New games related to old and new sequences

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Abstract

We define an infinite class of 2-pile subtraction games, where the amount that can be subtracted from both piles simultaneously, is a function f of the size of the piles. Wythoff's game is a special case. For each game, the 2nd player winning positions are a pair of complementary sequences, some of which are related to well-known sequences, but most are new. The main result is a theorem giving necessary and sufficient conditions on f so that the sequences are 2nd player winning positions. Sample games are presented, strategy complexity questions are discussed, and possible further studies are indicated.

Keywords: 2-pile subtraction games, complexity of games, integer sequences

1 Introduction

We begin with an example. Given two piles of tokens (x, y) of sizes x, y, with $0 \le x \le y < \infty$. Two players alternate removing tokens from the piles according to the following two rules. A player, at his turn, can choose precisely one of them, as he sees fit.

(a) Remove any positive number of tokens from a single pile, possibly the entire pile.

(b) Remove a positive number of tokens from each pile, say k, ℓ, so that $|k-\ell|$ isn't too large with respect to the position (x_1, y_1) moved to from (x_0, y_0), namely, $|k - \ell| < x_1 + 1 \ \ (x_1 \le y_1)$.

The player making the move after which both piles are empty (a *leaf* of the game), wins; the opponent loses. Thus, $(11, 15) \to (3, 4)$ or to $(2, 4)$ are legal moves, but $(11, 15) \to (2, 3)$ or to $(0, 3)$ are not. The position $(0, 0)$ is the only leaf of this and our following games.

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Table 1. The first few P-positions for $\mathbf{G}_1$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a_n	0	1	3	4	5	7	8	9	10	12	13	14	15	16	18	19	20
b_n	0	2	6	11	17	25	34	44	55	68	82	97	113	130	149	169	190

For any acyclic combinatorial game without ties, such as $\mathbf{G}_1$, a position $u = (x, y)$ is labeled N (*Next* player win) if the player moving from u has a winning strategy; otherwise it's a P-position (*Previous* player win). Denote by $\mathcal{P}$ the set of all P-positions, by $\mathcal{N}$ the set of all N-positions, and by $F(u)$ the set of all game positions that are reachable from u in a single move. It is easy to see that for any acyclic game,

$$u \in \mathcal{P} \quad \text{if and only if} \quad F(u) \subseteq \mathcal{N} \ , \tag{1}$$

$$u \in \mathcal{N} \quad \text{if and only if} \quad F(u) \cap \mathcal{P} \neq \emptyset \ . \tag{2}$$

Indeed, player I, beginning from an N-position, will move to a P-position, which exists by (2), and player II has no choice but to go to an N-position, by (1). Since the game is finite and acyclic, player I will eventually win by moving to a leaf, which is clearly a P-position.

The partitioning of the game's positions into the sets $\mathcal{P}$ and $\mathcal{N}$ is unique for every acyclic combinatorial game without ties.

Let $\mathbb{Z}$ denote the set of integers. Let $S \subset \mathbb{Z}_{\geq 0}$, $S \neq \mathbb{Z}_{\geq 0}$, and $\overline{S} = \mathbb{Z}_{\geq 0} \setminus S$. The *minimum excluded value* of S is

$$\operatorname{mex} S = \min \overline{S} = \text{ least nonnegative integer not in } S.$$

Note that mex of the empty set is 0.

Table 1 portrays the first few P-positions (a_n, b_n) of $\mathbf{G}_1$. The reader is encouraged to verify that the first few entries of the table are indeed P-positions of the game. For a technical reason we put $b_{-1} = -1$. In §4 we prove, as a simple corollary to a considerably more general result,

Theorem 1. *For* $\mathbf{G}_1$, $\mathcal{P} = \cup_{i=0}^{\infty}(a_i, b_i)$, *where, for all* $n \in \mathbb{Z}_{\geq 0}$,

$$a_n = \operatorname{mex}\Big\{\{a_i : 0 \leq i < n\} \bigcup \{b_i : 0 \leq i < n\}\Big\}, \tag{3}$$

$$b_n = b_{n-1} + a_n + 1. \tag{4}$$

The game $\mathbf{G}_1$ is a member of the following new family of combinatorial games defined on two piles of finitely many tokens, with two types of moves: a

Table 2. The first few P-positions for $\mathbf{G}_2$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a_n	0	1	3	4	5	7	9	11	12	13	15	16	17	19	20	21	23
b_n	0	2	6	8	10	14	18	22	24	26	30	32	34	38	40	42	46

Table 3. The first few P-positions for $\mathbf{G}_3$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a_n	0	1	3	4	5	7	8	9	10	12	13	14	15	16	17	18	19
b_n	0	2	6	11	20	38	71	136	265	523	1036	2061	4110	8207	16400	32785	65554

move of type (a), and a more general move of type (b), namely, $|k-\ell|$ depends on the present and next position. Denote the present position by (x_0, y_0) and the position moved to by (x_1, y_1). We then require,

$$|(y_0 - y_1) - (x_0 - x_1)| = |(y_0 - x_0) - (y_1 - x_1)| < f(x_1, y_1, x_0), \tag{5}$$

where f is a real *constraint function* depending on x_1, y_1, x_0. If also $(y_0 - x_0) \geq (y_1 - x_1)$, then requirement (5) becomes $y_0 < f(x_1, y_1, x_0) + y_1 - x_1 + x_0$. The type (b) move defined for $\mathbf{G}_1$ is the special case $f = x_1 + 1$. Here are descriptions of two additional games.

$\mathbf{G}_2$ is the same as $\mathbf{G}_1$, except that in (b), $|k-\ell| < x_1 + 1$ is replaced by $|k - \ell| < x_0 - x_1$.

$\mathbf{G}_3$ is the same as $\mathbf{G}_1$, except that in (b), $|k-\ell| < x_1 + 1$ is replaced by $|k - \ell| < y_1 - x_1 + 1$.

The first few P-positions for $\mathbf{G}_2$ and $\mathbf{G}_3$ are listed in Tables 2 and 3 respectively. In §4 we also prove, as a corollary to the master theorem (Theorem 3),

Theorem 2. *For $\mathbf{G}_2$ and $\mathbf{G}_3$, $\mathcal{P} = \cup_{i=0}^{\infty}(a_i, b_i)$, where, for all $n \in \mathbb{Z}_{\geq 0}$, a_n is given by* (3). *For $\mathbf{G}_2$: $b_n = 2a_n$; and for $\mathbf{G}_3$: $b_0 = 0$, and for $n \in \mathbb{Z}_{\geq 1}$, $b_n = a_n + 2^n - 1$.*

Each of our games is associated with a pair of complementary sequences

$$A = \{a_i\}_{i=1}^{\infty}, \qquad B = \{b_i\}_{i=1}^{\infty}.$$

A special case is the well-known (classical) Wythoff game [37]. See also [4], [5], [7], [8], [13], [17], [18], [23], [24], [32], [33], [34], [38]. In fact, the classical Wythoff game is the case $f(x_1, y_1, x_0) = 1$, and the generalization considered in [17] is the case $f(x_1, y_1, x_0) = t$ for any fixed $t \in \mathbb{Z}_{>0}$. Whereas the winning strategy of Wythoff's game is associated with sequences related to algebraic integers of the form $(2 - t + \sqrt{t^2 + 4})/2$ (this is the golden section when $t = 1$), our games give rise to an infinity of sequences, some well-known, but mostly new ones.

In §2 we shall see that the pair of sequences of P-positions associated with $\mathbf{G}_1$ is related to a "self-generating" sequence of Hofstadter (see Sloane [35]). In §3 we indicate how the P-positions of $\mathbf{G}_2$ are related to another well-known sequence. The central result appears in §4, where a general theorem is formulated and proved, that yields winning strategies for a large class of 2-pile subtraction games. Roughly speaking it states that for every 2-pile subtraction game, if its constraint function f is "positive", "monotone" and "semi-additive", then it has P-positions (A, B), where a_n satisfies (3), and b_n has an explicit form depending on f. In a complementary proposition we show that positivity, monotonicity and semi-additivity are also necessary, in the sense that if any one of them is dropped, then there are constraint functions and their associated games G, such that the positions claimed to be P-positions by the central result, are not P-positions for these G. Theorems 1 and 2 are then deduced as simple corollaries of the central result. In §5 we give an assortment of sample games with their P-positions that can be produced from the central theorem. Questions of complexity and related issues are discussed in §6. The epilogue in §7 wraps up with some concluding remarks and indications for further study.

2 The Gödel, Escher, Bach Connection

On p. 73 of Hofstadter's famous book [30] the reader is asked to characterize the following sequence:

$$B'_{n\geq 0} = \{1,\ 3,\ 7,\ 12,\ 18,\ 26,\ 35,\ 45,\ 56, \ldots\}.$$

Answer: the sequence $\{2,\ 4,\ 5,\ 6,\ 8,\ 9,\ 10,\ 11, \ldots\}$ constitutes the set of differences of consecutive terms of $B'_{n\geq 0}$, as well as the complement with respect to $\mathbb{Z}_{>0}$ of $B'_{n\geq 0}$. For our purposes it is convenient to preface 0 to the latter sequence, so we define

$$A'_{n\geq 0} = \{0,\ 2,\ 4,\ 5,\ 6,\ 8,\ 9,\ 10,\ 11, \ldots\},$$

which is the complement with respect to $\mathbb{Z}_{\geq 0}$ of $B'_{n\geq 0}$. Now $a'_{10} = \text{mex}\{a'_i, b'_i : 0 \leq i < 10\} = 13$, so $b'_{10} = 56 + 13 = 69$. We see that in general, for all $n \in \mathbb{Z}_{\geq 0}$,

$$a'_n = \text{mex}\left\{\{a'_i : 0 \leq i < n\} \bigcup \{b'_i : 0 \leq i < n\}\right\}, \tag{6}$$

which has the form (3), and

$$b'_{-1} = 1, \qquad b'_n = b'_{n-1} + a'_n, \tag{7}$$

which is similar to (4). Moreover, the following proposition shows that there is a very close relationship between the P-positions of the game $\mathbf{G}_1$ and Hofstadter's sequence $B'_{n\geq 0}$, namely, b'_n exceeds b_n by 1. This can be observed by comparing the bottom row of Table 1 with $B'_{n\geq 0}$.

Proposition 1. $a'_n = a_n + 1 \ \ (n \geq 1), \quad b'_n = b_n + 1 \ \ (n \geq 0)$, *where* a'_n, b'_n *are given by* (6), (7) *respectively, and* a_n, b_n *by* (3), (4) *respectively.*

Proof. We see that the assertions are true for small n. Suppose they hold for all $i \leq n$. Then

$$a'_{n+1} = \text{mex}\{a'_i, b'_i : 0 \leq i \leq n\} = \text{mex}\{0, a_i + 1, b_i + 1 : 0 \leq i \leq n\}.$$

Let $S'_n = \{0, a_i + 1, b_i + 1 : 0 \leq i \leq n\}$, $S_n = \{a_i, b_i : 0 \leq i \leq n\}$. If, say, the integer interval $[0, k]$ is in S_n for some $k \in \mathbb{Z}_{>0}$ and $k + 1 \notin S_n$, then the integers in the interval $[0, k + 1]$ are in S'_n and $k + 2 \notin S'_n$. It follows that $\text{mex}\, S'_n = a_{n+1} + 1$. Also, $b'_{n+1} = b'_n + a'_{n+1} = b_n + 1 + a_{n+1} + 1 = b_{n+1} + 1$. ■

Thus the P-positions of G_1 constitute an "offset by 1" of the Hofstadter sequence and its complement.

3 Prouhet-Thue-Morse

It is not hard to see that the sequence $A_{n\geq 1}$ of $\mathbf{G}_2$ contains precisely all positive integers whose binary representation ends in an even number of zeros. The sequence $A_{n\geq 1}$ is also lexicographically minimal with respect to the property that the parity of the number of 1's in the binary expansion alternates. Furthermore, it is lexicographically minimal with respect to the property that the complement is the double of the sequence. If m appears in A, then $2m$ appears in B. In particular, $B_{n\geq 1}$ contains precisely all positive integers whose binary representation ends in an odd number of zeros [6]. The sequence

$$\begin{aligned} C_n &= 0^{a_1 - a_0} 1^{a_2 - a_1} 0^{a_3 - a_2} \dots 0^{a_{2n+1} - a_{2n}} 1^{a_{2n+2} - a_{2n+1}} \dots \\ &= 011010011001011010010 \dots, \end{aligned}$$

is the Prouhet-Thue-Morse sequence, which arises in many different areas of mathematics. See the charming paper [3], which also contains the sequence A, for many further properties of these sequences.

4 A Master Theorem

The three previously described games $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{G}_3$, are members of an infinite family of games that we now formulate. We will then provide a general winning strategy for this family of games and prove its validity.

General 2-pile subtraction games

Given two piles of tokens (x, y) of sizes x, y, with $0 \le x \le y < \infty$. Two players alternate removing tokens from the piles:

(aa) Remove any positive number of tokens from a single pile, possibly the entire pile.

(bb) Remove a positive number of tokens from each pile, say k, ℓ, so that $|k - \ell|$ isn't too large with respect to the position (x_1, y_1) moved to from (x_0, y_0), namely, $|k - \ell| < f(x_1, y_1, x_0)$; equivalently:

$$|(y_0 - y_1) - (x_0 - x_1)| = |(y_0 - x_0) - (y_1 - x_1)| < f(x_1, y_1, x_0), \qquad (8)$$

where the constraint function $f(x_1, y_1, x_0)$ is integer-valued and satisfies:

- Positivity:
$$f(x_1, y_1, x_0) > 0 \;\; \forall y_1 \ge x_1 \ge 0 \;\; \forall x_0 > x_1.$$
- Monotonicity:
$$x_0' < x_0 \Longrightarrow f(x_1, y_1, x_0') \le f(x_1, y_1, x_0).$$
- Semi-additivity (or generalized triangle inequality) on the P-positions, namely: for all $n > m \ge 0$,
$$\sum_{i=0}^{m} f(a_{n-1-i}, b_{n-1-i}, a_{n-i}) \ge f(a_{n-m-1}, b_{n-m-1}, a_n).$$

The player making the move after which both piles are empty wins; the opponent loses.

In view of (8), positivity is a natural condition. Without positivity, a move of type (bb) isn't even possible. Monotonicity appears to be a minimal requirement to enforce positivity. Semi-additivity is a convenient condition to have, and many functions are semi-additive. Whereas positivity and monotonicity are defined on any game positions, semi-additivity is defined on the candidate P-positions. All three conditions are applied to P-positions. If all three are satisfied, then the candidates are indeed P-positions, as enunciated in Theorem 3 below. The function f depends on three independent variables x_1, y_1, x_0 or a_{n-1}, b_{n-1}, a_n, and the dependent variable b_n — so that $(a_n, b_n) \in \mathcal{P}$ — is then computed by f.

Note that $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{G}_3$ clearly satisfy positivity and monotonicity; $\mathbf{G}_1$ and $\mathbf{G}_3$, in whose functions f there is no a_n, are clearly semi-additive; and $\mathbf{G}_2$ is semi-additive with equality. (See also the proof of Theorems 1 and 2 at the end of this section.)

Theorem 3. *Let $\mathcal{S} = \cup_{i=0}^{\infty}(a_i, b_i)$, where, for all $n \in \mathbb{Z}_{\ge 0}$, a_n is given by* (3), *$b_0 = 0$, and for all $n \in \mathbb{Z}_{>0}$,*

$$b_n = f(a_{n-1}, b_{n-1}, a_n) + b_{n-1} + a_n - a_{n-1}. \qquad (9)$$

If f is positive, monotone and semi-additive, then $\mathcal{S}$ is the set of P-positions of a general 2-pile subtraction game with constraint function f, and the sequences A, B share the following common features: (i) they partition $\mathbb{Z}_{\geq 1}$; (ii) $b_{n+1}-b_n \geq 2$ for all $n \in \mathbb{Z}_{\geq 0}$; (iii) $a_{n+1} - a_n \in \{1, 2\}$ for all $n \in \mathbb{Z}_{\geq 0}$.

Proof. The definition of a_n implies directly,

$$a_n > a_{n-1} \tag{10}$$

for all $n \in \mathbb{Z}_{>0}$. From (9) we have, for all $n \in \mathbb{Z}_{>0}$,

$$b_n - b_{n-1} = f(a_{n-1}, b_{n-1}, a_n) + a_n - a_{n-1}, \tag{11}$$

$$b_n - a_n = f(a_{n-1}, b_{n-1}, a_n) + b_{n-1} - a_{n-1}. \tag{12}$$

Now $f(a_0, b_0, a_1) > 0$ by positivity, so $b_1 - b_0 \geq 2$ by (10), (11). Hence we get, by induction on n,

$$b_n - b_m \geq 2 \text{ for all } n > m \geq 0. \tag{13}$$

Similarly we get from (12),

$$b_n - a_n > b_m - a_m \geq 0 \text{ for all } n > m \geq 0. \tag{14}$$

Now A and B are *complementary* sets of integers, i.e., $A \cup B = \mathbb{Z}_{\geq 1}$ (by (3)), and $A \cap B = \emptyset$. Indeed, if $a_n = b_m$, then $n > m$ implies that a_n is the mex of a set containing $b_m = a_n$, a contradiction to the mex definition; and $1 \leq n \leq m$ is impossible since

$$\begin{aligned} b_m &= f(a_{m-1}, b_{m-1}, a_m) + b_{m-1} + a_m - a_{m-1} \\ &\geq f(a_{m-1}, b_{m-1}, a_n) + b_{n-1} + a_n - a_{n-1} \\ &\quad \text{(by (10), (14) and monotonicity)} \\ &> a_n \text{ (by positivity).} \end{aligned}$$

Since $b_n - b_{n-1} \geq 2$ for all $n \geq 1$ by (13), and since A and B are complementary,

$$a_n - a_{n-1} \in \{1, 2\} \tag{15}$$

for all $n \in \mathbb{Z}_{>0}$. For the remainder of the proof it is useful to denote the set $\mathcal{S}$ defined in the statement of the theorem by $\mathcal{P}'$. Also let $\mathcal{N}' = \mathbb{Z}_{\geq 0} \setminus \mathcal{P}'$. For showing that $\mathcal{P}' = \mathcal{P}$ and $\mathcal{N}' = \mathcal{N}$, it evidently suffices to show two things:

I. Every move from any $(a_n, b_n) \in \mathcal{P}'$ results in a position in the complement $\mathcal{N}'$.

II. From every position (x, y) in the complement $\mathcal{N}'$, there is a move to some $(a_n, b_n) \in \mathcal{P}'$.

(It is useful to note that these two conditions are also necessary: (1) implies that *all* positions reachable in one move from a P-position are N-positions; whereas (2) shows that at least one P-position is reachable in one move from an N-position.)

I. A move of type (aa) from $(a_n, b_n) \in \mathcal{P}'$ has the form (x, b_n) or (a_n, y) $(x < a_n,\ y < b_n)$. Both are in $\mathcal{N}'$ since the sequences A and B are strictly increasing. Suppose there is a move of type (bb): $(a_n, b_n) \to (a_j, b_j) \in \mathcal{P}'$. Then $j < n$. Note that

$$\begin{aligned} & |(b_n - b_j) - (a_n - a_j)| \\ = \; & |(b_n - a_n) - (b_j - a_j)| = (b_n - a_n) - (b_j - a_j) \end{aligned}$$

by (14). By iterating (9) we have,

$$\begin{aligned} & (b_n - a_n) - (b_j - a_j) \\ & = f(a_{n-1}, b_{n-1}, a_n) + (b_{n-1} - a_{n-1}) - (b_j - a_j) \\ & = f(a_{n-1}, b_{n-1}, a_n) + f(a_{n-2}, b_{n-2}, a_{n-1}) \\ & \qquad\qquad\qquad + (b_{n-2} - a_{n-2}) - (b_j - a_j) \\ & \vdots \\ & = \sum_{i=0}^{n-j-1} f(a_{n-i-1}, b_{n-i-1}, a_{n-i}) \ge f(a_j, b_j, a_n), \end{aligned}$$

where the inequality follows from semi-additivity. Thus

$$|(b_n - b_j) - (a_n - a_j)| \ge f(a_j, b_j, a_n),$$

contradicting condition (bb).

II. Let $(x, y) \in \mathcal{N}'$ $(0 \le x \le y)$. Since A and B are complementary, every $n \in \mathbb{Z}_{>0}$ appears exactly once in exactly one of A and B. Therefore we have either $x = b_n$ or else $x = a_n$ for some $n \ge 0$.

(i) $x = b_n$. Then move $y \to a_n$. This is always possible since if $n = 0$, then $y > a_0 = b_0$; whereas $a_n < b_n$ for $n \ge 1$ by (14).

(ii) $x = a_n$. If $y > b_n$, move $y \to b_n$. So suppose that $a_n \le y < b_n$. Then $n \ge 1$. For any $m \in \{0, \ldots, n-1\}$ we have by (9) and by monotonicity,

$$\begin{aligned} (b_{m+1} - a_{m+1}) - (b_m - a_m) &= f(a_m, b_m, a_{m+1}) \\ &\le f(a_m, b_m, a_n). \end{aligned}$$

Thus $b_m - a_m + f(a_m, b_m, a_n) \ge b_{m+1} - a_{m+1}$. Therefore the intervals $[b_m - a_m,\ b_m - a_m + f(a_m, b_m, a_n))$ (closed on the left, open on the right) cover $\mathbb{Z}_{\ge 0}$ for $m \ge 0$. Hence

$$y - a_n \in [b_m - a_m,\ b_m - a_m + f(a_m, b_m, a_n)) \tag{16}$$

for a smallest $m \in \mathbb{Z}_{\ge 0}$. We then move $(x, y) \to (a_m, b_m)$. This move is legal, since:

Table 4. The first few values of $\mathcal{S}$ for $f = (x_0 - x_1)^2$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a_n	0	1	3	4	5	6	7	9	11	13	15	17	18	19	20	21	23
b_n	0	2	8	10	12	14	16	22	28	34	40	46	48	50	52	54	60

- $m < n$. Indeed, $y - a_n < b_n - a_n = f(a_{n-1}, b_{n-1}, a_n) + b_{n-1} - a_{n-1}$. Thus $m \leq n - 1$ by (16).
- $y > b_m$. By (16), $y - a_n \geq b_m - a_m$. Hence $y - b_m \geq a_n - a_m > 0$.
- The move satisfies (bb):

$$\begin{aligned} |(y - b_m) - (x - a_m)| &= |(y - a_n) - (b_m - a_m)| \\ &= (y - a_n) - (b_m - a_m) \end{aligned}$$

  where the last equality follows from (16) and our choice of m. We thus have $|(y - a_n) - (b_m - a_m)| = (y - a_n) - (b_m - a_m) < f(a_m, b_m, a_n)$ by (16). ■

We now show that if any of the three conditions of Theorem 3 is dropped, then there are games for which its conclusion fails.

Proposition 2. *There exist* 2*-pile subtraction games with constraint functions* f *which lack precisely one of positivity, monotonicity or semi-additivity, such that* $\mathcal{S} \neq \mathcal{P}$*, where* $\mathcal{S} = \cup_{i=0}^{\infty}(a_i, b_i)$*, and* a_i *satisfies* (3) $(i \in \mathbb{Z}_{\geq 0})$*;* $b_0 = 0$*,* b_i *satisfies* (9) $(i \in \mathbb{Z}_{>0})$*.*

Proof. Consider the function $f(x_1, y_1, x_0) = (x_0 - x_1)^2$. It is clearly positive and monotone. However, $(a_n - a_{n-1})^2 + (a_{n-1} - a_{n-2})^2 < (a_n - a_{n-2})^2$, no matter whether $a_n - a_{n-1} = a_{n-1} - a_{n-2} = 1$ or otherwise, so f is not semi-additive. From (9) we get, $b_n = b_{n-1} + (a_n - a_{n-1})(a_n - a_{n-1} + 1)$, where a_n satisfies (3). The first few values of (a_n, b_n) are depicted in Table 4. Note that these are not P-positions: we can move $(a_n, b_n) \to (a_i, b_i)$ in many ways; e.g., $(4, 10) \to (0, 0)$ satisfies (bb).

The function $f(x_1, y_1, x_0) = \lfloor (x_1 + 1)/x_0 \rfloor + 1$ is positive. Since

$$\left(\left\lfloor \frac{a_{n-1}+1}{a_n} \right\rfloor + 1\right) + \left(\left\lfloor \frac{a_{n-2}+1}{a_{n-1}} \right\rfloor + 1\right) > \left\lfloor \frac{a_{n-2}+1}{a_n} \right\rfloor + 1 = 1,$$

it is also semi-additive. But it is not monotone. From (9), $b_n = b_{n-1} - a_{n-1} + a_n + \lfloor (a_{n-1}+1)/a_n \rfloor + 1$. The first few values of $\mathcal{S} = \cup_{n=0}^{\infty}(a_n, b_n)$ are shown in

Table 5. The first few values of $\mathcal{S}$ for $f = \lfloor (x_1+1)/x_0 \rfloor + 1$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a_n	0	1	2	4	5	7	8	10	11	13	14	16	17	19	20	22	23
b_n	0	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48

Table 6. The first few values of $\mathcal{S}$ for $f = \left(1 + (-1)^{y_1+1}\right) x_1/2$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a_n	0	1	2	4	5	6	8	9	10	11	14	15	16	17	18	19	20
b_n	0	1	3	7	12	13	21	30	31	42	45	60	61	78	79	98	99

Table 5. The game-position $(4, 7) \notin \mathcal{S}$, but it cannot be moved into $\mathcal{S}$. Hence $\mathcal{S} \neq \mathcal{P}$. (Incidentally, note that the sequence B consists of all nonnegative multiples of 3.)

Lastly, consider $f(x_1, y_1, x_0) = \left(1 + (-1)^{y_1+1}\right) x_1/2$. We see easily that f is semi-additive, and it's trivially monotone. But whenever y_1 is even, f is not positive. We have, $b_n = a_n + b_{n-1} - \left(1 + (-1)^{b_{n-1}}\right) a_{n-1}/2$. Table 6 shows the first few $\mathcal{S}$-positions. These are not P-positions: The position $(10, 29) \notin \mathcal{S}$, cannot be moved into any position in $\mathcal{S}$. ■

Proof of Theorems 1 and 2. The function $f(x_1, y_1, x_0) = x_1 + 1$ is clearly positive. Monotonicity is satisfied trivially. It's also clear that f is semi-additive. The function $f(x_1, y_1, x_0) = x_0 - x_1$ is positive, since $x_0 > x_1$. It's also monotone. Since $(a_{n+1} - a_n) + (a_n - a_{n-1}) = a_{n+1} - a_{n-1}$, we see that f is semi-additive. Finally, the function $f(x_1, y_1, x_0) = y_1 - x_1 + 1$ is positive for all $x_1 \leq y_1$ and is trivially monotone. It's also semi-additive. Thus by Theorem 3 we have for $\mathbf{G}_1$, $b_n = a_{n-1} + 1 + b_{n-1} - a_{n-1} + a_n = b_{n-1} + a_n + 1$, as stated in Theorem 1. For $\mathbf{G}_2$, (9) implies, $b_n = a_n - a_{n-1} + b_{n-1} - a_{n-1} + a_n = 2a_n - 2a_{n-1} + b_{n-1} = 2a_n$, where the last equality follows by induction on n. For $\mathbf{G}_3$, $b_n = b_{n-1} - a_{n-1} + 1 + b_{n-1} - a_{n-1} + a_n = 2(b_{n-1} - a_{n-1}) + a_n + 1 = a_n + 2^n - 1$. Again the last equality follows by induction. ■

Table 7. The first few values of $\mathcal{S}$ for $f = x_1 - \lfloor (x_1+1)/x_0 \rfloor + 2$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a_n	0	1	3	4	5	6	8	9	10	11	13	14	15	16	17	19	20
b_n	0	2	7	12	18	25	35	45	56	68	83	98	114	131	149	170	191

Table 8. The first few values of $\mathcal{S}$ for $f = x_0 - x_1 + 2$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a_n	0	1	2	3	5	6	7	9	10	11	13	14	15	16	17	19	20
b_n	0	4	8	12	18	22	26	32	36	40	46	50	54	58	62	68	72

5 Further Sample Games

For the examples below, we leave it to the reader to verify positivity, monotonicity and semi-additivity of f. Some of these examples are elaborated on in the next two sections.

Example 1. $f(x_1, y_1, x_0) = x_1 - \lfloor (x_1+1)/x_0 \rfloor + 2$. Then $b_n = b_{n-1} + a_n - \lfloor (a_{n-1}+1)/a_n \rfloor + 2$. The first few P-positions are depicted in Table 7.

Example 2. $f(x_1, y_1, x_0) = x_0 - x_1 + 2$. Then $b_n = b_{n-1} + 2(a_n - a_{n-1} + 1) = 2(a_n + n)$. See Table 8 for the first few P-positions.

Example 3. $f(x_1, y_1, x_0) = (-1)^{y_1} - (-1)^{x_1} + 3$. Then $b_n = b_{n-1} - a_{n-1} + a_n + (-1)^{b_{n-1}} - (-1)^{a_{n-1}} + 3$. See Table 9 for the first few P-positions.

Example 4. $f(x_1, y_1, x_0) = x_1 (1 + (-1)^{x_1}) + 1$. This leads to $b_n = b_{n-1} + (-1)^{a_{n-1}} a_{n-1} + a_n + 1$. Table 10 exhibits the first few P-positions.

6 Computational Complexity Issues

What is the computational complexity of computing the winning strategy for our games? Given a position (x, y) with $0 \le x \le y < \infty$, the *statement* of Theorem 3 enables us to compute the table of P-positions. It suffices to compute it up to the smallest $n = n_0$ such that $a_{n_0} \ge x$, and thus determine whether

Table 9. The first few values of $\mathcal{S}$ for $f = (-1)^{y_1} - (-1)^{x_1} + 3$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a_n	0	1	2	3	5	6	7	8	9	11	12	13	15	16	17	18	19
b_n	0	4	10	14	21	25	27	31	33	38	44	48	55	59	61	65	67

Table 10. The first few values of $\mathcal{S}$ for $f = x_1\,(1 + (-1)^{x_1}) + 1$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a_n	0	1	3	4	6	8	9	10	11	12	13	14	15	16	17	19	20
b_n	0	2	5	7	18	33	42	44	66	68	94	96	136	138	172	175	177

$(x, y) \in \mathcal{P}$ or in $\mathcal{N}$. The *proof* of Theorem 3 then enables us, if $(x, y) \in \mathcal{N}$, to make a winning move to a position in $\mathcal{P}$. The latter part of the strategy, that of making a winning move, is clearly polynomial. The first part, determining whether or not $(x, y) \in \mathcal{P}$ is linear in x, since $a_{n_0} \le 2x$ by (15).

Our games, however, are *succinct*, i.e., the input size is $\Omega(\log x)$ rather than $\Omega(x)$ (assuming that y is bounded by a polynomial in x). Thus their complexity isn't obvious a priori. Even if the sequence B grows exponentially, polynomiality of the strategy doesn't necessarily follow. For example, I don't know whether the sequence B of $\mathbf{G}_3$ can be computed polynomially.

Special sequences are known to be computable polynomially. For example, consider the numeration system with bases defined by the recurrence $u_n = (s+t-1)u_{n-1} + su_{n-2}$ $(n \ge 1)$, where $s, t \in \mathbb{Z}_{>0}$, with initial conditions $u_{-1} = 1/s$, $u_0 = 1$. It follows from [19] that every positive integer N has a unique representation of the form $N = \sum_{i \ge 0} d_i u_i$, with digits $d_i \in \{0, \ldots, s+t-1\}$, such that $d_{i+1} = s+t-1 \Longrightarrow d_i < s$ for all $i \in \mathbb{Z}_{\ge 0}$. The representation of the first few entries for the special case $s = 2$, $t = 2$, is depicted in Table 10.

If we compare Table 11 with Table 8, we might note the following two properties:

- All the a_n have representations ending in an *even* number of 0s, and all the b_n have representations ending in an *odd* number of 0s.
- For every $(a_n, b_n) \in \mathcal{P}$, the representation of b_n is the "left shift" of the

Table 11. Representation of the first few integers in a special numeration system.

50	14	4	1	n	14	4	1	n
	2	0	3	31			1	1
	2	1	0	32			2	2
	2	1	1	33			3	3
	2	1	2	34		1	0	4
	2	1	3	35		1	1	5
	2	2	0	36		1	2	6
	2	2	1	37		1	3	7
	2	2	2	38		2	0	8
	2	2	3	39		2	1	9
	2	3	0	40		2	2	10
	2	3	1	41		2	3	11
	3	0	0	42		3	0	12
	3	0	1	43		3	1	13
	3	0	2	44	1	0	0	14
	3	0	3	45	1	0	1	15
	3	1	0	46	1	0	2	16
	3	1	1	47	1	0	3	17
	3	1	2	48	1	1	0	18
	3	1	3	49	1	1	1	19
1	0	0	0	50	1	1	2	20
1	0	0	1	51	1	1	3	21
1	0	0	2	52	1	2	0	22
1	0	0	3	53	1	2	1	23
1	0	1	0	54	1	2	2	24
1	0	1	1	55	1	2	3	25
1	0	1	2	56	1	3	0	26
1	0	1	3	57	1	3	1	27
1	0	2	0	58	2	0	0	28
1	0	2	1	59	2	0	1	29
1	0	2	2	60	2	0	2	30

representation of a_n.

Thus $(1,4)$ of Table 8 has representation $(1,10)$, and $(6,22)$ has representation $(12,120)$: 10 is the "left shift" of 1, 120 the left shift of 12.

These properties hold, in fact, in general for Example 2, which is a member of another family of sequences and games analyzed in [21]. They enable one to win in polynomial time for that family.

However, we don't even know whether there are NP-hard sequences. A case in point is the infinite family of octal games [29], [4] ch. 4, even for the subfamily where there are only finitely many nonzero octal digits. Some octal games have been shown to have polynomial strategies, (see e.g., [27]), but the complexity of most is unknown.

We mention very briefly other relevant complexities. They include Kolmogorov complexity, subword complexity, palindrome complexity, and, we might add, squares complexity. The *subword complexity* $c(n)$ of a sequence S is the number of distinct words of length n occurring in S. In [2], this notion is attributed to [14]. Surveys can be found in [1], [15], [16]. The *palindrome complexity* $p(n)$ of S is the number of distinct palindromes of length n in S. See e.g., [12], [2]. Define the *squares complexity* $s(n)$ of S as the number of distinct squares of length n in S. Thus the result of [26] implies that there are binary sequences for which $s(2) = 2$, $s(4) = 1$, $s(2k) = 0$ for all $k > 2$. There is also the notion of *program complexity* [9], [10], [11] concerning the complexity of computing a sequence, which is related to Kolmogorov complexity [31].

7 Epilogue

We have defined an infinite class of 2-pile subtraction games with two types of moves: (aaa) remove any positive number from a single pile; (bbb) remove $k > 0$ from one pile, $\ell > 0$ from the other. This move is restricted by the requirement $|k - \ell| < f$, where f is a positive real-valued function. We have shown that a pair A, B of judiciously chosen complementary sequences constitutes the set of P-positions if and only if f is monotone and semi-additive.

As we have pointed out, the generalized Wythoff game [17] is a member of the family of games considered here. It has the property that a polynomial strategy can be given by using a special numeration system, and noting that the elements of A are characterized by ending in an even number of 0s in that representation, and those of B are their left shifts. A similar situation exists for $\mathbf{G}_2$, but with the standard binary representation as numeration system. With the game in Example 2, an essentially different numeration system (see [21]) can be associated to the same effect.

Further studies

1. With which games can we associate an appropriate numeration system so as to establish a polynomial strategy?

2. Extend the games in a natural way to handle more than two piles. This seems to be difficult for Wythoff's game, for which I have a conjecture; see [20] §6(2), [28] Problem 53, [22] §5, [36].

3. Compute the Sprague-Grundy function g for the games, which will enable to play *sums* of games. For Wythoff's game this is an as yet unsolved problem, though eventual additive periodicity has been proved [13], [32].

4. Compute a strategy for the games when played in misère version, i.e., the player making the last move loses. This is easy for Wythoff's game. See [4], ch. 13.

5. We have already mentioned the question of the polynomiality of the strategy. Is there a 2-pile subtraction game that's Pspace-complete?

6. Computation of complexities of P-positions sequences, such as Kolmogorov-, program-, subword-, palindrome-, squares-complexities. For the sequence A of Example 2, the subword complexity was computed in [25].

7. Make an about-face: begin with pairs of known complementary sequences, and design matching 2-pile subtraction games.[1]

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