New forms of the Taylor's remainder

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Abstract

We present two new forms of the remainder in Taylor's formula involving a generalization of the Taylor-Langrange formula. An asymptotic formula of the Taylor's remainder for real analytic functions is given as application.

Keywords : *Taylor's remainder, harmonic alternating series, real analytic functions.*

1. Introduction

Let us denote by $(\{c,d\})$ the open interval $(\min\{c,d\},\max\{c,d\})$, and by $[\{c,d\}]$ the closed interval $[\min\{c,d\},\max\{c,d\}]$ for all $c,d\in\mathbb{R}$ with $c\neq d$.

The most popular forms of remainder in Taylor's formula are the classical well known integral, Langrange's and the Cauchy's forms of remainder. The Langrange's and Cauchy's forms are special cases of the Schloemilch-Roeche's remainder:

Theorem 1. *Let $a,b\in\mathbb{R}$ such that $a\neq b$. Let $f:[\{a,b\}]\to\mathbb{R}$ be a mapping, such that $f\in C^n([\{a,b\}])$, $f^{(n+1)}$ exists on $(\{a,b\})$ and $f^{(n+1)}(t)\neq 0$ for*

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all $t \in (\{a,b\})$. *Then for a positive integer* p *not greater than* $n+1$, *there is one* $\xi \in (\{a,b\})$ *such that*

$$R_n(f;a,b) = \frac{(b-a)^p(b-\xi)^{n+1-p}}{n!p} f^{(n+1)}(\xi)\,,$$

where $R_n(f;a,b) := f(b) - \sum_{k=0}^{n} \frac{(b-a)^k}{k!} f(k)(a)$.

Setting $p = n+1$ in the previous form of remainder, we obtain the Langrange's remainder.

$$R_n(f;a,b) = \frac{(b-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)\,, \text{ for some } \xi \in (\{a,b\})\,, \tag{1.1}$$

while for $p = 1$ follows the Cauchy's remainder

$$R_n(f;a,b) = \frac{(b-a)(b-\xi)^n}{n!} f^{(n+1)}(\xi), \text{ for some } \xi \in (\{a,b\})\,.$$

Also, G.A. Anastassiou and S.S. Dragomir have given some new bounds for their remainder [2].

Many other researchers have developed different forms of Taylor's remainder in order to improve the bounds of error. In the literature (see for example [1], [4], [5], [6], [8], [9], [10]) many forms of the remainder in the Taylor's formula are given.

In this paper, we present a new form of remainder in Taylor's formula, namely:

Theorem 2. *If* $g \in C^{n+1}([\{a,b\}])$, $g^{(n+2)}$ *exists on* $(\{a,b\})$, $g^{(n+1)}(a) \neq 0$, *and* $g^{(n+2)}(t) \neq 0$ *for all* $t \in (\{a,b\})$, *then there is a number* $\xi \in (\{a,b\})$ *such that*

$$R_n(g,a;b) = \frac{g^{(n+1)}(a)(b-a)^{n+1}}{(n+1)!} \cdot \frac{(n+2)g^{(n+1)}(\xi) + (\xi-a)g^{(n+2)}(\xi)}{(n+2)g^{(n+1)}(\xi) + (\xi-b)g^{(n+2)}(\xi)}\,. \tag{1.2}$$

In many cases the remainder which is defined in (1.2) gives essentially better bounds of error than all other known forms of remainder. With the following example on the harmonic alternating series $1 - \frac{1}{2} + \frac{1}{3} - \dots$ we would like to show the significance of our form (1.2):

For any integer $n \geq 1$ the following estimation holds

$$\frac{n+2}{(n+1)(2n+3)} < \left| \sum_{k=n+1}^{\infty} \frac{(-1)^{k+1}}{k} \right| < \frac{n+3}{(2n+4)(n+1)} . \tag{1.3}$$

Remark 1. The main result of László Tóth and József Bukor in paper [7] is the following inequality

$$\frac{1}{2n+a} < \left| \sum_{k=n+1}^{\infty} \frac{(-1)^{k+1}}{k} \right| < \frac{1}{2n+1} , \quad (n \geq 1) \tag{1.4}$$

where $a = \dfrac{1}{1-\ln 2} - 2 \simeq 1.258891$. A simple calculation yields $\dfrac{1}{2n+a} < \dfrac{n+2}{(n+1)(2n+3)}$ for all $n \geq 2$. Thus the lower bound in our estimation (1.3) is sharper than in (1.4).

Also, another interesting application of the form of remainder in Theorem 2 allow us to study the behavior of the Taylor's remainder $R_n(f,a;b)$ as $n \to \infty$, for real analytic functions.

The paper is organized as follows: In Section 2, we prove the Theorem 2 and the inequality (1.3) (see Example 1) and give one more example. The Section 3 is devoted to the study of asymptotic behavior of Taylor's remainder for real analytic functions as well as to the behavior of ξ in the Langrange's form of remainder (1.1). The obtained results are applied to real exponential polynomials. In the last section we present another form of remainder in Taylor's formula, which in some cases gives essentially better bounds of error than the Langrange's form.

2. Proof of Theorem 2 and examples

For our purpose, we need the following generalization of Taylor's formula [3, Theorem 5.20, p. 113]:

Theorem 3. *If $f^{(n)}$, $g^{(n)}$ are continuous on $[\{a,b\}]$, and $f^{(n+1)}$, $g^{(n+1)}$ exist on $(\{a,b\})$, and if $g^{(n+1)}(t) \neq 0$ for any $t \in (\{a,b\})$, then there is a number $\xi \in (\{a,b\})$ such that*

$$\frac{R_n(f;a,b)}{R_n(g;a,b)} = \frac{f^{(n+1)}(\xi)}{g^{(n+1)}(\xi)}.$$

Theorem 3 is very useful for the achievement of new forms of the remainder in Taylor's formula by suitable choice of the functions f and g.

The next Lemma 1 is an application of Theorem 3.

Lemma 1. *Let m, n be positive integers. If $g^{(m+n)}$ is continuous on $[\{a,b\}]$, $g^{(m+n+1)}$ exists on $(\{a,b\})$ and $g^{(m+n+1)}(t) \neq 0$ for all $t \in (\{a,b\})$, then there is a number $\xi \in (\{a,b\})$ such that*

$$\frac{R_n(g;a,b)}{R_{n+m}(g;a,b)} \frac{((\xi-a)^m g(\xi))^{(m+n+1)}}{(b-a)^m g^{(m+n+1)}(\xi)} .$$

Proof. Let f be a function defined on $[\{a,b\}]$ by

$$f(x) = (x-a)^m g(x) . \tag{2.1}$$

According to Theorem 3 there is a number $\xi \in (\{a,b\})$ such that

$$\frac{R_{n+m}(f;a,b)}{R_{n+m}(g;a,b)} = \frac{f^{(n+m+1)}(\xi)}{g^{(n+m+1)}(\xi)} . \tag{2.2}$$

Using Leibnitz derivative formula on (2.1), we calculate

$$f^{(k)}(a) = 0 , \quad \text{for } 0 < k < m , \tag{2.3}$$

and

$$f^{(k)}(a) = \frac{k!}{(k-m)!} g^{k-m}(a) , \quad \text{for } m \le k \le n+m+1 . \tag{2.4}$$

Using (2.1), (2.3), (2.4) in (2.2), we obtain

$$\frac{(b-a)^m g(b) - \sum_{k=m}^{m+n} \frac{(b-a)^k}{(k-m)!} g^{(k-m)}(a)}{R_{n+m}(g;a,b)} = \frac{((\xi-a)^m g(\xi))^{(n+m+1)}}{g^{(n+m+1)}(\xi)} .$$

Substituting $k - m = j$, we obtain

$$\frac{(b-a)^m \left(g(b) - \sum_{j=0}^{n} \frac{(b-a)^j}{j!} g^{(j)}(a) \right)}{R_{n+m}(g;a,b)} = \frac{((\xi-a)^m g(\xi))^{(n+m+1)}}{g^{(n+m+1)}(\xi)} ,$$

which proves Lemma 1. □

***Proof of Theorem* 2**. Clearly the function g satisfies the assumptions of Lemma 1 with $m = 1$. Thus, according to Lemma 1, there is a ξ in $(\{a,b\})$ such that

$$\frac{R_n(g;a,b)}{R_{n+1}(g;a,b)} = \frac{((\xi-a)g(\xi))^{(n+2)}}{(b-a)g^{(n+2)}(\xi)} .$$

Using the Leibnitz derivative formula in the numerator on the right part fraction, we take

$$\frac{R_n(g;a,b)}{R_{n+1}(g;a,b)} = \frac{(n+2)g^{(n+1)}(\xi) + (\xi - a)g^{(n+2)}(\xi)}{(b-a)g^{(n+2)}(\xi)} .$$

This can be written

$$\begin{aligned}&((n+2)g^{(n+1)}(\xi) + (\xi - a)g^{(n+2)}(\xi))R_{n+1}(g;a,b)\\ &= \left(R_{n+1}(g;a,b) - \frac{(b-a)^{n+1}g^{(n+1)}(a)}{(n+1)!}\right)(b-a)g^{(n+2)}(\xi) ,\end{aligned}$$

or equivalently

$$\begin{aligned}&((n+2)g^{(n+1)}(\xi) + (\xi - b)g^{(n+2)}(\xi))R_{n+1}(g;a,b)\\ &= \frac{(b-a)^{n+2}g^{(n+1)}(a)}{(n+1)!}g^{(n+2)}(\xi) . \end{aligned} \tag{2.5}$$

According to the assumptions of this theorem, we conclude that the right part of (2.5) is non zero. Consequently

$$(n+2)g^{(n+1)}(\xi) + (\xi - b)g^{(n+2)}(\xi) \neq 0 . \tag{2.6}$$

Now, from (2.5), follows

$$\begin{aligned}&((n+2)g^{(n+1)}(\xi) + (\xi - b)g^{(n+2)}(\xi))\left(R_n(g;a,b) + \frac{(b-a)^{n+1}g^{(n+1)}(a)}{(n+1)!}\right)\\ &= \frac{(b-a)^{n+2}g^{(n+1)}(a)}{(n+1)!}g^{(n+2)}(\xi) ,\end{aligned}$$

or equivalently

$$\begin{aligned}&((n+2)g^{(n+1)}(\xi) + (\xi - b)g^{(n+2)}(\xi))R_n(g;a,b)\\ &= \frac{(b-a)^{n+2}g^{n+1}(a)}{(n+1)!}(((n+2)g^{(n+1)}(\xi) + (\xi - a)g^{(n+2)}(\xi)) .\end{aligned}$$

From this, and from (2.6), follows the conclusion. □

With the next two examples, we want to show that the Taylor's remainder of the Theorem 2 in many cases gives essentially finer bounds of error than the Langrange's remainder.

Example 1. Applying Theorem 2 to $g(x) = \ln(1+x)$, $-1 < x$, $x \neq 0$, we find out easily that there is at least one $\xi \in (\{0, x\})$ such that

$$R_n(g;0,x) := \ln(1+x) - \sum_{k=1}^{n} \frac{(-1)^{k-1}x^k}{k}$$

$$= \frac{(-1)^n x^{n+1}}{k+1} \cdot \frac{n+2+\xi}{(n+2)(x+1)+\xi-x}.$$

Therefore, easily we obtain, the following asymptotic formula

$$|R_n(g;0,x)| \sim \frac{|x|^{n+1}}{(n+1)(x+1)}, \text{ as } n \to \infty,$$

and the following estimation

$$\frac{(n+2)|x|^{n+1}}{((n+1)x+n+2)(n+1)} < \left| \sum_{k=n+1}^{\infty} \frac{(-1)^{k+1}x^k}{k} \right| < \frac{(n+2+x)|x|^{n+1}}{((n+2)(x+1))(n+1)}. \tag{2.7}$$

Thus the difference of bounds of estimation (2.7) is

$$A := \frac{|x|^{n+3}}{(n+2)(n(x+1)+x+2)(x+1)},$$

while the corresponding difference by using the Langrange's form of remainder is

$$B := \frac{|(1+x)^{n+1}-1|\,|x|^{n+1}}{(n+1)(1+x)^{n+1}}.$$

It is obvious that A is essentially smaller than B. Now, setting $x = 1$ in (2.7) we obtain inequality (1.3).

Example 2. Applying Theorem 2 to $g(x) = (1+x)^s$, $s \notin \mathbb{Z}$, $x \neq -1$, $x \neq 0$, then there is one $\xi \in (\{0, x\})$ such that

$$R_n(g;0,x) := (1+x)^s - \sum_{k=0}^{n} \frac{s(s-1)\dots(s-k+1)x^k}{k!} = \frac{s(s-1)\dots(s-n)(n+2+\xi(s+1))x^{n+1}}{(n+1)!(n+2+\xi(s+1)+x(n+1-s))}.$$

From this, we obtain the following asymptotic formula

$$R_n(g;0,x) \sim \frac{s(s-1)\dots(s-n)x^{n+1}}{(n+1)!(1+x)}, \text{ as } n \to \infty,$$

and for all $n > s$, the following estimation

$$\frac{|s(s-1)\dots(s-n)|\,(n+2+x(s+1))|x|^{n+1}}{(x+1)(n+2)} \frac{|x|^{n+1}}{(n+1)!} \leq \left| (1+x)^s - \sum_{k=0}^{n} \frac{s(s-1)\dots(s-k+1)x^k}{k!} \right|$$

$$\leq \frac{|s(s-1)\dots(s-n)|\,(n+2)}{(n+2)(x+1)-x-s} \cdot \frac{|x|^{n+1}}{(n+1)!}\,.$$

It is obvious that the previous estimation is better than the corresponding, which is obtained by using the Langrange's form of remainder.

3. The Taylor's remainder of real analytic functions

In this section, we will study the asymptotic behavior of Langrange's remainder in (1.1) as $n \to \infty$, for real analytic functions

Theorem 4. *If the radius of convergence of the Maclaurin's series expansion of a function g is infinite, and there is a positive integer n_0, such that $g^{(n)}(t) \neq 0$ for all integers $n \geq n_0$ and all $t \in [\{a,b\}]$, then*

$$R_n(g;a,b) \sim \frac{g^{(n+1)}(a)(b-a)^{n+1}}{(n+1)!}\,.$$

Proof. Let t be any number in $[\{a,b\}]$. Then the radius of convergence of the Taylor's series expansion of g about t,

$$g(x) = \sum_{n=0}^{\infty} \frac{(x-t)^n}{n!} g^{(n)}(t)\,,$$

is infinite, and hence, according to D'Alembert's formula for the radius of convergence, is valid

$$\lim_{n\to\infty} \frac{g^{(n+1)}(t)}{(n+1)g^{(n)}(t)} = 0\,.$$

Now, since the function g satisfies the assumptions of Theorem 2, we can use the form of $R_n(g;a,b)$, which is given in Theorem 2. Thus

$$\lim_{n\to\infty} \frac{R_n(g;a,b)}{\dfrac{g^{(n+1)}(a)(b-a)^{n+1}}{(n+1)!}} = \lim_{n\to\infty} \frac{1+(\xi-a)\dfrac{g^{(n+2)}(\xi)}{(n+2)g^{(n+1)}(\xi)}}{1+(\xi-b)\dfrac{g^{(n+2)}(\xi)}{(n+2)g^{(n+1)}(\xi)}}$$

$$= 1\,. \qquad \square$$

Theorem 4 gives us the motivation to search the behavior of ξ in the Langrange's form of the remainder in Taylor's formula (1.1):

Theorem 5. *Let $f : [\{a,b\}] \to \mathbb{R}$ be a mapping with the assumptions of Theorem 1. Moreover if $f^{(n+2)} \in C^{n+2}([\{a,b\}])$, and $f^{(n+2)}(x) \neq 0$ for all*

$x \in [\{a,b\}]$, *then holds*

$$|\xi - a| \leq \frac{|b-a|}{n+2} \frac{\max\limits_{x\in[\{a,b\}]} |f^{n+2}(x)|}{\min\limits_{x\in[\{a,b\}]} |f^{n+2}(x)|} \,. \tag{3.1}$$

Proof. From (1.1), we have

$$R_{n+1}(f,a;b) = \frac{(b-a)^{n+1}(f^{(n+1)}(\xi) - f^{(n+1)}(a))}{(n+1)!} \,. \tag{3.2}$$

In (3.2), we apply the mean value theorem to $f^{(n+1)}(\xi)$:

There is a number ρ in $(\{a,\xi\})$ such that

$$R_{n+1}(f,a;b) = (\xi - a) f^{(n+2)}(\rho) \frac{(b-a)^{n+1}}{(n+1)!} \,. \tag{3.3}$$

On the other hand, by the Taylor-Langrange formula, we have that for some $\sigma \in (\{a,b\})$

$$R_{n+1}(f,a;b) = \frac{(b-a)^{(n+2)} f^{(n+2)}(\sigma)}{(n+2)!} \,. \tag{3.4}$$

Combining (3.3) with (3.4), we obtain

$$(\xi - a) f^{(n+2)}(\rho) = \frac{b-a}{n+2} f^{(n+2)}(\sigma) \,,$$

and since $f^{(n+2)}(x) \neq 0$ for all $x \in [\{a,b\}]$, we get (3.1). □

Now, let us denote by $\mathcal{L}_{[\{a,b\}]}$ the set of all functions $f \in C^\infty([\{a,b\}])$ with the assumptions:

There is one positive integer n_0 such that $f^{(n)}(x) \neq 0$ for all $n \geq n_0$ and all $x \in [\{a,b\}]$, and

$$\lim_{n\to\infty} \frac{\max\{|f^{(n)}(a)|, |f^{(n)}(b)|\}}{\min\{|f^{(n)}(a)|, |f^{(n)}(b)|\}} = 0 \,.$$

Corollary 1. *Let* $f \in \mathcal{L}_{[\{a,b\}]}$, *and* ξ *be as in Theorem* 5. *Then* $\xi \to a$ *as* $n \to \infty$.

Proof. Using Theorem 5 we get immediately the conclusion. □

Remark 2. Let

$$P(x) = \sum_{k=1}^{m} c_k e^{\lambda_k x}$$

($c_k \in \mathbb{R}/\{0\}$, $\lambda_k \in \mathbb{R}$, with $\lambda_1 < \lambda_2 < \ldots < \lambda_m$) be any real exponential polynomial. Then we have

$$\begin{aligned} P^{(n)}(x) &= \sum_{k=1}^{m} \lambda_k^n c_k e^{\lambda_k x} \\ &= \lambda_m^n \left(c_m e^{\lambda_m x} + \sum_{k=1}^{m-1} \left(\frac{\lambda_k}{\lambda_m} \right)^n c_\lambda e^{\lambda_k x} \right) . \end{aligned}$$

Thus,

$$\begin{aligned} |P^{(n)}(x)| &\geq |\lambda_m|^n \left(|c_m| e^{\lambda_m x} - \sum_{k=1}^{m-1} \left| \frac{\lambda_k}{\lambda_m} \right|^n |c_\lambda| e^{\lambda_k x} \right) \\ &> |\lambda_m|^n \left(|c_m| e^{\lambda_m x} - \left| \frac{\lambda_{m-1}}{\lambda_m} \right|^n \sum_{k=1}^{m-1} |c_\lambda| e^{\lambda_k x} \right) . \end{aligned}$$

Then, it is straight forward to verify that for all $n \geq n_0$ and all $x \in [\{a, b\}]$, holds

$$P^{(n)}(x) \neq 0 ,$$

where

$$n_0 := \left\lceil \frac{\ln \dfrac{\max\limits_{x\in[a,b]} \sum\limits_{k=1}^{m} |c_k| e^{\lambda_k x}}{|c_m| \min\{e^{\lambda_m a}, e^{\lambda_m b}\}}}{\ln \left| \dfrac{\lambda_m}{\lambda_{m-1}} \right|} \right\rceil .$$

So, according to Theorem 5, we have that

$$R_n(P; a, b) \sim \frac{P^{(n+1)}(a)(b-a)^{n+1}}{(n+1)!} .$$

Further, a simple calculation yields

$$\begin{aligned} \lim_{n\to\infty} \frac{\max\{|P^{(n)}(a)|, |f^{(n)}(b)|}{n \min\{|f^{(n)}(a)|, |f^{(n)}(b)|} &= \lim_{n\to\infty} \frac{\max\{e^{\lambda_m a}, e^{\lambda_m b}\}}{n \min\{e^{\lambda_m a}, e^{\lambda_m b}\}} \\ &= 0 . \end{aligned}$$

Therefore, $P \in \mathcal{L}_{[\{a,b\}]}$. So, according to Corollary 1 we have that $\xi \to a$ as $n \to \infty$, where ξ is defined via $R_n(P; a, b) = \dfrac{P^{(n+1)}(\xi)(b-1)^{n+1}}{(n+l)!}$.

4. Another form of the Taylor's remainder

Using Theorem 3 we find out another form of the Taylor's remainder:

Theorem 6. *Let $a \in \mathbb{R}$, $r \in \mathbb{R}^+$, and let f be a function differentiable of order $n+1$ on the open interval $(a-r, a+r)$. Then, for any $b \in (a-r, a+r)$ with $b \neq a$, there is at least one $\xi \in (\{a,b\})$ such that*

$$R_n(f;a,b) = \frac{(b-a)^{n+1}}{(n+1)!} \frac{\left(1 - \frac{|\xi - a|}{r}\right)^{n+2}}{\left(1 - \frac{|b-a|}{r}\right)} f^{(n+1)}(\xi). \tag{4.1}$$

Proof. We consider the mapping $g(x) = \left(1 - \frac{x-a}{r}\right)^{-1}$ defined on $(a-r, a+r)$. Then, for all non negative integers k, a simple calculation yields

$$g^{(k)}(x) = \frac{k!}{r^k}\left(1 - \frac{x-a}{r}\right)^{-(k+1)},$$

and hence

$$\begin{aligned} R_n(g;a,b) &= \left(\sum_{k=0}^{\infty}\left(\frac{x-a}{r}\right)^k\right) - \left(\sum_{k=0}^{n}\left(\frac{x-a}{r}\right)^k\right) \\ &= \left(\frac{x-a}{r}\right)^{n+1} \frac{1}{1 - \frac{x-a}{r}}. \end{aligned}$$

Applying Lemma 1 to $g(x) = \left(1 - \frac{x-a}{r}\right)^{-1}$, and using the above formulas, we have, that there exists one $\xi \in (\{a,b\})$ such that

$$R_n(f;a,b) = \frac{(b-a)^{n+1}\left(1 - \frac{\xi - a}{r}\right)^{n+2} f^{(n+1)}(\xi)}{\left(1 - \frac{b-a}{r}\right)(n+1)!}.$$

Therefore, for any b with $a < b < a + r$, there is a number $\xi \in (a,b)$ such that

$$R_n(f;a,b) = \frac{(b-a)^{n+1}\left(1 - \frac{|\xi - a|}{r}\right)^{n+2} f^{(n+1)}(\xi)}{\left(1 - \frac{|b-a|}{r}\right)(n+1)!}.$$

Now, we consider the mapping $g(x) = \left(1+\frac{x-a}{r}\right)^{-1}$ defined on $(a-r, a+r)$. It can be verified that for any $k \in \mathbb{N}$, holds

$$g^{(k)}(x) = (-1)^k \frac{k!}{r^k}\left(1+\frac{x-a}{r}\right)^{-(k+1)},$$

and

$$\begin{aligned} R_n(g;a,b) &= \left(\sum_{k=0}^{\infty}(-1)^k\left(\frac{x-a}{r}\right)^k\right) - \left(\sum_{k=0}^{n}(-1)^k\left(\frac{x-a}{r}\right)^k\right) \\ &= (-1)^{n+1}\left(\frac{x-a}{r}\right)^{n+1}\frac{1}{1-\frac{x-a}{r}}. \end{aligned}$$

Applying Lemma 1 to $g(x) = \left(1+\frac{x-a}{r}\right)^{-1}$ and repeating the above steps we obtain, that there exists one $\xi \in (\{a,b\})$, such that

$$R_n(f;a,b) = \frac{(b-a)^{n+1}\left(1+\frac{\xi-a}{r}\right)^{n+2} f^{(n+1)}(\xi)}{\left(1+\frac{b-a}{r}\right)(n+1)!}.$$

Therefore, for any b with $a-r<b<a$, $b \neq a$, there is one $\xi \in (a-r,a)$ such that

$$R_n(f;a,b) = \frac{(b-a)^{n+1}\left(1-\frac{|\xi-a|}{r}\right)^{n+2} f^{(n+1)}(\xi)}{\left(1-\frac{|b-a|}{r}\right)(n+1)!}.$$

Finally, for any $b \in (a-r,a+r)$ with $b \neq a$ there is one $\xi \in (\{a,b\})$ such that

$$R_n(f;a,b) = \frac{(b-a)^{n+1}\left(1-\frac{|\xi-a|}{r}\right)^{n+2} f^{(n+1)}(\xi)}{\left(1-\frac{|b-a|}{r}\right)(n+1)!}. \qquad \square$$

Remark 3. Let a,b,r,f be as in Theorem 6. Suppose that the mapping $f_{n+1} : (a-r,a+r) \to \mathbb{R}$ defined via $f_{n+1}(x) := \left(1-\frac{|x-a|}{r}\right)^{n+2}|f^{(n+1)}(x)|$ is increasing and bounded on $(\{a,b\})$. Then from (4.1) easily we get the following estimation

$$|R_n(f;a,b)| \le \frac{(b-a)^{n+1}\left(1-\frac{|b-a|}{r}\right)^{n+1}}{(n+1)!} \sup_{x\in(a,b)} |f^{(n+1)}(x)|. \quad (4.2)$$

Moreover, from the assumption $b \in (a-r, a+r)$, we have $0 < 1 - \frac{|b-a|}{r} < 1$. Therefore the estimation (4.2) gives essentially better bounds of error than the corresponding, which are resulting by using the Langrange's form of remainder.

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