}\right], \\
\overline{Q}_{0,i}^2 &= -\frac{\alpha_i^{(1)}\varepsilon^2}{2\alpha_i^4}\left[(e^{-2r_i} - e^{-r_i} + r_i e^{-r_i})(r_i^2 + 2r_i) - r_i^3 e^{-r_i}\right], \\
\widehat{Q}_{1,i}^2 &= -\frac{\varepsilon^2}{\alpha_i^3}\left(1 - e^{-r_i} - r_i e^{-r_i} - \frac{r_i^2}{2!}e^{-r_i}\right), \\
\overline{Q}_{1,i}^2 &= -\frac{\varepsilon^2}{\alpha_i^3}\left(e^{-2r_i} - e^{-r_i} + r_i e^{-r_i} - \frac{r_i^2}{2!}e^{-r_i}\right).
\end{aligned}$$

Therefore, the second NFD scheme is

$$A_1 U_{i-1} + A_2 U_i + A_3 U_{i+1} = Y_0 f_i + Y_1 f_i^{(1)}, \tag{2.15}$$

where

$$\begin{aligned}
A_1 &= e^{-r_i}(\widehat{P}_i^1 + \widehat{P}_i^2), \\
A_2 &= e^{-r_i}(\widehat{P}_i^1 + \widehat{P}_i^2) - e^{-r_i}(\overline{P}_i^1 + \overline{P}_i^2), \\
A_3 &= e^{-r_i}(\overline{P}_i^1 + \overline{P}_i^2), \\
Y_0 &= (\overline{P}_i^1 + \overline{P}_i^2)(\widehat{Q}_{0,i}^1 + \widehat{Q}_{0,i}^2) - (\widehat{P}_i^1 + \widehat{P}_i^2)(\overline{Q}_{0,i}^1 + \overline{Q}_{0,i}^2), \\
Y_1 &= (\overline{P}_i^1 + \overline{P}_i^2)\widehat{Q}_{1,i}^2 - (\widehat{P}_i^1 + \widehat{P}_i^2)\overline{Q}_{1,i}^2.
\end{aligned}$$

The third NFD scheme are constructed by collecting the terms with respect to $1/\varepsilon^{n-1}$, $1/\varepsilon^{n-2}$, $1/\varepsilon^{n-3}$ on the right-hand side of (2.10). And the scheme is

$$A_1U_{i-1}+A_2U_i+A_3U_{i+1}=Y_0f_i+Y_1f_i^{(1)}+Y_2f_i^{(2)}, \tag{2.16}$$

where

$$\begin{aligned}
A_1&=e^{-r_i}(\widehat{P}_i^1+\widehat{P}_i^2+\widehat{P}_i^3),\\
A_2&=e^{-r_i}(\widehat{P}_i^1+\widehat{P}_i^2+\widehat{P}_i^3)-e^{-r_i}(\overline{P}_i^1+\overline{P}_i^2+\overline{P}_i^3),\\
A_3&=e^{-r_i}(\overline{P}_i^1+\overline{P}_i^2+\overline{P}_i^3),\\
Y_0&=(\overline{P}_i^1+\overline{P}_i^2+\overline{P}_i^3)(\widehat{Q}_{0,i}^1+\widehat{Q}_{0,i}^2+\widehat{Q}_{0,i}^3)-(\widehat{P}_i^1+\widehat{P}_i^2+\widehat{P}_i^3)(\overline{Q}_{0,i}^1+\overline{Q}_{0,i}^2+\overline{Q}_{0,i}^3),\\
Y_1&=(\overline{P}_i^1+\overline{P}_i^2+\overline{P}_i^3)(\widehat{Q}_{1,i}^2+\widehat{Q}_{1,i}^3)-(\widehat{P}_i^1+\widehat{P}_i^2+\widehat{P}_i^3)(\overline{Q}_{1,i}^2+\overline{Q}_{1,i}^3),\\
Y_2&=(\overline{P}_i^1+\overline{P}_i^2+\overline{P}_i^3)\widehat{Q}_{2,i}^3-(\widehat{P}_i^1+\widehat{P}_i^2+\widehat{P}_i^3)\overline{Q}_{2,i}^3,
\end{aligned}$$

and

$$\begin{aligned}
\widehat{P}_i^3&=\frac{\alpha_i^{(2)}\varepsilon^3}{6\alpha_i^4}\left[(r_i^3-3r_i^2+6r_i-6)+6e^{-r_i})\right]\\
&\quad+\frac{(\alpha_i^{(1)})^2\varepsilon^3}{8\alpha_i^5}\left[(r_i^4-4r_i^3+12r_i^2-24r_i+24)-24e^{-r_i})\right],\\
\widehat{Q}_{0,i}^3&=-\frac{\alpha_i^{(2)}\varepsilon^3}{6\alpha_i^5}\left[(r_i^3-3r_i^2+6r_i-12)+e^{-r_i}(r_i^3+3r_i^2+6r_i+12))\right]\\
&\quad-\frac{(\alpha_i^{(1)})^2\varepsilon^3}{8\alpha_i^6}(r_i^4-4r_i^3+8r_i^2-16r_i+40)\\
&\quad+\frac{(\alpha_i^{(1)})^2\varepsilon^3}{8\alpha_i^6}\left[e^{-r_i}\left(\frac{8}{3}r_i^3+12r_i^2+24r_i+40\right)\right],\\
\widehat{Q}_{1,i}^3&=-\frac{\alpha_i^{(1)}\varepsilon^3}{2\alpha_i^5}\left[(r_i^2-2r_i-4)+e^{-r_i}\left(\frac{2}{3}r_i^3+3r_i^2+6r_i+4\right)\right],\\
\widehat{Q}_{2,i}^3&=-\frac{\varepsilon^3}{\alpha^4}\left(1-e^{-r_i}-r_ie^{-r_i}-\frac{r_i^2}{2!}e^{-r_i}-\frac{r_i^3}{3!}e^{-r_i}\right),\\
\overline{P}_i^3&=\frac{\alpha_i^{(2)}\varepsilon^3}{6\alpha_i^4}\left[e^{-2r_i}(-r_i^3-3r_i^2-6r_i-6)+6e^{-r_i}\right]\\
&\quad+\frac{(\alpha_i^{(1)})^2\varepsilon^3}{8\alpha_i^5}\left[e^{-2r_i}(r_i^4+4r_i^3+12r_i^2+24r_i+24)-24e^{-r_i}\right],\\
\overline{Q}_{0,i}^3&=-\frac{\alpha_i^{(2)}\varepsilon^3}{6\alpha_i^5}\left[e^{-2r_i}(-r_i^3-3r_i^2-6r_i-12)+e^{-r_i}(-r_i^3+3r_i^2-6r_i+12)\right]\\
&\quad-\frac{(\alpha_i^{(1)})^2\varepsilon^3}{8\alpha_i^6}\left[e^{-2r_i}(r_i^4+4r_i^3+8r_i^2+16r_i+40)\right]
\end{aligned}$$

$$+\frac{(\alpha_i^{(1)})^2\varepsilon^3}{8\alpha_i^6}\left[e^{-r_i}\left(-\frac{8}{3}r_i^3+12r_i^2-24r_i+40\right)\right],$$

$$\overline{Q}_{1,i}^3=-\frac{\alpha_i^{(1)}\varepsilon^3}{2\alpha_i^5}\left[e^{-2r_i}(r_i^2+2r_i-4)+e^{-r_i}\left(-\frac{2}{3}r_i^3+3r_i^2-6r_i+4\right)\right],$$

$$\overline{Q}_{2,i}^3=-\frac{\varepsilon^3}{\alpha^4}\left[e^{-2r_i}+e^{-r_i}\left(-1+r_i-\frac{r_i^2}{2!}+\frac{r_i^3}{3!}\right)\right].$$

Applying the same process above, other higher order schemes can also be deduced when more terms with respect to $1/\varepsilon^k$ on the right-hand side of (2.10) are taken into calculation.

Remark 2.3. It's easy to find that if we set $\alpha_i^{(k)}=0$ $(k\ge 1)$ in the NFD schemes of 1D equations with variable coefficient, we can get the corresponding NFD schemes for 1D equations with constant coefficient. Furthermore, for the case of constant coefficient, as we can see in (2.9), the right-hand terms of the NFD schemes only contain the source term f and its derivatives $f_i^{(n)}$. Thus, if the original equation is homogeneous or $f_i^{(n)}$ is zero for some $n\ge N$ $(N\in\mathbb{Z}^+)$, the numerical solution we obtain is the exact solution actually when we use the corresponding schemes.

2.3. Error estimate

In this subsection, we will derive the error estimate for the NFD schemes for 1D convection-diffusion equations. And we will take the NFD scheme (2.9) with $n=2$ for example to complete it. For other cases, the estimates can be derive in a similar way.

First, we give some notes which will be frequently used in the following. Setting

$$V_h=\big\{v\mid v=\{v_i\mid 0\le i\le N\}\big\},$$
$$V_h^0=\big\{v\mid v=\{v_i\mid 0\le i\le N\}\in V_h, v_0=v_N=0\big\}.$$

For all $v\in V_h^0$, we define

$$D^+v_i=\frac{1}{h}(v_{i+1}-v_i),\qquad D^-v_i=\frac{1}{h}(v_i-v_{i-1}),$$
$$\delta_x v_i=\frac{1}{2h}(v_{i+1}-v_{i-1}),\qquad \delta_x^2 v_i=\frac{1}{h^2}(v_{i-1}-2v_i+v_{i+1}),$$
$$\|v\|_\infty=\max_{0\le i\le N}|v_i|,\qquad \|v\|_2=\sqrt{h\left(\frac{1}{2}v_0^2+\sum_{i=1}^{N-1}v_i^2+\frac{1}{2}v_N^2\right)},$$
$$|v|_1=\sqrt{h\sum_{i=1}^{N-1}(\delta_x v_{i-1/2})^2},\qquad \|v\|_1=\sqrt{\|v\|_2+|v|_1},$$

where D^+ and D^- are forward and backward difference operators respectively; δ_x, δ_x^2 are standard central difference operators; $\|v\|_\infty$, $\|v\|_2$, $|v|_1$, $\|v\|_1$ denote L^∞ norm, L^2 norm, H^1 semi-norm and H^1 norm respectively.

Lemma 2.4. [22] *Assume* $v = \{v_i \mid 0 \le i \le N\} \in V_h^0$, *then the following conclusions hold*

$$h \sum_{i=1}^{N-1} (-\delta_x^2 v_i) v_i = |v|_1^2, \quad \|v\|_\infty \le \frac{\sqrt{b-a}}{2} |v|_1, \quad \|v\|_2 \le \frac{b-a}{\sqrt{6}} |v|_1,$$

where a, b are the left and right boundary point of computational domain.

Before deriving the error estimate, we need to define some new finite difference operators for the NFD scheme (2.9) with $n = 2$ which is deduced from the following two formulas according to (2.6) and (2.7)

$$\begin{aligned} &e^{-r}(u_{i+1} - u_i) \\ &= \frac{\varepsilon}{\alpha}(1 - e^{-r}) u_i^{(1)} - \sum_{n=2}^{4} \frac{\varepsilon^{n-1}}{\alpha^n} \left[1 - e^{-r} \sum_{l=0}^{n-1} \frac{r^l}{l!} \right] f_i^{(n-2)} + \widehat{R}_i, \end{aligned} \tag{2.17}$$

$$\begin{aligned} &e^{-r}(u_{i-1} - u_i) \\ &= \frac{\varepsilon}{\alpha}(e^{-2r} - e^{-r}) u_i^{(1)} - \sum_{n=2}^{4} \frac{\varepsilon^{n-1}}{\alpha^n} \left[e^{-2r} - e^{-r} \sum_{l=0}^{n-1} \frac{(-r)^l}{l!} \right] f_i^{(n-2)} + \overline{R}_i, \end{aligned} \tag{2.18}$$

where $r = \alpha h / \varepsilon$ and

$$\begin{aligned} \widehat{R}_i &= -\frac{\varepsilon^4}{\alpha^5} \left[1 - e^{-r} - e^{-r} \left(r + \frac{r^2}{2!} + \frac{r^3}{3!} + \frac{r^4}{4!} \right) \right] f_i^{(3)}, \\ \overline{R}_i &= -\frac{\varepsilon^4}{\alpha^5} \left[e^{-2r} - e^{-r} + e^{-r} \left(r - \frac{r^2}{2!} + \frac{r^3}{3!} - \frac{r^4}{4!} \right) \right] f_i^{(3)}. \end{aligned}$$

And $\widehat{R}_i$, $\overline{R}_i$ are regarded as reminder terms, the rest terms that with respect to ε^m $(m \ge 5)$ are neglected due to ε considered is sufficient small.

Adding (2.17) and (2.18) and subtracting (2.18) from (2.17) respectively, we have

$$e^{-r}(u_{i-1} - 2u_i + u_{i+1}) = \widehat{P} u_i^{(1)} + \widehat{Q}_0 f_i + \widehat{Q}_1 f_i^{(1)} + \widehat{Q}_2 f_i^{(2)} + \widehat{R}_i + \overline{R}_i, \tag{2.19}$$

$$e^{-r}(u_{i+1} - u_{i-1}) = \overline{P} u_i^{(1)} + \overline{Q}_0 f_i + \overline{Q}_1 f_i^{(1)} + \overline{Q}_2 f_i^{(2)} + \widehat{R}_i - \overline{R}_i, \tag{2.20}$$

where

$$\begin{aligned} \widehat{P} &= \frac{\varepsilon}{\alpha}(1 + e^{-2r} - 2e^{-r}), & \overline{P} &= \frac{\varepsilon}{\alpha}(1 - e^{-2r}), \\ \widehat{Q}_0 &= -\frac{\varepsilon}{\alpha^2}(1 + e^{-2r} - 2e^{-r}), & \overline{Q}_0 &= -\frac{\varepsilon}{\alpha^2}(1 - e^{-2r} - 2re^{-r}), \\ \widehat{Q}_1 &= -\frac{\varepsilon^2}{\alpha^3}(1 + e^{-2r} - 2e^{-r} - r^2 e^{-r}), & \overline{Q}_1 &= -\frac{\varepsilon^2}{\alpha^3}(1 - e^{-2r} - 2re^{-r}), \\ \widehat{Q}_2 &= -\frac{\varepsilon^3}{\alpha^4}(1 + e^{-2r} - 2e^{-r} - r^2 e^{-r}), & \overline{Q}_2 &= -\frac{\varepsilon^3}{\alpha^4}\left(1 - e^{-2r} - 2re^{-r} - \frac{1}{3} r^3 e^{-r}\right), \end{aligned}$$

$$\widehat{R}_i + \overline{R}_i = -\frac{\varepsilon^4}{\alpha^5}\left(1 + e^{-2r} - 2e^r - r^2e^{-r} - \frac{1}{12}r^4e^{-r}\right) f_i^{(3)},$$
$$\widehat{R}_i - \overline{R}_i = -\frac{\varepsilon^4}{\alpha^5}\left(1 - e^{-2r} - 2re^{-r} - \frac{1}{3}r^3e^{-r}\right) f_i^{(3)}.$$

On the one hand, from (2.19) and the original equation (2.1), we get

$$\begin{aligned} u_i^{(2)} = & \frac{e^{-r}\alpha^2(u_{i-1} - 2u_i + u_{i+1})}{\varepsilon^2(1 + e^{-2r} - 2e^{-r})} + D_0^2 f_i + D_1^2 f_i^{(1)} + D_2^2 f_i^{(2)} \\ & - \frac{\alpha^2(\widehat{R}_i + \overline{R}_i)}{\varepsilon^2(1 + e^{-2r} - 2e^{-r})}, \end{aligned} \tag{2.21}$$

where

$$D_0^2 = 0, \quad D_1^2 = \frac{1 + e^{-2r} - 2e^{-r} - r^2e^{-r}}{\alpha(1 + e^{-2r} - 2e^{-r})}, \quad D_2^2 = \frac{\varepsilon(1 + e^{-2r} - 2e^{-r} - r^2e^{-r})}{\alpha^2(1 + e^{-2r} - 2e^{-r})}.$$

On the other hand, (2.20) also suggests that

$$u_i^{(1)} = \frac{\alpha e^{-r}(u_{i+1} - u_{i-1})}{\varepsilon(1 - e^{-2r})} + D_0^1 f_i + D_1^1 f_i^{(1)} + D_2^1 f_i^{(2)} - \frac{\alpha(\widehat{R}_i - \overline{R}_i)}{\varepsilon(1 - e^{-2r})}, \tag{2.22}$$

where

$$D_0^1 = \frac{1 - e^{-2r} - 2re^{-r}}{\alpha(1 - e^{-2r})}, \quad D_1^1 = \frac{\varepsilon(1 - e^{-2r} - 2re^{-r})}{\alpha^2(1 - e^{-2r})},$$
$$D_2^1 = \frac{\varepsilon^2\left(1 - e^{-2r} - 2re^{-r} - \frac{1}{3}r^3e^{-r}\right)}{\alpha^3(1 - e^{-2r})}.$$

Thus, overlooking the remainder terms in (2.21) and (2.22), we can define two new difference operators as follows

$$\Pi_x u_i = \frac{\alpha e^{-r}(u_{i+1} - u_{i-1})}{\varepsilon(1 - e^{-2r})} + D_0^1 f_i + D_1^1 f_i^{(1)} + D_2^1 f_i^{(2)}, \tag{2.23}$$

$$\Pi_x^2 u_i = \frac{\alpha^2 e^{-r}(u_{i-1} - 2u_i + u_{i+1})}{\varepsilon^2(1 + e^{-2r} - 2e^{-r})} + D_0^2 f_i + D_1^2 f_i^{(1)} + D_2^2 f_i^{(2)}. \tag{2.24}$$

Substituting (2.23) and (2.24) into the original equation (2.1), we have

$$-\varepsilon\Pi_x^2 U_i + \alpha\Pi_x U_i = f_i. \tag{2.25}$$

And it is very easy to verify that (2.25) which is used to error estimates is equivalent to the NFD scheme (2.9) with $n = 2$.

The above deducing indicates that we can obtain all the NFD schemes by constructing new difference operators for $u_i^{(2)}$ and $u_i^{(1)}$. And from (2.25), we can also observe that different NFD schemes need different new operators for the reason that the new operators contain the source term f_i and its derivatives.

Substituting (2.21) and (2.22) into the equation (2.1), we have

$$-\varepsilon\Pi_x^2 u_i + \alpha\Pi_x u_i = f_i + R_i, \tag{2.26}$$

where

$$R_i = \frac{\alpha^2(\widehat{R}_i - \overline{R}_i)}{\varepsilon(1-e^{-2r})} - \frac{\alpha^2(\widehat{R}_i + \overline{R}_i)}{\varepsilon(1+e^{-2r}-2e^{-r})}.$$

Then, subtracting (2.25) from (2.26) and setting $e_i = u_i - U_i$, we get

$$-\varepsilon\widetilde{\Pi}_x^2 e_i + \alpha\widetilde{\Pi}_x e_i = R_i, \tag{2.27}$$

where

$$\widetilde{\Pi}_x e_i = \frac{\alpha e^{-r}(u_{i+1}-u_{i-1})}{\varepsilon(1-e^{-2r})}, \quad \widetilde{\Pi}_x^2 e_i = \frac{\alpha^2 e^{-r}(u_{i-1}-2u_i+u_{i+1})}{\varepsilon^2(1+e^{-2r}-2e^{-r})}.$$

Comparing with the central difference operators δ_x and δ_x^2, we have

$$\widetilde{\Pi}_x e_i = \frac{2re^{-r}}{1-e^{-2r}}\delta_x e_i, \tag{2.28}$$

$$\widetilde{\Pi}_x^2 e_i = \frac{r^2 e^{-r}}{1+e^{-2r}-2e^{-r}}\delta_x^2 e_i. \tag{2.29}$$

Putting (2.28) and (2.29) into (2.27), we have

$$C_1(-\delta_x^2 e_i) + C_2(\delta_x e_i) = R_i = C_3 f_i^{(3)}, \tag{2.30}$$

where

$$C_1 = \frac{\varepsilon r^2 e^{-r}}{1+e^{-2r}-2e^{-r}}, \quad C_2 = \frac{2\alpha r e^{-r}}{1-e^{-2r}},$$
$$C_3 = \frac{\varepsilon^3}{\alpha^3}\left[\frac{e^{-r}\left(2r+\frac{1}{3}r^3\right)}{1-e^{-2r}} - \frac{e^{-r}\left(r^2+\frac{1}{12}r^4\right)}{1+e^{-2r}-2e^{-r}}\right].$$

Theorem 2.5. *Let $u(x)$ $(0 \le x \le 1)$ be the solution of equation* (2.1) *$(\alpha > 0$ is a constant) with Dirichlet boundary condition, $\{U_i \mid 0 \le i \le N\}$ be the solution of the NFD scheme* (2.9) *$(n = 2)$ or* (2.25)*. The local error at the grid point x_i be $e_i = u_i - U_i$, then, we have*

$$\|e\|_\infty \le \frac{Mh^2}{24\sqrt{6}\alpha},$$

where $M = \max_{0\le i\le N} |f_i^{(3)}|$.

Proof. For every interior point x_i $(1 \le i \le N-1)$, multiplying (2.30) by he_i and then summing for i from 1 to $N-1$, we get

$$C_1 h\sum_{i=1}^{N-1}(-\delta_x^2 e_i)e_i + C_2 h\sum_{i=1}^{N-1}(\delta_x e_i)e_i = h\sum_{i=1}^{N-1} R_i e_i. \tag{2.31}$$

Noticing that the original equation has Dirichlet boundary condition, so $e = \{e_i \mid 0 \le i \le N\} \in V_h^0$. Then according to Lemma 2.4, we have

$$h \sum_{i=1}^{N-1} (-\delta_x^2 e_i) e_i = |e|_1^2. \tag{2.32}$$

After simple calculation, it holds

$$\begin{aligned} \sum_{i=1}^{N-1} (\delta_x e_i) e_i &= \frac{1}{2h} \left[(e_2 - e_0) e_1 + (e_3 - e_1) e_2 + \cdots + (e_N - e_{N-2}) e_{N-1} \right] \\ &= \frac{1}{2h} (e_N e_{N-1} - e_1 e_0) \\ &= 0. \end{aligned} \tag{2.33}$$

And the following formula is obvious as well

$$h \sum_{i=1}^{N-1} R_i e_i \le \|R\|_2 \|e\|_2. \tag{2.34}$$

Substituting (2.32)–(2.34) into (2.31), we have

$$C_1 |e|_1^2 \le \|R\|_2 \|e\|_2.$$

Using the inequations in Lemma 2.4, we get

$$C_1 |e|_1^2 \le \frac{1}{\sqrt{6}} \|R\|_2 |e|_1,$$

and

$$\|e\|_\infty \le \frac{1}{2} |e|_1 \le \frac{1}{C_1 2\sqrt{6}} \|R\|_\infty. \tag{2.35}$$

Due to expression of R_i in (2.30), we have

$$\|R\|_\infty = \max_{0 \le i \le N} |C_3 f_i^{(3)}| \le M |C_3|, \tag{2.36}$$

where

$$M = \max_{0 \le i \le N} |f_i^{(3)}|.$$

Then, after substituting (2.36) into (2.35), we have

$$\|e\|_\infty \le \frac{M |C_3|}{2\sqrt{6} C_1}.$$

For $\alpha > 0$, $C_3 < 0$, then we get

$$\begin{aligned}\frac{|C_3|}{C_1} &= \frac{\varepsilon^3}{\alpha^3}\left[\frac{e^{-r}\left(r^2+\frac{1}{12}r^4\right)}{1+e^{-2r}-2e^{-r}} - \frac{e^{-r}\left(2r+\frac{1}{3}r^3\right)}{1-e^{-2r}}\right]\frac{1+e^{-2r}-2e^{-r}}{\varepsilon r^2 e^{-r}} \\ &= \frac{\varepsilon^2}{\alpha^3}\left(1+\frac{1}{12}r^2\right) - \frac{1+e^{-2r}-2e^{-r}}{1-e^{-2r}}\frac{\varepsilon^2}{\alpha^3}\left(\frac{2}{r}+\frac{r}{3}\right).\end{aligned}$$

And it is obvious that $2/r + r/3 \geq 2\sqrt{2/3}$ for $r = \alpha h/\varepsilon > 0$, thus

$$\begin{aligned}\frac{|C_3|}{C_1} &\leq \frac{\varepsilon^2}{\alpha^3}\left(1+\frac{\alpha^2 h^2}{12\varepsilon^2}\right) - 2\frac{1+e^{-2r}-2e^{-r}}{1-e^{-2r}}\sqrt{\frac{2}{3}}\frac{\varepsilon^2}{\alpha^3} \\ &= \frac{h^2}{12\alpha} + \left[1-2\frac{1+e^{-2r}-2e^{-r}}{1-e^{-2r}}\sqrt{\frac{2}{3}}\right]\frac{\varepsilon^2}{\alpha^3} \\ &\leq \frac{h^2}{12\alpha}.\end{aligned}$$

Thus, the proof is completed. □

Remark 2.6. (1) If the convection coefficient $\alpha < 0$ for all $x \in \Omega$, by multiplying (2.6), (2.7) and other formulas by e^r ($r = \alpha h/\varepsilon$), the similar new finite difference (NFD) schemes can also be got via the same processes. And the error estimates can also be done in the same way.

(2) If the convection coefficient $\alpha(x) = 0$, (2.1) will reduce to the diffusion equation. And the following formula which is similar with (2.5) will be obtained as well

$$u^{(n)} = -\frac{1}{\varepsilon}f^{(n-2)}, \quad n \geq 2.$$

Applying the similar process above, we have

$$u_{i+1} + u_{i-1} - 2u_i = -\frac{2}{\varepsilon}\sum_{m=1}^{+\infty}\frac{(-h)^{2m}}{(2m)!}f_i^{(2m-2)}.$$

Thus, a kind of new schemes can be also derived in this case when finite terms are considered in the right-hand side of the above formula.

3. New scheme in 2D

In the previous sections, a family of NFD schemes for 1D convection-diffusion equations have been proposed. Next, we will extend this idea to the 2D equation (1.1).

Suppose that Ω has been divided into $N_x \times N_y$ parts with $h_x = 1/N_x$, $h_y = 1/N_y$ being the mesh size in x and y directions respectively, (x_i, y_j) ($0 \leq i \leq N$, $0 \leq j \leq M$) are the mesh points. Rewrite the 2D equation as follows

$$-\varepsilon\frac{\partial^2 u}{\partial x^2}+\alpha(x,y)\frac{\partial u}{\partial x}=\widetilde{f}(x,y), \tag{3.1}$$

$$-\varepsilon\frac{\partial^2 u}{\partial y^2}+\beta(x,y)\frac{\partial u}{\partial y}=\widehat{f}(x,y), \tag{3.2}$$

where

$$\widetilde{f}(x,y)=f(x,y)+\varepsilon\frac{\partial^2 u}{\partial y^2}-\beta(x,y)\frac{\partial u}{\partial y},$$
$$\widehat{f}(x,y)=f(x,y)+\varepsilon\frac{\partial^2 u}{\partial x^2}-\alpha(x,y)\frac{\partial u}{\partial x}.$$

Obviously, (3.1) and (3.2) can be regarded as two 1D problems, thus the all NFD schemes achieved in Section 2 can be used to deduce the NFD schemes for 2D equations.

For example, applying the scheme (2.15) to them, it follows that

$$\frac{A_1^x}{Y_0^x}U_{i-1,j}+\frac{A_2^x}{Y_0^x}U_{i,j}+\frac{A_3^x}{Y_0^x}U_{i+1,j}=\widetilde{f}_{i,j}+\frac{Y_1^x}{Y_0^x}\widetilde{f}_{xi,j}, \tag{3.3}$$

$$\frac{A_1^y}{Y_0^y}U_{i,j-1}+\frac{A_2^y}{Y_0^y}U_{i,j}+\frac{A_3^y}{Y_0^y}U_{i,j+1}=\widehat{f}_{i,j}+\frac{Y_1^y}{Y_0^y}\widehat{f}_{yi,j}, \tag{3.4}$$

where $U_{i,j}$ denotes the approximation of $u_{i,j}=u(x_i,y_j)$, A_k^x $(k=1,2,3)$, Y_0^x, Y_1^x and A_k^y $(k=1,2,3)$, Y_0^y, Y_1^y denote the coefficients in x and y directions respectively, which are analog to that in (2.15), and

$$\widetilde{f}_{xi,j}=\frac{\partial\widetilde{f}}{\partial x}\bigg|_{i,j},\quad \widehat{f}_{yi,j}=\frac{\partial\widehat{f}}{\partial y}\bigg|_{i,j}.$$

Using the standard second order finite difference scheme to approximate $\partial u/\partial x$, $\partial^2 u/\partial x^2$ and $\partial u/\partial y$, $\partial^2 u/\partial y^2$, it is valid that

$$\begin{aligned}\widetilde{f}_{xi,j}=f_{xi,j}&+\Phi_1^1U_{i-1,j-1}+\Phi_2^1U_{i,j-1}+\Phi_3^1U_{i+1,j-1}+\Phi_4^1U_{i-1,j}+\Phi_5^1U_{i,j}\\&+\Phi_6^1U_{i+1,j}+\Phi_7^1U_{i-1,j+1}+\Phi_8^1U_{i,j+1}+\Phi_9^1U_{i+1,j+1},\end{aligned} \tag{3.5}$$

$$\begin{aligned}\widehat{f}_{yi,j}=f_{yi,j}&+\Psi_1^1U_{i-1,j-1}+\Psi_2^1U_{i,j-1}+\Psi_3^1U_{i+1,j-1}+\Psi_4^1U_{i-1,j}+\Psi_5^1U_{i,j}\\&+\Psi_6^1U_{i+1,j}+\Psi_7^1U_{i-1,j+1}+\Psi_8^1U_{i,j+1}+\Psi_9^1U_{i+1,j+1},\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
&\Phi_1^1=-\frac{\varepsilon}{2h_xh_y^2}-\frac{\beta_{i,j}}{4h_xh_y}, &&\Phi_2^1=\frac{\beta_{xi,j}}{2h_y}, &&\Phi_3^1=\frac{\varepsilon}{2h_xh_y^2}+\frac{\beta_{i,j}}{4h_xh_y},\\
&\Phi_4^1=\frac{\varepsilon}{h_xh_y^2}, &&\Phi_5^1=0, &&\Phi_6^1=-\frac{\varepsilon}{h_xh_y^2},\\
&\Phi_7^1=-\frac{\varepsilon}{2h_xh_y^2}+\frac{\beta_{i,j}}{4h_xh_y}, &&\Phi_8^1=-\frac{\beta_{xi,j}}{2h_y}, &&\Phi_9^1=\frac{\varepsilon}{2h_xh_y^2}-\frac{\beta_{i,j}}{4h_xh_y},
\end{aligned}$$

and

$$\Psi_1^1 = -\frac{\varepsilon}{2h_x^2 h_y} - \frac{\alpha_{i,j}}{4h_x h_y}, \qquad \Psi_2^1 = \frac{\varepsilon}{h_x^2 h_y}, \qquad \Psi_3^1 = -\frac{\varepsilon}{2h_x^2 h_y} + \frac{\alpha_{i,j}}{4h_x h_y},$$
$$\Psi_4^1 = \frac{\alpha_{yi,j}}{2h_x}, \qquad \Psi_5^1 = 0, \qquad \Psi_6^1 = -\frac{\alpha_{yi,j}}{2h_x},$$
$$\Psi_7^1 = \frac{\varepsilon}{2h_x^2 h_y} + \frac{\alpha_{i,j}}{4h_x h_y}, \qquad \Psi_8^1 = -\frac{\varepsilon}{h_x^2 h_y}, \qquad \Psi_9^1 = \frac{\varepsilon}{2h_x^2 h_y} - \frac{\alpha_{i,j}}{4h_x h_y}.$$

Then adding (3.3) and (3.4), and substituting (3.5) and (3.6) into it, a nine-point scheme for the 2D convection-diffusion equation with variable coefficients is obtained

$$\overline{A} \cdot \overline{U} = f_{i,j} + \frac{Y_1^x}{Y_0^x} f_{xi,j} + \frac{Y_1^y}{Y_0^y} f_{yi,j}, \tag{3.7}$$

where

$$\begin{aligned}
\overline{A} &= (A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9),\\
\overline{U} &= (U_{i-1,j-1}, U_{i,j-1}, U_{i+1,j-1}, U_{i-1,j}, U_{i,j}, U_{i+1,j}, U_{i-1,j+1}, U_{i,j+1}, U_{i+1,j+1}),\\
A_k &= -\frac{Y_1^x}{Y_0^x}\Phi_k^1 - \frac{Y_1^y}{Y_0^y}\Psi_k^1 \quad (k = 1, 3, 7, 9),\\
A_2 &= -\frac{Y_1^x}{Y_0^x}\Phi_2^1 - \frac{Y_1^y}{Y_0^y}\Psi_2^1 + \frac{A_1^y}{Y_0^y},\\
A_4 &= -\frac{Y_1^x}{Y_0^x}\Phi_4^1 - \frac{Y_1^y}{Y_0^y}\Psi_4^1 + \frac{A_1^x}{Y_0^x},\\
A_5 &= -\frac{Y_1^x}{Y_0^x}\Phi_5^1 - \frac{Y_1^y}{Y_0^y}\Psi_5^1 + \frac{A_2^y}{Y_0^y} + \frac{A_2^x}{Y_0^x},\\
A_6 &= -\frac{Y_1^x}{Y_0^x}\Phi_6^1 - \frac{Y_1^y}{Y_0^y}\Psi_6^1 + \frac{A_3^x}{Y_0^x},\\
A_8 &= -\frac{Y_1^x}{Y_0^x}\Phi_8^1 - \frac{Y_1^y}{Y_0^y}\Psi_8^1 + \frac{A_3^y}{Y_0^y}.
\end{aligned}$$

Similarly, applying the scheme (2.16) to (3.1) and (3.2), we can get a more accuracy scheme as follows

$$\overline{A} \cdot \overline{U} = f_{i,j} + \frac{Y_1^x}{Y_0^x} f_{xi,j} + \frac{Y_1^y}{Y_0^y} f_{yi,j} + \frac{Y_2^x}{Y_0^x} f_{xxi,j} + \frac{Y_2^y}{Y_0^y} f_{yyi,j}, \tag{3.8}$$

where

$$\begin{aligned}
A_k &= -\frac{Y_1^x}{Y_0^x}\Phi_k^1 - \frac{Y_1^y}{Y_0^y}\Psi_k^1 - \frac{Y_2^x}{Y_0^x}\Phi_k^2 - \frac{Y_2^y}{Y_0^y}\Psi_k^2 \quad (k = 1, 3, 7, 9),\\
A_2 &= -\frac{Y_1^x}{Y_0^x}\Phi_2^1 - \frac{Y_1^y}{Y_0^y}\Psi_2^1 - \frac{Y_2^x}{Y_0^x}\Phi_2^2 - \frac{Y_2^y}{Y_0^y}\Psi_2^2 + \frac{A_1^y}{Y_0^y},\\
A_4 &= -\frac{Y_1^x}{Y_0^x}\Phi_4^1 - \frac{Y_1^y}{Y_0^y}\Psi_4^1 - \frac{Y_2^x}{Y_0^x}\Phi_4^2 - \frac{Y_2^y}{Y_0^y}\Psi_4^2 + \frac{A_1^x}{Y_0^x},
\end{aligned}$$

$$
\begin{aligned}
A_5 &= -\frac{Y_1^x}{Y_0^x}\Phi_5^1 - \frac{Y_1^y}{Y_0^y}\Psi_5^1 - \frac{Y_2^x}{Y_0^x}\Phi_5^2 - \frac{Y_2^y}{Y_0^y}\Psi_5^2 + \frac{A_2^y}{Y_0^y} + \frac{A_2^x}{Y_0^x},\\
A_6 &= -\frac{Y_1^x}{Y_0^x}\Phi_6^1 - \frac{Y_1^y}{Y_0^y}\Psi_6^1 - \frac{Y_2^x}{Y_0^x}\Phi_6^2 - \frac{Y_2^y}{Y_0^y}\Psi_6^2 + \frac{A_3^x}{Y_0^x},\\
A_8 &= -\frac{Y_1^x}{Y_0^x}\Phi_8^1 - \frac{Y_1^y}{Y_0^y}\Psi_8^1 - \frac{Y_2^x}{Y_0^x}\Phi_8^2 - \frac{Y_2^y}{Y_0^y}\Psi_8^2 + \frac{A_3^y}{Y_0^y},
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi_1^2 &= \frac{\varepsilon}{h_x^2h_y^2} + \frac{\beta_{i,j}}{2h_x^2h_y} - \frac{\beta_{xi,j}}{2h_xh_y}, & \Phi_2^2 &= -\frac{2\varepsilon}{h_x^2h_y^2} - \frac{\beta_{i,j}}{h_x^2h_y} + \frac{\beta_{xxi,j}}{2h_y},\\
\Phi_3^2 &= \frac{\varepsilon}{h_x^2h_y^2} + \frac{\beta_{i,j}}{2h_x^2h_y} + \frac{\beta_{xi,j}}{2h_xh_y}, & \Phi_4^2 &= -\frac{2\varepsilon}{h_x^2h_y^2},\\
\Phi_5^2 &= \frac{4\varepsilon}{h_x^2h_y^2}, & \Phi_6^2 &= -\frac{2\varepsilon}{h_x^2h_y^2},\\
\Phi_7^2 &= \frac{\varepsilon}{h_x^2h_y^2} - \frac{\beta_{i,j}}{2h_x^2h_y} + \frac{\beta_{xi,j}}{2h_xh_y}, & \Phi_8^2 &= -\frac{2\varepsilon}{h_x^2h_y^2} + \frac{\beta_{i,j}}{h_x^2h_y} - \frac{\beta_{xxi,j}}{2h_y},\\
\Phi_9^2 &= \frac{\varepsilon}{h_x^2h_y^2} - \frac{\beta_{i,j}}{2h_x^2h_y} - \frac{\beta_{xi,j}}{2h_xh_y}, & \Psi_1^2 &= \frac{\varepsilon}{h_x^2h_y^2} + \frac{\alpha_{i,j}}{2h_xh_y^2} - \frac{\alpha_{yi,j}}{2h_xh_y},\\
\Psi_2^2 &= -\frac{2\varepsilon}{h_x^2h_y^2}, & \Psi_3^2 &= \frac{\varepsilon}{h_x^2h_y^2} - \frac{\alpha_{i,j}}{2h_xh_y^2} + \frac{\alpha_{yi,j}}{2h_xh_y},\\
\Psi_4^2 &= -\frac{2\varepsilon}{h_x^2h_y^2} - \frac{\alpha_{i,j}}{h_xh_y^2} + \frac{\alpha_{yyi,j}}{2h_x}, & \Psi_5^2 &= \frac{4\varepsilon}{h_x^2h_y^2},\\
\Psi_6^2 &= -\frac{2\varepsilon}{h_x^2h_y^2} + \frac{\alpha_{i,j}}{h_xh_y^2} - \frac{\alpha_{yyi,j}}{2h_x}, & \Psi_7^2 &= \frac{\varepsilon}{h_x^2h_y^2} + \frac{\alpha_{i,j}}{2h_xh_y^2} + \frac{\alpha_{yi,j}}{2h_xh_y},\\
\Psi_8^2 &= -\frac{2\varepsilon}{h_x^2h_y^2}, & \Psi_9^2 &= \frac{\varepsilon}{h_x^2h_y^2} - \frac{\alpha_{i,j}}{2h_xh_y^2} - \frac{\alpha_{yi,j}}{2h_xh_y}.
\end{aligned}
$$

4. Numerical experiments

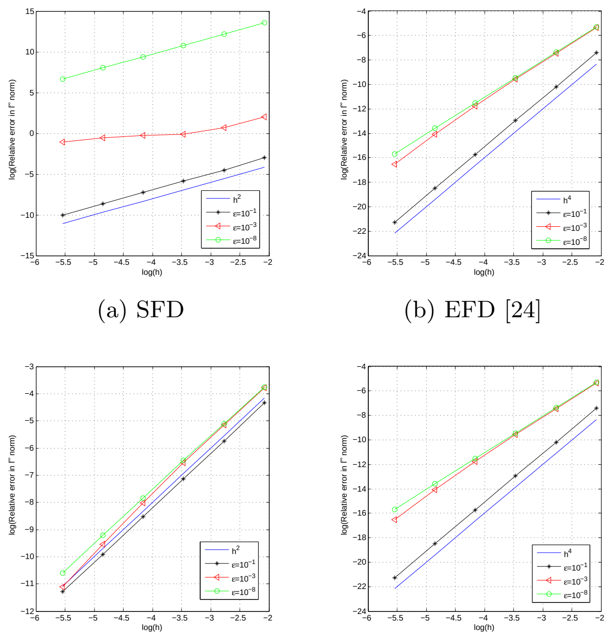
To test the effectiveness of the proposed schemes in above two sections, several numerical experiments are presented in this section. Attention will be focus on the problems with small diffusion coefficients. Moreover, not only the numerical results of the NFD schemes based on uniform mesh are shown, but also we test the NFD schemes based on Shishkin mesh. Using Shishkin mesh, we can simulate the solution inside the boundary layer more accurate and obtain the convergence order in the whole interval.

The Shishkin mesh we used could be constructed as follows. Let N be a positive even integer which denotes the total number of subintervals in Shishkin mesh. Set $\sigma = \min\{1/2, 2\varepsilon \ln N\}$, and the transition point λ used in Shishkin mesh is $1-\sigma$ when $\alpha > 0$ which case is considered in the following numerical experiments. Thus, the mesh sizes outside and inside the boundary layer are

$$h_1 = \frac{2\lambda}{N}, \quad h_2 = \frac{2(1-\lambda)}{N}.$$

And the mesh points of Shishkin mesh are

$$x_i = \begin{cases} ih_1 & i = 0, 1, 2, \dots, N/2, \\ \lambda + (i - N/2)h_2 & i = N/2+1, \dots, N. \end{cases}$$



(a) SFD (b) EFD [24]

(c) NFD (2.9) with $n = 1$ (d) NFD (2.9) with $n = 2$

Figure 4.1: Convergence order of different schemes for Problem 1.

Problem 1. 1D problem with constant coefficient

Set $\alpha = 1$ and the exact solution of (2.1) be $u(x) = \sin \pi x + (\exp(x/\varepsilon) - 1)/(\exp(1/\varepsilon) - 1)$ (see [19]). For the uniform mesh, we first compare the convergence order of the NFD scheme (2.9) ($n = 1, 2$) with the standard finite difference (SFD) scheme and the exponential finite difference (EFD) method in [24] with $\varepsilon = 10^{-1}$, 10^{-3}, 10^{-8} (see Figure 4.1). It can be found that the NFD schemes can keep the convergence order stable even the diffusion coefficient ε is very small. Then, let $h = \varepsilon$, we investigate the development of the error in l^∞-norm for four schemes in Figure 4.2. We can see that the error of the SFD scheme almost doesn't change as ε decreases, but the errors got from other three schemes decrease as ε decreases. The interesting thing is that the scheme (2.9) with $n = 2$ and the

EFD scheme in [24] have almost the same computational accuracy for the problem with constant coefficients.

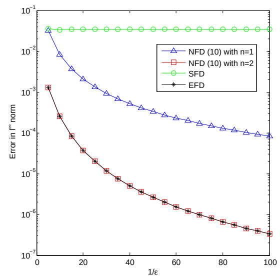


Figure 4.2: Development of the maximum error for Problem 1 with $h = \varepsilon$.

Figures 4.3 and 4.4 exhibit the numerical solutions of the NFD scheme (2.9) with $n = 2$ based on the unform mesh and Shishkin mesh, respectively. We can see that there is no numerical oscillation near the boundary layer despite using the uniform mesh. The NFD scheme performs very well in this case. Since less information is captured in the boundary layer when uniform mesh is used, there is no surprise that the maximum error based on Shishkin mesh is slightly larger than that based on the uniform mesh.

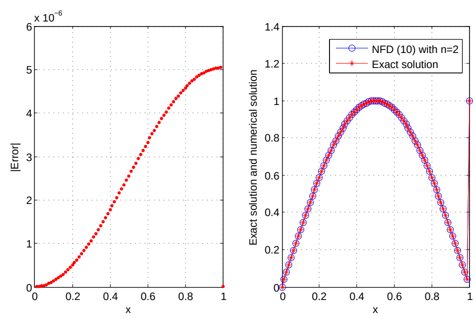


Figure 4.3: Numerical solution and error of the scheme (2.9) with $n = 2$ based on uniform mesh for Problem 1 ($\varepsilon = 10^{-8}$, $N = 80$).

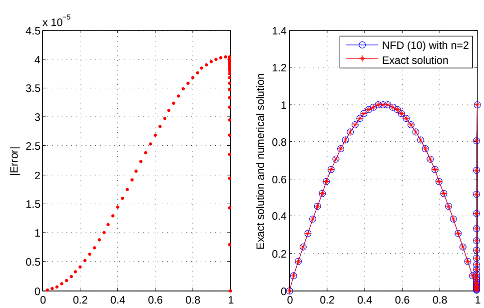
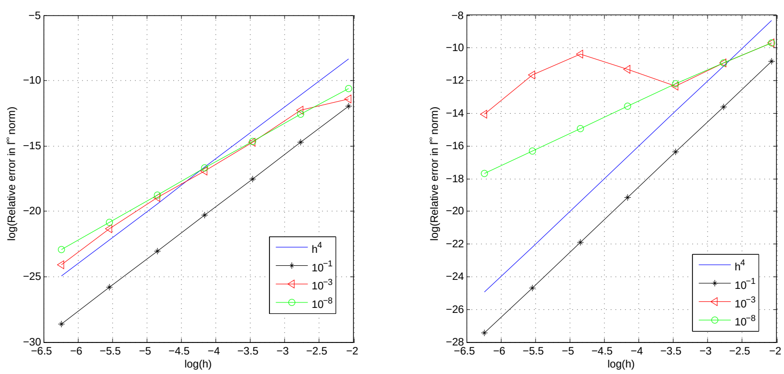


Figure 4.4: Numerical solution and error of the scheme (2.9) with $n = 2$ based on Shishkin mesh for Problem 1 ($\varepsilon = 10^{-8}$, $N = 80$).

Problem 2. 1D problem with variable coefficient

Then, let $\alpha(x) = 1/(1+x)$ and the exact solution $u(x) = e^x + 2^{-1/\varepsilon}(1+x)^{1+1/\varepsilon}$ (see [24]), we consider the 1D equation with a variable coefficient. In Figure 4.5, we compare the convergence order and the computational accuracy of the EFD methods in [24] and the NFD scheme (2.16) with $\varepsilon = 10^{-1}$, 10^{-3}, 10^{-8}, respectively. As exhibiting in this figure, when ε decreases, the new proposed scheme can keep the convergence rate stable better and reach much higher accuracy. Furthermore, we perform the development of the error in l^∞-norm with $h = \varepsilon$ in Figure 4.6 as well. It can be found that the maximum error almost does not change as ε is dropping for the SFD scheme and the CFD scheme. However, for the NFD scheme (2.16) and the EFD scheme in [24], the maximum errors are decreasing and the former decreases more sharply. On the other hand, from Figures 4.7–4.10, we can see that the NFD schemes could achieve high accuracy based on both the uniform mesh and Shishkin mesh in the whole computational interval even though $\varepsilon = 10^{-8}$ and $N = 80$.



(a) NFD (2.16) (b) EFD [24]

Figure 4.5: Convergence order of different schemes for Problem 2.

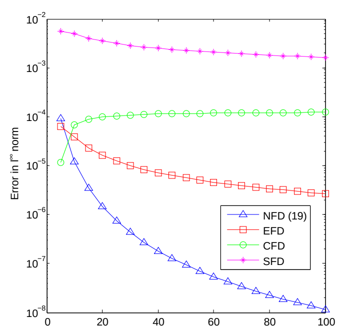


Figure 4.6: Development of the maximum error for Problem 2 with $h = \varepsilon$.

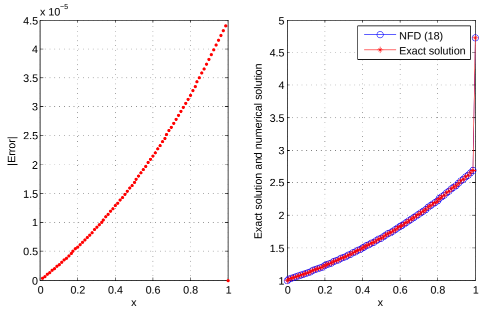


Figure 4.7: Numerical solution and error of the scheme (2.15) based on uniform mesh for Problem 2 ($\varepsilon = 10^{-8}$, $N = 80$).

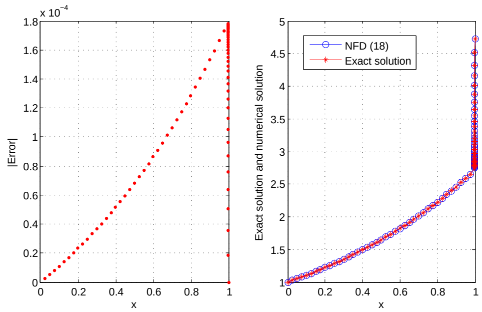


Figure 4.8: Numerical solution and error of the scheme (2.15) based on Shishkin mesh for Problem 2 ($\varepsilon = 10^{-8}$, $N = 80$).

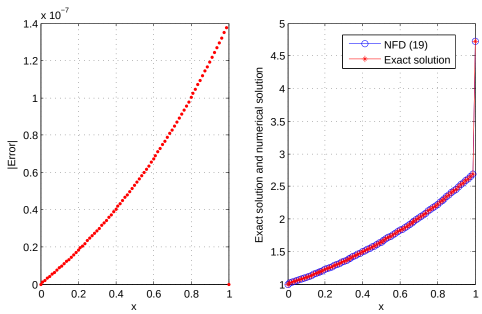


Figure 4.9: Numerical solution and error of the scheme (2.16) based on uniform mesh for Problem 2 ($\varepsilon = 10^{-8}$, $N = 80$).

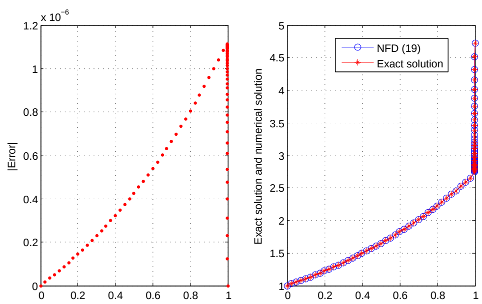


Figure 4.10: Numerical solution and error of the scheme (2.16) based on Shishkin mesh for Problem 2 ($\varepsilon = 10^{-8}$, $N = 80$).

N	NFD (2.15)	Order	NFD (2.16)	Order	SFD	Order
$\varepsilon = 10^{-1}$						
8	1.1178e-03	-	7.7762e-06	-	3.3512e-02	-
16	2.7729e-04	2.0112	5.1966e-07	3.9034	8.4051e-03	1.9954
32	6.9211e-05	2.0023	3.3620e-08	3.9502	2.0794e-03	2.0151
64	1.7310e-05	1.9994	2.1384e-09	3.9747	5.1851e-04	2.0037
128	4.3277e-06	2.0000	1.3483e-10	3.9873	1.2959e-04	2.0004
$\varepsilon = 10^{-3}$						
8	3.5721e-03	-	1.4123e-05	-	1.0202e+01	-
16	6.0209e-04	2.5687	7.1407e-06	0.9839	2.7982e+00	1.8663
32	1.7871e-04	1.7524	7.1086e-07	3.3284	1.5577e+00	0.8451
64	3.0068e-05	2.5713	7.5017e-08	3.2443	1.1888e+00	0.3899
128	2.6891e-06	3.4831	8.5108e-09	3.1398	6.8794e-01	0.7891
$\varepsilon = 10^{-5}$						
8	3.7548e-03	-	2.7291e-05	-	1.0437e+03	-
16	1.0260e-03	1.8717	4.0343e-06	2.7580	2.6041e+02	2.0029
32	2.6748e-04	1.9395	5.5058e-07	2.8733	6.5049e+01	2.0012
64	6.8166e-05	1.9723	6.3234e-08	3.1222	1.6312e+01	1.9956
128	1.7721e-05	1.9436	9.5414e-09	2.7284	4.3591e+00	1.9038
$\varepsilon = 10^{-7}$						
8	3.7572e-03	-	2.7322e-05	-	1.0440e+05	-
16	1.0273e-03	1.8708	4.0447e-06	2.7560	2.6056e+04	2.0024
32	2.6813e-04	1.9378	5.5641e-07	2.8618	6.5112e+03	2.0006
64	6.8465e-05	1.9695	7.3253e-08	2.9252	1.6276e+03	2.0002
128	1.7296e-05	1.9849	9.4068e-09	2.9611	4.0689e+02	2.0001
$\varepsilon = 10^{-9}$						
8	3.7572e-03	-	2.7322e-05	-	1.0440e+07	-
16	1.0273e-03	1.8708	4.0448e-06	2.7559	2.6056e+06	2.0024
32	2.6814e-04	1.9378	5.5644e-07	2.8618	6.5113e+05	2.0006
64	6.8469e-05	1.9695	7.3260e-08	2.9251	1.6277e+05	2.0002
128	1.7298e-05	1.9849	9.4098e-09	2.9608	4.0690e+04	2.0000

Table 4.1: Convergence order based on uniform mesh in l^∞-norm for Problem 2.

Furthermore, the convergence orders are compared among the NFD scheme (2.15), (2.16) and the SFD scheme based on the uniform mesh and Shishkin mesh in Tables 4.1 and 4.2, respectively. It is shown that two NFD schemes can achieve their uniform convergence order in both cases, respectively.

N	NFD (2.15)	Order	NFD (2.16)	Order	SFD	Order
$\varepsilon = 10^{-1}$						
8	1.0518e-03	-	2.9727e-05	-	2.1580e-02	-
16	3.0628e-04	1.7799	2.4918e-06	3.5765	1.0672e-02	1.0159
32	1.1443e-04	1.4204	3.9895e-07	2.6429	4.2527e-03	1.3273
64	4.4461e-05	1.3639	5.7873e-08	2.7852	1.5256e-03	1.4790
128	1.6242e-05	1.4529	6.7988e-09	3.0895	4.9809e-04	1.6149
$\varepsilon = 10^{-3}$						
8	1.5859e-02	-	9.5848e-04	-	5.9210e-01	-
16	4.0155e-03	1.9817	6.8910e-05	3.7980	2.2446e-01	1.3994
32	7.7199e-04	2.3789	2.2299e-05	1.6278	7.1098e-02	1.6586
64	1.8009e-04	2.0998	1.8894e-06	3.5610	2.4556e-02	1.5337
128	2.9835e-05	2.5937	1.9881e-07	3.2485	9.2483e-03	1.4088
$\varepsilon = 10^{-5}$						
8	1.6639e-02	-	1.0523e-03	-	6.4086e-01	-
16	4.3230e-03	1.9445	1.3584e-04	2.9536	2.8346e-01	1.1769
32	1.0990e-03	1.9758	1.7199e-05	2.9815	1.3245e-01	1.0977
64	2.7669e-04	1.9899	2.1451e-06	3.0033	6.2647e-02	1.0801
128	6.9307e-05	1.9972	2.2673e-07	3.2420	2.8230e-02	1.1500
$\varepsilon = 10^{-7}$						
8	1.6647e-02	-	1.0532e-03	-	6.4137e-01	-
16	4.3266e-03	1.9440	1.3604e-04	2.9526	2.8417e-01	1.1744
32	1.1007e-03	1.9749	1.7251e-05	2.9793	1.3361e-01	1.0888
64	2.7744e-04	1.9881	2.1709e-06	2.9904	6.4695e-02	1.0462
128	6.9638e-05	1.9942	2.7223e-07	2.9954	3.1783e-02	1.0254
$\varepsilon = 10^{-9}$						
8	1.6647e-02	-	1.0532e-03	-	6.4138e-01	-
16	4.3266e-03	1.9440	1.3604e-04	2.9526	2.8418e-01	1.1744
32	1.1007e-03	1.9748	1.7252e-05	2.9793	1.3362e-01	1.0887
64	2.7745e-04	1.9881	2.1692e-06	2.9915	6.4716e-02	1.0459
128	6.9641e-05	1.9942	2.7318e-07	2.9893	3.1823e-02	1.0241

Table 4.2: Convergence order based on Shishkin mesh in l^{∞}-norm for Problem 2.

In particular, the NFD scheme (2.16) based on Shishkin mesh is third order, which is much better than the almost second ones in the literatures. All of these results show that the NFD schemes are very efficiency.

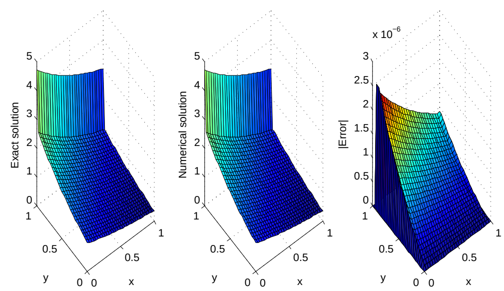


Figure 4.11: Numerical solution and error of the scheme (3.8) based on uniform mesh for Problem 3 ($\varepsilon = 10^{-8}$, $N_x = N_y = 30$).

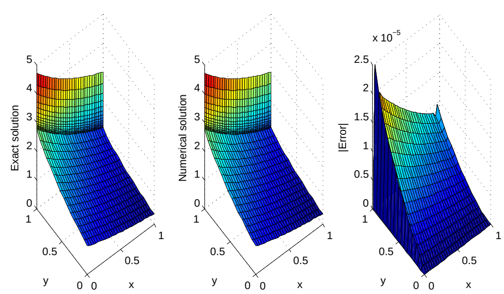


Figure 4.12: Numerical solution and error of the scheme (3.8) based on Shishkin mesh for Problem 3 ($\varepsilon = 10^{-8}$, $N_x = N_y = 30$).

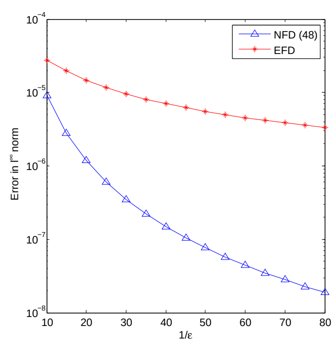


Figure 4.13: Development of the maximum error for Problem 3 with $h_x = h_y = \varepsilon$.

Problem 3. 2D problem with variable coefficient

In this subsection, we will verify the proposed new scheme for the convection-diffusion problem in 2D. Assume that $\alpha(x, y) = 0$, $\beta(x, y) = 1/(1 + y)$ in (1.1), and the exact solution $u(x) = \exp(y - x) + 2^{-1/\varepsilon}(1 + y)^{1+1/\varepsilon}$ (see [24]). The source term f is determined by (1.1). Due to $\alpha(x,y)=0$, the NFD scheme for x direction (3.3) will reduce to the one mentioned in Remark 2.6.

$N_x \times N_y$	NFD (3.8)	Order	EFD [24]	Order
$\varepsilon = 10^{-1}$				
10×10	9.2127e-06	-	2.7214e-05	-
20×20	5.8688e-07	3.9725	1.7155e-06	3.9877
40×40	3.6636e-08	4.0017	1.0745e-07	3.9969
80×80	2.2895e-09	4.0002	6.7194e-09	3.9992
$\varepsilon = 10^{-3}$				
10×10	1.5268e-05	-	1.5538e-04	-
20×20	1.0106e-05	0.5954	4.6988e-05	1.7254
40×40	8.0871e-07	3.6434	1.4614e-05	1.6850
80×80	9.8258e-08	3.0410	1.0982e-04	-2.9098
$\varepsilon = 10^{-5}$				
10×10	5.6088e-05	-	1.5526e-04	-
20×20	8.0633e-06	2.7982	5.2017e-05	1.5776
40×40	1.0574e-06	2.9308	1.4899e-05	1.8037
80×80	7.1938e-08	3.8777	3.9769e-06	1.9055
$\varepsilon = 10^{-7}$				
10×10	5.6223e-05	-	1.5522e-04	-
20×20	8.1239e-06	2.7909	5.1993e-05	1.5779
40×40	1.0997e-06	2.8851	1.4884e-05	1.8046
80×80	1.4506e-07	2.9223	3.9730e-06	1.9054
$\varepsilon = 10^{-9}$				
10×10	2.0081e-04	-	1.5522e-04	-
20×20	1.0825e-05	4.2135	5.1993e-05	1.5779
40×40	8.8350e-07	3.6149	1.4883e-05	1.8046
80×80	1.6068e-07	2.4591	3.9728e-06	1.9055

Table 4.3: Convergence order based on uniform mesh in l^∞-norm for Problem 3.

We first show the numerical solutions and the absolute errors for the new scheme (3.8) based on the unform mesh and Shishkin mesh in Figures 4.11 and 4.12 respectively when $\varepsilon = 10^{-8}$ and $N_x = N_y = 30$. It is obvious that no numerical oscillation can be found. Figure 4.13 is also devoted to the development of the error in l^∞-norm when $h_x = h_y = \varepsilon$ based on the uniform mesh. The similar results with that for the 1D equation are obtained:

the maximum error obtained from the NFD scheme decreases more sharply than that of the EFD scheme. Then, in Table 4.3, for series of fixed ε, the maximum errors and convergence rates are compared between the NFD scheme (3.8) and the EFD scheme.

$N_x \times N_y$	NFD (3.7)	Order	NFD (3.8)	Order
$\varepsilon = 10^{-1}$				
10×10	4.7257e-04	-	7.0962e-06	-
20×20	1.6449e-04	1.5226	9.3634e-07	2.9219
40×40	6.1951e-05	1.4088	1.2866e-07	2.8635
80×80	2.2536e-05	1.4589	3.5285e-08	1.8664
$\varepsilon = 10^{-3}$				
10×10	8.6043e-03	-	7.8792e-04	-
20×20	2.2807e-03	1.9156	1.3722e-05	5.8434
40×40	4.1783e-04	2.4485	9.7671e-06	0.4905
80×80	9.3091e-05	2.1662	7.7608e-07	3.6536
$\varepsilon = 10^{-5}$				
10×10	9.7846e-03	-	8.9399e-04	-
20×20	2.6423e-03	1.8887	8.9094e-05	3.3268
40×40	6.8022e-04	1.9577	1.0128e-05	3.1370
80×80	1.7266e-04	1.9781	1.1557e-06	3.1316
$\varepsilon = 10^{-7}$				
10×10	9.7977e-03	-	8.9442e-04	-
20×20	2.6527e-03	1.8850	8.9675e-05	3.3182
40×40	6.8925e-04	1.9444	1.0336e-05	3.1170
80×80	1.7556e-04	1.9731	1.2572e-06	3.0394
$\varepsilon = 10^{-9}$				
10×10	9.7978e-03	-	8.6099e-03	-
20×20	2.6528e-03	1.8849	1.7403e-04	5.6286
40×40	6.8934e-04	1.9442	1.1075e-05	3.9740
80×80	1.7566e-04	1.9726	1.6324e-06	2.7622

Table 4.4: Convergence order based on Shishkin mesh in l^∞-norm for Problem 3.

It can be observed that the new method has much higher computational accuracy and convergence order although ε is very small. Based on Shishkin mesh, the maximum errors and convergence orders of the NFD schemes (3.7) and (3.8) are also shown in Table 4.4. Again, the NFD schemes work very well in this case. Moreover, the NFD scheme (3.8) achives third convergence order which, to the best of our knowledge, is the first time to get this accuracy for the two-dimensional singularly perturbed convection-diffusion equations on Shishkin mesh.

5. Conclusions

A series of new finite difference methods are constructed for the 1D and 2D convection-dominate diffusion equations with constant and variable coefficients. For 1D problems, although the new schemes are high order methods, they have the same structure linear system as the standard difference scheme. The new schemes have the attractive advantages that there are no numerical oscillation and much higher accuracy than other methods when solving the singularly perturbed problems. Moreover, the new methods can keep the convergence order stable much better than others when the diffusion coefficient becomes smaller. Since the schemes in 2D are directly derived from the schemes for 1D case, these advantages are also hold when solving the 2D problems. Although better simulation results can be got by applying the special mesh technique to the NFD scheme, the new scheme is constructed based on the stability of the analytical solution, not the local asymptotic behavior. It can be easily extended to other types of linear and nonlinear singular perturbed problems [25, 26, 30, 31] whose oscillation location and asymptotic behavior are usually very complicated. These will be considered in the future.

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Xuefei He and Kun Wang
College of Mathematics and Statistics, Chongqing University, Chongqing 401331, China and Institute of Computing and Data Sciences, Chongqing University, Chongqing 400044, China
E-mail address: xuefei@cqu.edu.cn, kunwang@cqu.edu.cn