Hindawi
Journal of Probability and Statistics
Volume 2017, Article ID 2106748, 9 pages
https://doi.org/10.1155/2017/2106748

Research Article

New Generalizations of Exponential Distribution with Applications

N. A. Rather and T. A. Rather

Department of Mathematics, University of Kashmir, Srinagar 190006, India

Correspondence should be addressed to T. A. Rather; tariqrather@rediffmail.com

Received 5 December 2016; Revised 26 March 2017; Accepted 7 June 2017; Published 6 July 2017

Academic Editor: Chin-Shang Li

The main purpose of this paper is to present k-Generalized Exponential Distribution which among other things includes Generalized Exponential and Weibull Distributions as special cases. Besides, we also obtain three-parameter extension of Generalized Exponential Distribution. We shall also discuss moment generating functions (MGFs) of these newly introduced distributions.

1. Introduction

The gamma family distributions was discussed by Karl Pearson in 1895 as pointed out in Balakrishnan and Basu [1]. However, after a period of 35 years the Exponential Distribution is a special case of the gamma distribution to appear on its own. It is also related to Poisson process as it has been observed that the time between two successive Poisson events follows the Exponential Distribution. While discussing the sampling of standard deviation (SD), the Exponential Distribution was referred to by Kondo [2] as Pearson's Type X distribution. Steffensen [3], Teissier [4], and Weibull [5] proposed the applications of Exponential Distribution in actuarial, biological, and engineering problems, respectively.

An extension of Exponential Distribution was proposed by Weibull (1951). The Exponential Distribution is a special case wherein the shape parameter equals one. The Weibull distribution has many applications in survival analysis and reliability engineering; for reference see Lai et al. (2006). Some other applications in industrial quality control are discussed in Berrettoni (1964).

2. Some Basic Definitions and K-Generalized Exponential Distributions

We begin with some definitions which provide a base for the definition of K-Generalized Exponential Distributions.

The Euler gamma function $\Gamma(\alpha)$ is defined by the integral

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \quad R(\alpha) > 0. \tag{1}$$

A random variable X is said to have gamma distribution with parameter $\alpha > 0$, if its p.d.f. is given by

$$f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}, \quad 0 < x < \infty$$
$$f(x) = 0, \quad \text{elsewhere}. \tag{2}$$

Replacing x by x/λ, we get the following form of gamma distribution with parameters α, λ with $\alpha > 0$ and $\lambda > 0$:

$$f(x) = \frac{1}{\lambda^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\lambda},$$
$$0 < x < \infty, \ \alpha > 0, \ \lambda > 0. \tag{3}$$

The gamma distribution with parameters α, λ often arises in practices, as the distribution of time one has to wait until a fixed number of events have occurred.

The beta function of two variables m and n is defined by

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx,$$
$$\operatorname{Re}(m) > 0, \ \operatorname{Re}(n) > 0. \tag{4}$$

If we take $x = 1 - y$, then, $0 \le x \le 1$ implies $0 \le y \le 1$, and we get

$$B(m,n) = \int_0^1 (1-y)^{m-1} y^{n-1} dy = \int_0^1 y^{n-1} (1-y)^{m-1} dy = \int_0^1 x^{n-1} (1-x)^{m-1} dx = B(n,m). \quad (5)$$

In the literature (for reference see [6–8]), it is known that

$$B(n,m) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \quad (6)$$

In (5), if we take

$$x = \frac{1}{z+1} \quad (7)$$

then $0 \le x \le 1$ implies $0 \le z < \infty$, and we get

$$B(n,m) = \int_\infty^0 \left(\frac{1}{z+1}\right)^{n-1} \left(\frac{z}{z+1}\right)^{m-1} \left(-\frac{1}{(z+1)^2}\right) dz = \int_0^\infty \frac{z^{m-1}}{(z+1)^{m+n}} dz. \quad (8)$$

A continuous random variable X is said to have beta distribution with parameters m and n, if its p.d.f. is given by

$$f(x) = \frac{1}{B(m,n)} x^{m-1} (1-x)^{n-1}, \quad 0 \le x \le 1$$
$$f(x) = 0, \quad \text{elsewhere}. \quad (9)$$

This distribution is known as beta distribution of 1st kind (for reference see [7]).

The beta distribution has an application to model a random phenomenon whose set of possible values is a finite interval $[a,b]$, which by letting a denote the origin and taking $(b-a)$ as a unit measurement can be transformed into the interval $[0,1]$.

A continuous random variable X is said to have beta distribution of 2nd kind with parameters m and n, if its p.d.f. is given by

$$f(x) = \frac{1}{B(m,n)} \frac{z^{m-1}}{(1+z)^{m+n}}, \quad 0 \le z < \infty, \ m,n > 0,$$
$$f(x) = 0, \quad \text{elsewhere}. \quad (10)$$

More recently, Rahman et al. [7] (for more details see [8–16]) have defined k-gamma and k-beta distributions and their MGFs as follows.

For $k > 0$ and $z \in \mathbb{C}$, the k-gamma function is defined by the integral

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-t^k/k} dt. \quad (11)$$

For $R(m) > 0$, $R(n) > 0$, the k-beta function of two variables m and n is defined by

$$B_k(m,n) = \frac{1}{k} \int_0^1 t^{m/k-1} (1-t)^{n/k-1} dt. \quad (12)$$

It is implicit in the literature (for reference see [7]) that

$$B_k(m,n) = \frac{\Gamma_k(m)\Gamma_k(n)}{\Gamma_k(m+n)}. \quad (13)$$

For the sake of completeness, we present a very simple proof of the relation (13).

We have

$$B_k(m,n) = \frac{1}{k} \int_0^1 x^{m/k-1} (1-x)^{n/k-1} dx. \quad (14)$$

Put

$$x = \cos^2\theta; \quad (15)$$

then

$$dx = -2\cos\theta \sin\theta d\theta, \quad (16)$$

and we get

$$B_k(m,n) = \frac{2}{k} \int_0^{\pi/2} (\cos\theta)^{2m/k-1} (\sin\theta)^{2n/k-1} d\theta. \quad (17)$$

Now

$$\Gamma_k(m) = \int_0^\infty t^{m-1} e^{-t^k/k} dt. \quad (18)$$

Put

$$\frac{t^k}{k} = x^2 \quad (19)$$

or

$$t = \left(kx^2\right)^{1/k}, \quad (20)$$

so that

$$dt = \frac{2}{k} k^{1/k} x^{2/k-1} dx = 2k^{1/k-1} x^{2/k-1} dx; \quad (21)$$

we get

$$\Gamma_k(m) = 2k^{1/k-1} \int_0^\infty k^{(m-1)/k} x^{(2/k)(m-1)} e^{-x^2} x^{2/k-1} dx = 2k^{(m/k)-1} \int_0^\infty x^{2m/k-1} e^{-x^2} dx. \quad (22)$$

Since the integrals involved are convergent, we have

$$\Gamma_k(m)\Gamma_k(n) = \left(2k^{m/k-1} \int_0^\infty x^{2m/k-1} e^{-x^2} dx\right) \cdot \left(2k^{n/k-1} \int_0^\infty y^{2n/k-1} e^{-y^2} dy\right) = 4k^{(m+n)/k-2} \int_0^\infty \int_0^\infty x^{2m/k-1} y^{2n/k-1} e^{-(x^2+y^2)} dx\, dy. \quad (23)$$

Put

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{24}$$

so that

$$dxdy = rdrd\theta; \tag{25}$$

we get

$$\begin{aligned}\Gamma_k(m)\,\Gamma_k(n) &= 4k^{(m+n)/k-2}\int_0^\infty\int_0^{\pi/2}(\cos\theta)^{2m/k-1}\\ &\cdot(\sin\theta)^{2n/k-1}e^{-r^2}r^{2m/k+2n/k-1}dr\,d\theta = \Bigg\{\frac{2}{k}\\ &\cdot\int_0^{\pi/2}(\cos\theta)^{2m/k-1}(\sin\theta)^{2n/k-1}d\theta\Bigg\}\\ &\cdot\left\{2k^{(m+n)/k-1}\int_0^\infty r^{2(m+n)/k-1}e^{-r^2}dr\right\}.\end{aligned} \tag{26}$$

Using (17) and (22), we get

$$\Gamma_k(m)\,\Gamma_k(n) = B_k(m,n)\,\Gamma_k(m+n), \tag{27}$$

which gives

$$B_k(m,n) = \frac{\Gamma_k(m)\,\Gamma_k(n)}{\Gamma_k(m+n)}. \tag{28}$$

This completes the proof of the relation (13).

Corollary 1. *One has the following:*

(i) $\Gamma_k(k) = 1$.

(ii) $B_k(k/2, k/2) = \pi/k$.

(iii) $\Gamma_k(k/2) = \sqrt{\pi/k}$.

Proof. Put $m = k$ in (22); we get

$$\Gamma_k(k) = 2\int_0^\infty xe^{-x^2}dx = -\left[e^{-x^2}\right]_0^\infty = 1. \tag{29}$$

From (13), we have

$$B_k(m,n) = \frac{\Gamma_k(m)\,\Gamma_k(n)}{\Gamma_k(m+n)}. \tag{30}$$

We take $m = n = k/2$; we get

$$\begin{aligned}B_k\left(\frac{k}{2},\frac{k}{2}\right) &= \frac{\Gamma_k(k/2)\,\Gamma_k(k/2)}{\Gamma_k(k)} = \Gamma_k\left(\frac{k}{2}\right)\Gamma_k\left(\frac{k}{2}\right)\\ &= \left[\Gamma_k\left(\frac{k}{2}\right)\right]^2.\end{aligned} \tag{31}$$

This gives

$$\left[\Gamma_k\left(\frac{k}{2}\right)\right] = \sqrt{B_k\left(\frac{k}{2},\frac{k}{2}\right)}. \tag{32}$$

Putting $m = k/2$ and $n = k/2$ in (17), we get

$$\begin{aligned}B_k\left(\frac{k}{2},\frac{k}{2}\right) &= \frac{2}{k}\int_0^{\pi/2}d\theta = \frac{2}{k}\left[\theta\right]_0^{\pi/2} = \left(\frac{2}{k}\right)\left(\frac{\pi}{2}\right)\\ &= \frac{\pi}{k},\end{aligned} \tag{33}$$

which establishes (ii).

Putting the value of $B_k(k/2, k/2)$ in (32), we get

$$\Gamma_k\left(\frac{k}{2}\right) = \sqrt{\frac{\pi}{k}}, \tag{34}$$

which proves (iii). □

A random variable X of continuous type is said to have Weibull distribution if its probability density function is given by

$$\begin{aligned}f(x,\alpha,\beta,\nu) &= \frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1}e^{-((x-\nu)/\alpha)^\beta}, \quad x > \nu\\ f(x,\alpha,\beta,\nu) &= 0, \quad \text{elsewhere}, \ x \le \nu.\end{aligned} \tag{35}$$

Weibull distribution is widely used in engineering practice due to its versatility. It was originally proposed for the interpretation of fatigue data but now it is also used for many other problems in engineering. In particular, in the field of life phenomenon, it is used as the distribution of lifetime of some object, particularly when the "weakest link" model is appropriate for the model; that is, consider an object consisting of many parts and suppose that the object experiences death (failure) when any of its parts fail; it has been shown [6] (both oretically and empirically) under these conditions that Weibull distribution provides a close approximation to the distribution of the lifetime of the item.

The Gamma and Weibull distributions are commonly used for analyzing any lifetime data or skewed data. Both distributions have nice physical interpretation and several desirable properties. Unfortunately both distributions have drawbacks, one major disadvantage of the gamma distribution is that the distribution function or the survival function can not be computed easily if the shape parameter is not an integer. By using mathematical tables or computer software one obtains the distribution function, the survival function, or hazard function. This makes the gamma distribution unpopular as compared to Weibull distribution whose distribution function, hazard function, or survival function is easy to compute. It is well known that even though the Weibull distribution has convenient representation of distribution function, the distribution of the sum of independent and identically distributed (i.i.d) Weibull random variables is not simple to obtain. Therefore, the distribution of the mean of random sample from Weibull distribution is not easy to compute whereas the distribution of sum of independent and identically distributed (i.i.d) gamma random variables is well known. For more details see Mudholkar, Srivastava and Freimer (1995), Mudholkar and Srivastava [17], Gupta and Kundu [18], and Gupta et al. [19].

Recently R. D. Gupta and D. Kundu have introduced three-parameter Exponential Distribution (location, scale, and shape) and studied the theoretical properties of this family and compared them with respective good studies properties of Gamma and Weibull distributions. The increasing and decreasing hazard rate of the Generalized Exponential Distribution (GED) depends on the shape parameter. Generalized Exponential Distribution (GED) has several properties that are quite similar to gamma distribution but it has distribution function similar to that of the Weibull distribution which can be computed easily. Since the Generalized Exponential family has the likelihood ratio ordering on the shape parameter, one can construct a uniformly most powerful test for testing one sided hypothesis on the shape parameter when the scale and location parameters are known.

3. Generalized Exponential Distribution (GED)

A continuous random variable X whose p.d.f. is given by

$$f(x)=\begin{cases}\beta e^{-\beta x} & \text{if } x\geq 0\\ 0 & \text{if } x<0,\end{cases}\qquad \beta>0, \tag{36}$$

is said to have an Exponential Distribution.

The Generalized Exponential Distribution (GED) introduced by Mubeen et al. [15] has p.d.f.

$$f(x,\alpha,\beta)=\alpha\beta\left(1-e^{-\beta x}\right)^{\alpha-1}e^{-\beta x},\qquad x>0 \text{ for } \alpha,\beta>0 \tag{37}$$

$$f(x,\alpha,\beta)=0,\quad \text{otherwise.}$$

The MGF of (37) is given by

$$\begin{aligned}M_X(t)=E\left(e^{tX}\right)&=\int_0^\infty e^{tx}\alpha\beta\left(1-e^{-\beta x}\right)^{\alpha-1}e^{-\beta x}dx\\&=\alpha B\left(\alpha,1-\frac{t}{\beta}\right),\end{aligned} \tag{38}$$

where $B(m,n)=\int_0^1 y^{n-1}(1-y)^{m-1}dy$.

Replacing β by $1/\lambda$ and x by $(x-\mu)$ in (37), we get the following form of Generalized Exponential Distribution (GED):

$$f(x,\alpha,\lambda,\mu)=\frac{\alpha}{\lambda}\left(1-e^{-(x-\mu)/\lambda}\right)^{\alpha-1}e^{-(x-\mu)/\lambda},\qquad x>\mu,\ \alpha>0,\ \lambda>0. \tag{39}$$

4. Extensions of Generalized Exponential Distribution (GED)

The main aim of this paper is to present interesting extensions of Generalized Exponential Distribution (GED) in various ways and to study their moment generating functions (MGFs). We shall first define Generalized Exponential Distribution (GED) in terms of a new parameter $k>0$ and call it k-Generalized Exponential Distribution (k-GED). In fact, we prove the following result, which included Generalized Exponential Distribution as a special case.

Theorem 2. *Let X be a random variable of continuous type and let $\alpha>0$, $\beta>0$, and $k>0$ be the parameters; then the function*

$$f(x,\alpha,\beta,k)=\alpha\beta\left(1-e^{-\beta x^k/k}\right)^{\alpha-1}x^{k-1}e^{-\beta x^k/k},\qquad x>0 \tag{40}$$

$$f(x,\alpha,\beta,k)=0,\quad \textit{elsewhere}$$

is the p.d.f. of random variable X of continuous type.

Remark 3. If we take $k=1$, it reduces to Generalized Exponential Distribution.

Proof of Theorem 2. Clearly

$$f(x,\alpha,\beta,k)\geq 0\quad \forall x>0,\ \alpha>0,\ \beta>0,\ k>0. \tag{41}$$

Now

$$\begin{aligned}&\int_0^\infty f(x,\alpha,\beta,k)\,dx\\&=\int_0^\infty \alpha\beta\left(1-e^{-\beta x^k/k}\right)^{\alpha-1}x^{k-1}e^{-\beta x^k/k}dx\\&=\alpha\int_0^\infty\left(1-e^{-\beta x^k/k}\right)^{\alpha-1}\left(\beta x^{k-1}e^{-\beta x^k/k}\right)dx\\&=\alpha\left[\frac{\left(1-e^{-\beta x^k/k}\right)^{\alpha}}{\alpha}\right]_0^\infty=\left[\left(1-e^{-\beta x^k/k}\right)^{\alpha}\right]_0^\infty\\&=1-0=1.\end{aligned} \tag{42}$$

Hence $f(x,\alpha,\beta,k)$ is a p.d.f. of random variable X of continuous type. □

5. The Moment Generating Function (MGF) of Theorem 2

In this section, we derive MGF of the random variable X having k-Generalized Exponential Distribution in terms of new parameter $k>0$; we have

$$\begin{aligned}M_k(t)=E\left(e^{tX^k}\right)&=\int_0^\infty e^{tx^k}f(x,\alpha,\beta,k)\,dx\\&=k\alpha\beta\int_0^\infty e^{tx^k}\left(1-e^{-\beta^{x^k/k}}\right)^{\alpha-1}\left(x^{k-1}e^{-\beta x^k/k}\right)dx.\end{aligned} \tag{43}$$

Put

$$e^{-\beta x^k/k}=y; \tag{44}$$

then

$$e^{-\beta x^k/k}\left(-\beta x^{k-1}\right)dx = dy,\qquad e^{x^k} = y^{-k/\beta} \tag{45}$$

so that

$$e^{tx^k} = y^{-tk/\beta}. \tag{46}$$

Therefore,

$$\begin{aligned} M_k(t) &= \alpha\int_0^1 y^{-tk/\beta}(1-y)^{\alpha-1}dy \\ &= \alpha\int_0^1 y^{(1-tk/\beta)-1}(1-y)^{\alpha-1}dy \\ &= \alpha\int_0^1 (1-y)^{\alpha-1}y^{(1-tk/\beta)-1}dy \\ &= \alpha B\left(\alpha, 1-\left(\frac{tk}{\beta}\right)\right) = \frac{\alpha\Gamma(\alpha)\Gamma(1-(tk/\beta))}{\Gamma(\alpha+1-(tk/\beta))} \\ &= \frac{\Gamma(\alpha+1)\Gamma(1-(tk/\beta))}{\Gamma(\alpha+1-(tk/\beta))}. \end{aligned} \tag{47}$$

Our next theorem is also a generalization of Exponential Distribution in terms of new variable k, which includes Weibull distribution as a special case.

Theorem 4. *Let X be a random variable of continuous type and let $\alpha > 0$, $\beta > 0$, and $k > 0$ be the parameters; then the function*

$$\begin{aligned} f(x,\alpha,\beta,k) &= k\alpha\beta\left(1-e^{-\beta x^k}\right)^{\alpha-1}x^{k-1}e^{-\beta x^k}, \quad 0 < x < \infty \\ f(x,\alpha,\beta,k) &= 0, \quad \text{elsewhere}, \end{aligned} \tag{48}$$

is the p.d.f. of random variable X of continuous type.

Remark 5. For $k = 1$, K-Generalized Exponential Theorem 4 reduces to Classical Exponential Distribution.

Proof of Theorem 4. Clearly

$$f(x,\alpha,\beta,k) \geq 0 \quad \forall x > 0,\ \alpha > 0,\ \beta > 0,\ k > 0. \tag{49}$$

Now

$$\begin{aligned} &\int_0^\infty f(x,\alpha,\beta,k)\,dx \\ &= k\alpha\beta\int_0^\infty \left(1-e^{-\beta x^k}\right)^{\alpha-1}x^{k-1}e^{-\beta x^k}dx \\ &= \alpha\int_0^\infty \left(1-e^{-\beta x^k}\right)^{\alpha-1}\left(k\beta x^{k-1}\right)e^{-\beta x^k}dx \\ &= \alpha\left[\frac{\left(1-e^{-\beta x^k}\right)^\alpha}{\alpha}\right]_0^\infty = \left[\left(1-e^{-\beta x^k}\right)^\alpha\right]_0^\infty \\ &= 1-0 = 1. \end{aligned} \tag{50}$$

Hence $f(x,\alpha,\beta,k)$ is a p.d.f. of random variable X of continuous type. □

Remark 6 (Weibull distribution is a special case of Theorem 4). To see this, we take $\alpha = 1$, $\beta = 1$ in Theorem 4; it follows that

$$\begin{aligned} f(x,k) &= kx^{k-1}e^{-x^k} = e^{-x^k}\frac{d}{dx}\left(x^k\right), \quad x > 0 \\ f(x,k) &= 0, \quad x \leq 0. \end{aligned} \tag{51}$$

Replacing x by $(x-\mu)/\alpha$, we get

$$\begin{aligned} f(x,\alpha,k,\nu) &= e^{-((x-\nu)/\alpha)^k}\frac{d}{dx}\left(\frac{x-\nu}{\alpha}\right)^k, \quad \left(\frac{x-\nu}{\alpha}\right) > 0 \\ f(x,\alpha,k,\nu) &= 0, \quad \left(\frac{x-\nu}{\alpha}\right) \leq 0 \end{aligned} \tag{52}$$

or

$$\begin{aligned} f(x,\alpha,k,\nu) &= \frac{k}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{k-1}e^{-((x-\nu)/\alpha)^k}, \quad x > \nu, \\ f(x,\alpha,k,\nu) &= 0, \quad x \leq \nu, \end{aligned} \tag{53}$$

is a p.d.f. of X, which is clearly the density of Weibull Distribution.

6. The Moment Generating Function (MGF) of Theorem 4

In this section, we derive MGF of the random variable X having k-Generalized Exponential Distribution in terms of a new parameter $k > 0$; we have

$$\begin{aligned} M_k(t) &= E\left(e^{tX^k}\right) = \int_0^\infty e^{tx^k}f(x,\alpha,\beta,k)\,dx \\ &= k\alpha\beta\int_0^\infty e^{tx^k}\left(1-e^{-\beta x^k}\right)^{\alpha-1}x^{k-1}e^{-\beta x^k}dx. \end{aligned} \tag{54}$$

Put $e^{-\beta x^k} = y$; then

$$\begin{aligned} e^{x^k} &= y^{-1/\beta}, \\ e^{-\beta x^k}\left(-k\beta x^{k-1}\right)dx &= dy. \end{aligned} \tag{55}$$

Therefore,

$$\begin{aligned} M_k(t) &= \alpha \int_0^1 y^{-t/\beta}(1-y)^{\alpha-1}\,dy \\ &= \alpha \int_0^1 y^{(1-t/\beta)-1}(1-y)^{\alpha-1}\,dy \\ &= \alpha \int_0^1 (1-y)^{\alpha-1} y^{(1-t/\beta)-1} dy \\ &= \alpha B\left(\alpha, 1-\left(\frac{t}{\beta}\right)\right) = \frac{\alpha\Gamma(\alpha)\Gamma(1-(t/\beta))}{\Gamma(\alpha+1-(t/\beta))} \\ &= \frac{\Gamma(\alpha+1)\Gamma(1-(t/\beta))}{\Gamma(\alpha+1-(t/\beta))}. \end{aligned} \tag{56}$$

We also present the following three-parameter extension of Generalized Exponential Distribution (GED). In fact, we prove the following.

Theorem 7. *Let X be a continuous random variable; then the function*

$$f(x,\alpha,\beta,\delta) = \frac{\alpha\beta\delta}{1-(1-\delta)^{\alpha}}\left(1-\delta e^{-\beta x}\right)^{\alpha-1} e^{-\beta x}, \quad x>0,\ \alpha>0,\ \beta>0,\ 0<\delta\le 1, \tag{57}$$

$$f(x,\alpha,\beta,\delta) = 0, \quad \textit{otherwise}$$

is the p.d.f. of random variable X of continuous type.

Proof of Theorem 7. Clearly $f(x,\alpha,\beta,\delta) \ge 0$ for all $x>0,\ \alpha>0,\ \beta>0,\ \delta>0$.

Now

$$\begin{aligned} &\int_0^\infty f(x,\alpha,\beta,\delta)\,dx \\ &= \frac{\alpha\beta\delta}{1-(1-\delta)^{\alpha}}\int_0^\infty \left(1-\delta e^{-\beta x}\right)^{\alpha-1} e^{-\beta x}dx \\ &= \frac{\alpha}{1-(1-\delta)^{\alpha}}\int_0^\infty \left(1-\delta e^{-\beta x}\right)^{\alpha-1}\left(\beta\delta e^{-\beta x}\right)dx \\ &= \left(\frac{\alpha}{1-(1-\delta)^{\alpha}}\right)\left[\frac{\left(1-\delta e^{-\beta x}\right)^{\alpha}}{\alpha}\right]_0^\infty \\ &= \frac{\left(1-(1-\delta)^{\alpha}\right)}{\left(1-(1-\delta)^{\alpha}\right)} = 1. \end{aligned} \tag{58}$$

This shows that $f(x)$ is a p.d.f. of the random variable X. This proves Theorem 7. □

If we replace β by $1/\lambda$ and x by $(x-\mu)$ in Theorem 7, we get the following form of Generalized Exponential Distribution (GED):

$$\begin{aligned} &f(x,\alpha,\lambda,\mu,\delta) \\ &= \frac{\alpha\delta}{\lambda\left(1-(1-\delta)^{\alpha}\right)}\left(1-\delta e^{-(x-\mu)/\lambda}\right)^{\alpha-1} e^{-(x-\mu)/\lambda}, \\ &\qquad x>\mu,\ \alpha>0,\ \lambda>0,\ \mu>0,\ 0<\delta\le 1. \end{aligned} \tag{59}$$

$$f(x,\alpha,\lambda,\mu,\delta) = 0, \quad \text{otherwise}.$$

Remark 8. For $\delta = 1$, Theorem 7 reduced to Generalized Exponential Distribution (GED).

Remark 9. Taking $\delta = 1$ in (59), we get relation (39).

7. The Moment Generating Function (MGF) of Theorem 7

$$\begin{aligned} M_X(t) &= E\left(e^{tX}\right) = \int_0^\infty e^{tx} f(x,\alpha,\beta,\delta)\,dx \\ &= \frac{\alpha\beta\delta}{\left(1-(1-\delta)^{\alpha}\right)}\int_0^\infty e^{tx}\left(1-\delta e^{-\beta x}\right)^{\alpha-1} e^{-\beta x}dx \\ &= \frac{\alpha\delta}{\left(1-(1-\delta)^{\alpha}\right)}\int_0^\infty \left(1-\delta e^{-\beta x}\right)^{\alpha-1} e^{tx}\left(\beta e^{-\beta x}\right)dx, \end{aligned} \tag{60}$$

Put

$$e^{-\beta x} = y, \tag{61}$$

so that

$$-\beta e^{-\beta x}dx = dy, \qquad e^{tx} = y^{-t/\beta}; \tag{62}$$

we get

$$M(t) = \frac{\alpha\delta}{\left(1-(1-\delta)^{\alpha}\right)}\int_0^1 (1-\delta y)^{\alpha-1} y^{-t/\beta}dy, \quad 0<\delta\le 1. \tag{63}$$

$$M(t) = \frac{\alpha}{\left(1-(1-\delta)^{\alpha}\right)} B_\delta\left(\alpha, 1-\frac{t}{\beta}\right),$$

where

$$B_\delta = B_\delta(m,n) = \int_0^1 (1-\delta y)^{m-1} y^{n-1}dy, \quad m>0,\ n>0,\ 0<\delta\le 1. \tag{64}$$

Remark 10. For $\delta = 1$, we have

$$B_1(m,n) = \int_0^1 (1-y)^{m-1} y^{n-1}dy = B(m,n). \tag{65}$$

And (63) reduces to Remark 8.

Remark 11. If in $B_\delta(m,n) = \int_0^1 (1-\delta y)^{m-1} y^{n-1} dy$, we put $x = \delta y$;
then

$$dy = \frac{1}{\delta} dx \tag{66}$$

and we get

$$\begin{aligned} B_\delta(m,n) &= \int_0^\delta (1-x)^{m-1} \left(\frac{x}{\delta}\right)^{n-1} \frac{1}{\delta} dx \\ &= \frac{1}{\delta^n} \int_0^\delta (1-x)^{m-1} x^{n-1} dx, \quad 0 \le \delta \le 1. \end{aligned} \tag{67}$$

Remark 12. Letting $\delta \to 0$ in (57) and noting that

$$\underset{\delta\to 0}{Lt} \cdot \frac{\alpha\delta}{\left(1-(1-\delta)^\alpha\right)} = 1, \tag{68}$$

we get

$$f(x,\beta) = \beta e^{-\beta x}, \quad \text{if } x \ge 0,\ \beta > 0, \tag{69}$$

which is the p.d.f. of the Exponential Distribution.

Finally we present the following more general interesting result which among other things includes Weibull distribution as a limiting case.

Theorem 13. *Let X be a random variable of continuous type. If $\delta > 0$, $\beta > 0$, and $k > 0$ are the parameters, then the function*

$$\begin{aligned} f(x,\delta,\beta,k) &= \frac{k\delta\beta}{1-(1-\delta)^k} \left(1-\delta e^{-x^\beta}\right)^{k-1} x^{\beta-1} e^{-x^\beta}, \\ &\qquad x > 0,\ 0 < \delta < 1, \\ f(x,\delta,\beta,k) &= 0, \quad \textit{elsewhere},\ x \le 0, \end{aligned} \tag{70}$$

is the p.d.f. of the random variable X.

Proof of Theorem 13. Clearly

$$f(x,\delta,\beta,k) \ge 0 \quad \forall x > 0,\ \delta > 0,\ \beta > 0,\ k > 0. \tag{71}$$

Now

$$\begin{aligned} &\int_0^\infty f(x,\delta,\beta,k)\,dx \\ &= k\delta\beta \int_0^\infty \left(1-\delta e^{-x^\beta}\right)^{k-1} x^{\beta-1} e^{-x^\beta} dx \\ &= \frac{k}{1-(1-\delta)^k} \int_0^\infty \left(1-\delta e^{-x^\beta}\right)^{k-1} \left(\beta\delta e^{-x^\beta} x^{\beta-1}\right) dx \\ &= \frac{k}{1-(1-\delta)^k} \left[\frac{\left(1-\delta e^{-x^\beta}\right)^k}{k}\right]_0^\infty \\ &= \frac{k}{1-(1-\delta)^k} \left(\frac{\left(1-(1-\delta)^k\right)}{k}\right) = 1. \end{aligned} \tag{72}$$

This shows that $f(x,\alpha,\beta,k)$ is the p.d.f. of random variable X of continuous type. This proves Theorem 13. □

Remark 14. Weibull Distribution is the limiting case of Theorem 13. To see this, we let $\delta \to 0$ in Theorem 13 and note that

$$\lim_{\delta\to 0} \frac{k\delta}{1-(1-\delta)^k} = 1, \tag{73}$$

so that

$$\begin{aligned} f(x,\beta) &= \beta e^{-x^\beta} x^{\beta-1} = e^{-x^\beta} \frac{d}{dx}\left(x^\beta\right), \quad x > 0, \\ f(x,\beta) &= 0, \quad x \le 0, \end{aligned} \tag{74}$$

is the p.d.f. of random variable X of continuous type.

Replacing x by $(x-\nu)/\alpha$, it follows that

$$\begin{aligned} f(x,\alpha,\beta,\nu) &= \beta e^{-((x-\nu)/\alpha)^\beta} \frac{d}{dx}\left(\frac{x-\nu}{\alpha}\right)^\beta, \\ &\qquad \frac{x-\nu}{\alpha} > 0 \\ f(x,\alpha,\beta,\nu) &= 0, \quad \frac{x-\nu}{\alpha} \le 0. \end{aligned} \tag{75}$$

Equivalently

$$\begin{aligned} f(x,\alpha,\beta,\nu) &= \frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1} e^{-((x-\nu)/\alpha)^\beta}, \quad x > \nu, \\ f(x,\alpha,\beta,\nu) &= 0, \quad x \le \nu, \end{aligned} \tag{76}$$

is a p.d.f. of X, which is clearly density of Weibull Distribution.

8. Applications

The applications of Exponential Distribution have been widespread, which include models to determine bout criteria for analysis of animal behaviour [20]; design rainfall estimation in the Coast of Chiapas [21]; analysis of Los Angeles rainfall data [22]; software reliability growth models for vital quality metrics [23]; models for episode peak and duration for ecohydroclimatic applications [24]; estimating mean life of power system equipment with limited end-of-life failure data [25]; and cure rate modeling (Kannan et al. (2010). In related work, the closeness of the exponentiated Exponential Distribution with the Weibull, gamma, and lognormal distributions is studied in Gupta and Kundu (2003a), (2003b), (2004), (2006)) and [26]. Some generalizations of the exponentiated Exponential Distribution are discussed in Nadarajah and Kotz [27].

The Exponential Distribution is often used to model the reliability of electronic systems, which do not typically experience wear-out type failures. The distribution is called "memoryless," meaning that the calculated reliability for, say, a 10-hour mission is the same for a subsequent 10-hour mission, given that the system is working properly at the start of each mission. Given a hazard (failure) rate, λ, or mean time between failure (MTBF = $1/\lambda$), the reliability can be determined at a specific point in time (t). The distribution function, reliability function, and hazard rate of Theorem 2 are given by

$$F(x,\alpha,\beta,k) = P(X \le x) = \int_0^x f(x,\alpha,\beta,x)\,dx = \left(1 - e^{-\beta x^k/k}\right)^{\alpha}, \quad x > 0,\ \alpha > 0,\ \beta > 0. \tag{77}$$

The survival function $S(x,\alpha,\beta,k)$ is given by

$$S(x,\alpha,\beta,k) = 1 - F(x,\alpha,\beta,k) = 1 - \left(1 - e^{-\beta x^k/k}\right)^{\alpha}, \quad x > 0. \tag{78}$$

The hazard rate $h(x,\alpha,\beta,k)$ is given by

$$S(x,\alpha,\beta,k) = \frac{F(x,\alpha,\beta,k)}{S(x,\alpha,\beta,k)} = \frac{\left(1 - e^{-\beta x^k/k}\right)^{\alpha-1} x^{k-1} e^{-\beta x^k/k}}{1 - \left(1 - e^{-\beta x^k/k}\right)^{\alpha}}, \quad x > 0. \tag{79}$$

Similarly we can also obtain the distribution function, reliability function, and hazard rate of other theorems too.

9. Conclusions

In this paper the authors conclude the following.

(i) Weibull distribution is a special case of k-Generalized Exponential Distribution. Further if δ tends to 0, then our lastly proved more general result leads to Weibull distribution.

(ii) Also if δ tends to 1, then our newly introduced 3-parameter extension of Generalized Exponential Distribution (GED) reduced to classical Exponential Distribution.

(iii) The moment generating functions (MGFs) obtained in this paper generalize the classical moment generating functions (MGFs) of the given distributions.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] N. Balakrishnan and A. P. Basu, *Exponential Distribution Theory, Methods and Applications*, John Wiley, New York, NY, USA, 1995.

[2] T. Kondo, "A theory of the sampling distribution of standard deviation," *Journal of Biometrika*, vol. 22, 1/2, pp. 34–64, 1930.

[3] J. F. Steffensen, "Some recent research in the theory of statistics and actuarial science," *Institute of Actuaries*, 1930.

[4] G. Teissier, *Dysharmonies et Discontinuités dans la Croissance*, Act. Sci. et Industr., No. 95. (Exposés de Biometrie, 1), Hermann, Paris, 1934.

[5] W. Weibull, *A statistical theory of the strength of ma-terials*, vol. 151, Ingeniors Vetenskaps Akademiens, Stockholm, Sweden, 1939.

[6] S. C. Malik, *Mathematical analysis*, Wiley Eastern Limited, New Delhi, India, 1984.

[7] G. Rahman, S. Mubeen, A. Rehman, and M. Naz, "k-Gamma and k-Beta distributions and moment generating functions," *Journal of Probability and Statistics*, vol. 2014, Article ID 982013, 6 pages, 2014.

[8] M. G. Kendall and A. Stuart, *The Advanced Theory of Statistics*, vol. 2, Charles Griffin and Company, London, UK, 1961.

[9] S. Ross, *A First Course in Probability*, University of Southern California, 8th edition.

[10] C. G. Kokologiannaki, "Properties and inequalities of generalized k-gamma, beta and zeta functions," *International Journal of Contemporary Mathematical Sciences*, vol. 5, no. 13–16, pp. 653–660, 2010.

[11] S. Mubeen, M. Naz, and G. Rahman, "A note on k-hypergemetric differential equations," *Journal of Inequalities and Special Functions*, vol. 4, no. 3, pp. 38–43, 2013.

[12] S. Mubeen, G. Rahman, A. Rehman, and M. Naz, "Contiguous function relations for k-hypergeometric functions," *ISRN Mathematical Analysis*, vol. 2014, Article ID 410801, 6 pages, 2014.

[13] S. Mubeen, M. Naz, A. Rehman, and G. Rahman, "Solutions of k-hypergeometric differential equations," *Journal of Applied Mathematics*, vol. 2014, Article ID 128787, 13 pages, 2014.

[14] F. Merovci, "Power product inequalities for the Γ_k function," *International Journal of Mathematical Analysis*, vol. 4, no. 21, pp. 1007–1012, 2010.

[15] S. Mubeen, A. Rehman, and F. Shaheen, "Properties of k-gamma, k-beta and k-functions," *Bothalia Journal*, vol. 4, pp. 371–379, 2014.

[16] A. Rehman, S. Mubeen, N. Sadiq, and F. Shaheen, "Some inequalities involving k-beta functions with applications," *Journal of Inequalities and Applications*, 2014:224, 16 pages, 2014.

[17] G. S. Mudholkar and D. K. Srivastava, "Exponentiated Weibull family for analyzing bathtub failure-rate data," *IEEE Transactions on Reliability*, vol. 42, no. 2, pp. 299–302, 1993.

[18] R. D. Gupta and D. Kundu, "Exponentiated exponential family: an alternative to gamma and Weibull distributions," Dept. of Math., Stat. & Comp. Sci, University of New Brusnwick, Saint-John, Canada, 1997.

[19] R. C. Gupta, P. L. Gupta, and R. D. Gupta, "Modeling failure time data by Lehman alternatives," *Communications in Statistics—Theory and Methods*, vol. 27, no. 4, pp. 887–904, 1998.

[20] M. P. Yeates, B. J. Tolkamp, D. J. Allcroft, and I. Kyriazakis, "The use of mixed distribution models to determine bout criteria for analysis of animal behaviour," *Journal of Theoretical Biology*, vol. 213, no. 3, pp. 413–425, 2001.

[21] C. Escalante-Sandoval, "Design rainfall estimation using exponentiated and mixed exponentiated distributions in theCoast of Chiapas," *Ingenieria Hidraulica en Mexico*, vol. 22, pp. 103–113, 2007.

[22] M. T. Madi and M. Z. Raqab, "Bayesian prediction of rainfall records using the generalized exponential distribution," *Environmetrics*, vol. 18, no. 5, pp. 541–549, 2007.

[23] R. Subburaj, G. Gopal, and P. K. Kapur, "A software reliability growth model for vital quality metrics," *The South African Journal of Industrial Engineering*, vol. 18, pp. 93–108, 2007.

[24] F. Biondi, T. J. Kozubowski, A. K. Panorska, and L. Saito, "A new stochastic model of episode peak and duration for eco-hydro-climatic applications," *Ecological Modelling*, vol. 211, no. 3-4, pp. 383–395, 2008.

[25] J. E. Cota-Felix, F. Rivas-Davalos, and S. Maximov, "An alternative method for estimating mean life of power system equipment with limited end-oflife failure data," in *Proceedings of the 2009 IEEE Bucharest PowerTech: Innovative Ideas Toward the Electrical Grid of the Future*, July 2009.

[26] D. Kundu and B. Pradhan, "Bayesian inference and life testing plans for generalized exponential distribution," *Science in China. Series A. Mathematics*, vol. 52, no. 6, pp. 1373–1388, 2009.

[27] S. Nadarajah and S. Kotz, "The exponentiated type distributions," *Acta Applicandae Mathematicae*, vol. 92, no. 2, pp. 97–111, 2006.