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NEW FRACTIONAL INEQUALITIES OF OSTROWSKI-GRÜSS TYPE

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In this paper, we improve and further generalize some Ostrowski-Grüss type inequalities for the fractional integrals by using new Montgomery identities.

1. Introduction and preliminary results

The inequality of Ostrowski provides us an estimates for the deviation of the values of a smooth function from its mean value. More precisely, if $f:[a,b]\to\mathbb{R}$ is a differentiable function with bounded derivative, then

$$\left| f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\left\|f'\right\|_\infty$$

for every $x\in[a,b]$. Moreover the constant $1/4$ is the best possible. For some generalizations of this classical inequality see the book [12, p.468-484] by Mitrinovic, Pecaric and Fink. A simple proof of this inequality can be done by using the following identity [12]: If $f:[a,b]\to\mathbb{R}$ is differentiable on (a,b) with the first derivative f' integrable on $[a,b]$, then Montgomery identity holds:

$$f(x)=\frac{1}{b-a}\int_a^b f(t)dt+\int_a^b P_1(x,t)f'(t)dt,$$

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where $P_1(x,t)$ is the Peano kernel defined by

$$P_1(x,t) := \begin{cases} \dfrac{t-a}{b-a}, & a \leq t < x \\ \\ \dfrac{t-b}{b-a}, & x \leq t \leq b. \end{cases}$$

This inequality gives an upper bound for the approximation of integral mean of a function f by the functional value $f(x)$ at $x \in [a,b]$. In 2001, Cheng [1] proved the following Ostrowski-Grüss type integral inequality.

Theorem 1.1. *Let $I \subset \mathbb{R}$ be an open interval, $a,b \in I$, $a < b$. If $f : I \to \mathbb{R}$ is a differentiable function such that there exist constants $\gamma, \Gamma \in \mathbb{R}$, with $\gamma \leq f'(x) \leq \Gamma$, $x \in [a,b]$. Then have*

$$\left| \frac{1}{2} f(x) - \frac{(x-b)f(b) - (x-a)f(a)}{2(b-a)} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \tag{1}$$
$$\leq \frac{(x-a)^2 + (b-x)^2}{8(b-a)}(\Gamma - \gamma), \textit{for all } x \in [a,b].$$

Theorem 1.1 is a generalization of the following Ostrowski-Grüss type integral inequality, which was firstly by Dragomir and Wang in [2] and further improved by Matic et al. in [6].

Theorem 1.2. *Let the assumptions of Theorem 1.1 hold. Then for all $x \in [a,b]$, we have*

$$\left| f(x) - \frac{f(b) - f(a)}{b-a}\left(x - \frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{1}{4}(b-a)(\Gamma - \gamma) \tag{2}$$

The above two inequalities are both connections between the Ostrowski inequality [4] and the Grüss inequality [5] and can be applied to bound some special mean and some numerical quadrature rules. During the past few years many researchers have attracted considerable attention to the above inequalities and various generalizations, extensions and variants of these inequalities have appeared in the literature, see [1], [2], [6], [24] and the references cited therein. For recent results and generalizations regarding Ostrowski and Grüss inequalities, we refer the reader to the recent papers [1]-[6], [12]-[15], [17]-[24]. Now, we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [11], [16].

Definition 1.3. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ with $a \geq 0$ is defined as

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt,$$
$$J_a^0 f(x) = f(x).$$

The theory of fractional calculus has known an intensive development over the last few decades. It is shown that derivatives and integrals of fractional type provide an adequate mathematical modelling of real objects and processes see ([7], [8], [9], [10], [20], [21], [22]). Therefore, the study of fractional differential equations need more developmental of inequalities of fractional type. In [7] and [20], the authors established some inequalities for differentiable mappings which are connected with Ostrowski type inequality by used the Riemann-Liouville fractional integrals, and they used the following lemma to prove their results:

Lemma 1.4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable function on I° with $a, b \in I = [a,b]$ $(a < b)$ and $f' \in L_1[a,b]$, then, for $\alpha \geq 1$*

$$f(x) = \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha} J_a^\alpha f(b) - J_a^{\alpha-1}(P_2(x,b)f(b)) + J_a^\alpha (P_2(x,b)f'(b)) \quad (3)$$

where $P_2(x,t)$ is the fractional Peano kernel defined by

$$P_2(x,t) = \begin{cases} \dfrac{t-a}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), & a \leq t < x \\ \\ \dfrac{t-b}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), & x \leq t \leq b. \end{cases}$$

In this article, we use the Riemann-Liouville fractional integrals to establish some new integral inequalities of Ostrowski-Grüss type. From our results, the classical Ostrowski-Grüss type inequalities can be deduced as some special cases.

2. Main Results

Lemma 2.1. *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on (a,b) $(a < b)$, $\alpha \geq 1$ and $f' \in L_1[a,b]$, then the generalization of the Montgomery identity for* *fractional integrals holds:*

$$f(x) = (\alpha+1)\Gamma(\alpha)\frac{(b-x)^{1-\alpha}}{(b-a)}\boldsymbol{J}_a^{\alpha} f(b) - \boldsymbol{J}_a^{\alpha-1}(P_2(x,b)f(b)) - \frac{(b-x)^{2-\alpha}}{(b-a)}\Gamma(\alpha)\boldsymbol{J}_a^{\alpha-1} f(b) - \frac{(b-x)^{1-\alpha}(x-a)}{(b-a)^{2-\alpha}} f(a) + 2\boldsymbol{J}_a^{\alpha}\left(K_1(x,b) f'(b)\right), \quad (4)$$

where $K_1(x,t)$ is the fractional Peano kernel defined by

$$K_1(x,t) := \begin{cases} \left(t - \frac{a+x}{2}\right)\frac{(b-x)^{1-\alpha}}{b-a}\Gamma(\alpha), & a \le t < x \\ \left(t - \frac{b+x}{2}\right)\frac{(b-x)^{1-\alpha}}{b-a}\Gamma(\alpha), & x \le t \le b. \end{cases} \quad (5)$$

Proof. By definition of $K_1(x,t)$, we have

$$\begin{aligned} \boldsymbol{J}_a^{\alpha}\left(K_1(x,b) f'(b)\right) &= \frac{1}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1} K_1(x,t) f'(t)\,dt \\ &= \frac{(b-x)^{1-\alpha}}{b-a}\left[\int_a^x (b-t)^{\alpha-1}\left(t - \frac{a+x}{2}\right) f'(t)\,dt \right. \\ &\quad \left. + \int_x^b (b-t)^{\alpha-1}\left(t - \frac{b+x}{2}\right) f'(t)\,dt\right] \end{aligned}$$

that can be written as

$$\begin{aligned} &\boldsymbol{J}_a^{\alpha}\left(K_1(x,b) f'(b)\right) \\ &= \frac{1}{2}\boldsymbol{J}_a^{\alpha}(P_2(x,b)f'(b)) + \frac{(b-x)^{1-\alpha}}{2(b-a)}\int_a^b (b-t)^{\alpha-1}(t-x) f'(t)\,dt. \quad (6) \end{aligned}$$

For term in the right hand side of (6) integrating by parts implies that

$$\begin{aligned} &\int_a^b (b-t)^{\alpha-1}(t-x) f'(t)\,dt \\ &= (b-x)\int_a^b (b-t)^{\alpha-1} f'(t)\,dt - \int_a^b (b-t)^{\alpha} f'(t)\,dt \quad (7) \\ &= (x-a)(b-a)^{\alpha-1} f(a) + (b-x)\Gamma(\alpha)\boldsymbol{J}_a^{\alpha-1} f(b) - \Gamma(\alpha+1)\boldsymbol{J}_a^{\alpha} f(b) \end{aligned}$$

Substituting $J_a^{\alpha}(P_2(x,b)f^{'}(b))$ in the Lemma 1.4 and (7) in (6), we obtain that

$$\begin{aligned}
&J_a^{\alpha}\left(K_1(x,b)f^{'}(b)\right)\\
&=\frac{1}{2}f(x)-(\alpha+1)\Gamma(\alpha)\frac{(b-x)^{1-\alpha}}{2(b-a)}J_a^{\alpha}f(b)+\frac{1}{2}J_a^{\alpha-1}(P_2(x,b)f(b))\\
&+\frac{(b-x)^{1-\alpha}(x-a)}{2}(b-a)^{\alpha-2}f(a)+\frac{(b-x)^{2-\alpha}}{2(b-a)}\Gamma(\alpha)J_a^{\alpha-1}f(b).
\end{aligned}$$

The proof is completed. □

Remark 2.2. Letting $\alpha=1$, formula (4) reduces the following identity

$$\frac{1}{2}f(x)=\frac{1}{(b-a)}\int_a^b f(t)dt+\frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)}+\int_a^b K(x,t)f^{'}(t)dt$$

where

$$K(x,t):=\begin{cases}\left(t-\dfrac{a+x}{2}\right), & t\in[a,x)\\ \left(t-\dfrac{b+x}{2}\right), & t\in[x,b]\end{cases}$$

which was given by Tong and Guan in [24]. Using the above identity, the authors proved another simple proof of Theorem 1.1.

Now using the new Montgomery identity for fractional integrals (4) and the corresponding fractional Peano kernel (5), we derive a new Ostrowski-Grüss type inequality of fractional type.

Theorem 2.3. *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on (a,b) ($a<b$) and $|f'(x)|\leq M$ for any $x\in[a,b]$. Then the following fractional inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{2}f(x)-(\alpha+1)\Gamma(\alpha)\frac{(b-x)^{1-\alpha}}{2(b-a)}J_a^{\alpha}f(b)+\frac{1}{2}J_a^{\alpha-1}(P_2(x,b)f(b))\right.\\
&\left.+\frac{(b-x)^{2-\alpha}}{2(b-a)}\Gamma(\alpha)J_a^{\alpha-1}f(b)+\frac{(b-x)^{1-\alpha}(x-a)}{2(b-a)^{2-\alpha}}f(a)\right| \qquad (8)\\
&\leq\frac{M(b-x)^{1-\alpha}}{(b-a)}\left[\frac{(b-a)^{\alpha}(x-a)+(b-x)^{\alpha}(a+b-2x)}{2\alpha}\right]
\end{aligned}$$

for $\alpha\geq 1$ and $a\leq x<b$.

Proof. From Lemma 2.1, we have

$$\left|\frac{1}{2}f(x)-(\alpha+1)\Gamma(\alpha)\frac{(b-x)^{1-\alpha}}{2(b-a)}\boldsymbol{J}_a^{\alpha}f(b)+\frac{1}{2}\boldsymbol{J}_a^{\alpha-1}(P_2(x,b)f(b))\right.$$
$$\left.+\frac{(b-x)^{2-\alpha}}{2(b-a)}\Gamma(\alpha)\boldsymbol{J}_a^{\alpha-1}f(b)+\frac{(b-x)^{1-\alpha}(x-a)}{2(b-a)^{2-\alpha}}f(a)\right|$$
$$=\frac{1}{\Gamma(\alpha)}\left|\int_a^b(b-t)^{\alpha-1}K_1(x,t)f'(t)\,dt\right|.$$

Taking into account the assumptions on the function f, it yields

$$\left|\frac{1}{2}f(x)-(\alpha+1)\Gamma(\alpha)\frac{(b-x)^{1-\alpha}}{2(b-a)}\boldsymbol{J}_a^{\alpha}f(b)+\frac{1}{2}\boldsymbol{J}_a^{\alpha-1}(P_2(x,b)f(b))\right.$$
$$\left.+\frac{(b-x)^{2-\alpha}}{2(b-a)}\Gamma(\alpha)\boldsymbol{J}_a^{\alpha-1}f(b)+\frac{(b-x)^{1-\alpha}(x-a)}{2(b-a)^{2-\alpha}}f(a)\right|$$
$$\leq\frac{M(b-x)^{1-\alpha}}{(b-a)}\left[\int_a^x(b-t)^{\alpha-1}\left|t-\frac{a+x}{2}\right|dt\right.\tag{9}$$
$$\left.+\int_x^b(b-t)^{\alpha-1}\left|t-\frac{b+x}{2}\right|dt\right].$$

Noting the left hand side of (9) by J then integrating by parts the right hand side of (9), we obtain

$$J\leq\frac{M(b-x)^{1-\alpha}}{2(b-a)}\left[\frac{(b-a)^{\alpha}-(b-x)^{\alpha}}{\alpha}(x-a)+\frac{(b-x)^{\alpha+1}}{\alpha}\right].$$

Consequently

$$J\leq\frac{M(b-x)^{1-\alpha}}{(b-a)}\left[\frac{(b-a)^{\alpha}(x-a)+(b-x)^{\alpha}(a+b-2x)}{2\alpha}\right].$$

This completes the proof. □

Remark 2.4. Letting $\alpha=1$, formula (8) reduces the following inequality

$$\left|\frac{1}{2}f(x)-\frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)}-\frac{1}{(b-a)}\int_a^b f(t)dt\right|$$
$$\leq\frac{M}{(b-a)}\left[\frac{(x-a)^2+(b-x)^2}{2}\right].$$

which connected with Ostrowski-Grüss type integral inequality. If we take $x = \frac{a+b}{2}$ in this inequality, it follows that

$$\left|\frac{1}{2}f\left(\frac{a+b}{2}\right)+\frac{f(b)+f(a)}{4}-\frac{1}{(b-a)}\int_a^b f(t)dt\right| \leq \frac{M(b-a)}{4}.$$

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