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New Hermite–Hadamard type inequalities for n-polynomial harmonically convex functions

Muhammad Uzair Awan[1], Nousheen Akhtar[1], Sabah Iftikhar[2], Muhammad Aslam Noor[2] and Yu-Ming Chu[3,4*]

*Correspondence: chuyuming2005@126.com
[3]Department of Mathematics, Huzhou University, Huzhou, China
[4]Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, Changsha University of Science & Technology, Changsha, China
Full list of author information is available at the end of the article

Abstract

In the article, we introduce a class of n-polynomial harmonically convex functions, establish their several new Hermite–Hadamard type inequalities which are the generalizations and variants of the previously known results for harmonically convex functions.

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1 Introduction and preliminaries

Let $I \subseteq \mathbb{R}$ be an interval. Then a real-valued function $f : I \to \mathbb{R}$ is said to be convex (concave) if the inequality

$$f\big(\lambda x + (1-\lambda)y\big) \le (\ge) \lambda f(x) + (1-\lambda)f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

It is well known that the convexity (concavity) has wide applications in many branches of pure and applied mathematics [1–30], many inequalities can be derived via the convexity or concavity theory [31–56]. Recently, the generalizations, extensions, and variants for the convexity have attracted the attention of many researchers [57–65].

The classical Hermite–Hadamard inequality [66] is the most famous one in convex functions theory, which states that the double inequality

$$f\left(\frac{x+y}{2}\right) \le (\ge) \frac{1}{y-x}\int_x^y f(t)\,dt \le (\ge) \frac{f(x)+f(y)}{2}$$

holds for all $x, y \in I$ with $x \neq y$ if $f : I \to \mathbb{R}$ is a convex (concave) function.

In the past few decades, many generalizations, improvements, refinements, and variants for the Hermite–Hadamard inequality have been made by several researchers, we recommend the literature [67–72] to interested readers.

Now, we recall the definition of harmonically convex function [73] as follows.

Definition 1.1 (see [73]) Let $H \subseteq (0,\infty)$ be an interval. Then a real-valued function $f : H \to \mathbb{R}$ is said to be harmonically convex if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le tf(y) + (1-t)f(x)$$

for all $x, y \in H$ and $t \in [0,1]$.

İşcan [73] provided the Hermite–Hadamard type inequality for the harmonically convex function.

Theorem 1.2 (see [73]) *Let $H \subseteq (0,\infty)$ be an interval and $f : H \to \mathbb{R}$ be a harmonically convex function. Then the Hermite–Hadamard type inequality*

$$f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds for all $a, b \in H$ with $a < b$ if $f \in L[a,b]$.

Very recently, Toplu et al. [74] introduced and investigated a new class of n-polynomial convex functions and established several new Hermite–Hadamard type inequalities for this class of functions.

The main purpose of the article is to introduce the notion of n-polynomial harmonically convex functions, derive the variants of the classical Hermite–Hadamard inequality by use of the class of n-polynomial harmonically convex functions. We also discuss several new special cases for the obtained results which show that our obtained results are the generalizations and extensions of some previously known results.

2 Results and discussions

In this section, we first introduce the definition of n-polynomial harmonically convex function.

Definition 2.1 Let $n \in \mathbb{N}$ and $H \subseteq (0,\infty)$ be an interval. Then a nonnegative real-valued function $f : H \to [0,\infty)$ is said to be an n-polynomial harmonically convex function if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le \frac{1}{n}\sum_{s=1}^{n}\bigl(1-(1-t)^s\bigr)f(y) + \frac{1}{n}\sum_{s=1}^{n}\bigl(1-t^s\bigr)f(x)$$

for all $x, y \in H$ and $t \in [0,1]$.

From Definitions 1.1 and 2.1, we clearly see that the class of n-polynomial harmonically convex functions reduces to the class of harmonically convex functions if $n = 1$ and the 2-polynomial harmonically convex function f satisfies the inequality

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le \frac{3t-t^2}{2}f(y) + \frac{2-t-t^2}{2}f(x).$$

Theorem 2.2 *Let $b > a > 0, f_\alpha : [a,b] \to [0,\infty)$ be a family of n-polynomial harmonically convex functions and $f(u) = \sup_\alpha f_\alpha(u)$. Then f is an n-polynomial harmonically convex function if $J = \{x \in [a,b] : f(x) < \infty\}$ is an interval.*

Proof Let $t \in [0,1]$ and $x, y \in J$. Then we clearly see that

$$\begin{aligned} f\left(\frac{xy}{tx+(1-t)y}\right) &= \sup_\alpha f_\alpha\left(\frac{xy}{tx+(1-t)y}\right) \\ &\le \frac{1}{n}\sum_{s=1}^{n}\left(1-(1-t)^s\right)\sup_\alpha f_\alpha(y) + \frac{1}{n}\sum_{s=1}^{n}\left(1-t^s\right)\sup_\alpha f_\alpha(x) \\ &= \frac{1}{n}\sum_{s=1}^{n}\left(1-(1-t)^s\right)f(y) + \frac{1}{n}\sum_{s=1}^{n}\left(1-t^s\right)f(x) < \infty, \end{aligned}$$

which completes the proof. □

Theorem 2.3 *Let $f : [a,b] \subseteq (0,\infty) \to [0,\infty)$ be an n-polynomial harmonically convex function. Then one has*

$$\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx \le \left(\frac{f(a)+f(b)}{n}\right)\sum_{s=1}^{n}\frac{s}{s+1}$$

if $f \in L[a,b]$.

Proof It follows from the n-polynomial harmonic convexity of f that

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le \frac{1}{n}\sum_{s=1}^{n}\left[1-(1-t)^s\right]f(y) + \frac{1}{n}\sum_{s=1}^{n}\left(1-t^s\right)f(x),$$

which leads to

$$f\left(\frac{2xy}{x+y}\right) \le \frac{1}{n}\sum_{s=1}^{n}\left[1-\left(\frac{1}{2}\right)^s\right]f(y) + \frac{1}{n}\sum_{s=1}^{n}\left[1-\left(\frac{1}{2}\right)^s\right]f(x).$$

Using the change of variables, we have

$$f\left(\frac{2ab}{a+b}\right) \le \frac{1}{n}\sum_{s=1}^{n}\left[1-\left(\frac{1}{2}\right)^s\right]\left[f\left(\frac{ab}{ta+(1-t)b}\right) + f\left(\frac{ab}{tb+(1-t)a}\right)\right].$$

Integrating both sides of the above inequality with respect to t on $[0,1]$, we get

$$\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx. \tag{2.1}$$

Note that

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx = \int_o^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt$$

$$\leq \int_0^1 \left[\frac{1}{n} \sum_{s=1}^{n} [1-(1-t)^s] f(b) + \frac{1}{n} \sum_{s=1}^{n} [1-t^s] f(a) \right] dt$$
$$= \frac{f(b)}{n} \sum_{s=1}^{n} \int_0^1 [1-(1-t)^s]\, dt + \frac{f(a)}{n} \sum_{s=1}^{n} \int_0^1 [1-t^s]\, dt$$
$$= \left[\frac{f(a)+f(b)}{n} \right] \sum_{s=1}^{n} \frac{s}{s+1}. \tag{2.2}$$

Therefore, Theorem 2.3 follows from (2.1) and (2.2). □

Remark 2.4 Let $n = 1$. Then Theorem 2.3 leads to the Hermite–Hadamard inequality for harmonically convex functions of [73].

Lemma 2.5 *Let $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function. Then the identity*

$$\frac{\lambda f(a)+\mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\, dx$$
$$= \frac{ab(b-a)}{4} \int_0^1 \left[\frac{4(1-\lambda-t)}{((1-t)a+(1+t)b)^2} f'\left(\frac{2ab}{(1-t)a+(1+t)b}\right) + \frac{4(\mu-t)}{((2-t)a+tb)^2} f'\left(\frac{2ab}{(2-t)a+tb}\right) \right] dt$$

holds for $\lambda, \mu \in [0,1]$ if $f' \in L[a,b]$.

Proof Integrating by parts and changing variable of the definite integral give

$$I_1 = \int_0^1 \left[\frac{4(1-\lambda-t)}{((1-t)a+(1+t)b)^2} f'\left(\frac{2ab}{(1-t)a+(1+t)b}\right) \right] dt$$
$$= -\frac{2}{ab(b-a)} \left[(1-\lambda-t) f\left(\frac{2ab}{(1-t)a+(1+t)b}\right) \Big|_0^1 + \int_0^1 f\left(\frac{2ab}{(1-t)a+(1+t)b}\right) dt \right]$$
$$= \frac{2}{ab(b-a)} \left[\lambda f(a) + (1-\lambda) f\left(\frac{2ab}{a+b}\right) \right] - \frac{4}{(b-a)^2} \int_a^{\frac{2ab}{b-a}} \frac{f(x)}{x^2}\, dx.$$

Similarly, we have

$$I_2 = \int_0^1 \frac{4(\mu-t)}{((2-t)a+tb)^2} f'\left(\frac{2ab}{(2-t)a+tb)}\right) dt$$
$$= -\frac{2}{ab(b-a)} \left[(\mu-t) f\left(\frac{2ab}{(2-t)a+tb}\right) \Big|_0^1 + \int_0^1 f\left(\frac{2ab}{(2-t)a+tb}\right) dt \right]$$
$$= \frac{2}{ab(b-a)} \left[(1-\mu) f\left(\frac{2ab}{a+b}\right) + \mu f(b) \right] - \frac{4}{(b-a)^2} \int_{\frac{2ab}{a+b}}^{b} \frac{f(x)}{x^2}\, dx.$$

Multiplying I_1 and I_2 by $\frac{ab(b-a)}{4}$ and combining both equalities, we get the desired result. □

For the sake of simplicity, in what follows we denote

$$A_a = \bigl((1-t)a + (1+t)b\bigr)$$

and

$$A_b = \bigl((2-t)a + tb\bigr).$$

Before we give our next result, let us recall the definitions of the gamma function $\Gamma(\cdot)$, beta function $B(\cdot,\cdot)$, and hypergeometric function ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ as follows:

$$\begin{aligned}
\Gamma(x) &= \int_0^\infty e^{-t}t^{x-1}\,dt,\\
B(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}\,dt,\\
B(x,y) &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\\
{}_2F_1(x,y;c;z) &= \frac{1}{\mathrm{B}(y,c-y)}\int_0^1 t^{y-1}(1-t)^{c-y-1}(1-zt)^{-x}\,dt.
\end{aligned}$$

Theorem 2.6 *Let $p,q>1$ with $1/p+1/q=1$, $\lambda,\mu\in[0,1]$, and $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function such that $f'\in L[a,b]$ and $|f'|^q$ is an n-polynomial harmonically convex function. Then we have*

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2} + \frac{2-\lambda-\mu}{2}f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx\right|\\
&\quad\le \frac{ab(b-a)}{4}\bigl[\bigl\{\psi_1^{\frac{1}{p}}\bigl(C_1\bigl|f'(a)\bigr|^q + C_2\bigl|f'(b)\bigr|^q\bigr)^{\frac{1}{q}}\bigr\}\\
&\qquad + \bigl\{\psi_2^{\frac{1}{p}}\bigl(C_3\bigl|f'(a)\bigr|^q + C_4\bigl|f'(b)\bigr|^q\bigr)^{\frac{1}{q}}\bigr\}\bigr],
\end{aligned}$$

where

$$\psi_1 = 4\int_0^1 |1-\lambda-t|^p\,dt = 4\left(\frac{(1-\lambda)^{p+1}+(\lambda)^{p+1}}{p+1}\right), \tag{2.3}$$

$$\psi_2 = 4\int_0^1 |\mu-t|^p\,dt = 4\left(\frac{(1-\mu)^{p+1}+(\mu)^{p+1}}{p+1}\right), \tag{2.4}$$

$$\begin{aligned}
C_1 = \frac{1}{2n}\sum_{s=1}^{n}\biggl(&2(a+b)^{-2q}{}_2F_1\left(2q,1;2;\frac{a-b}{a+b}\right)\\
&- \frac{(a+b)^{-2q}}{s+1}{}_2F_1\left(2q,1;s+2;\frac{a-b}{a+b}\right)\biggr),
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
C_2 = \frac{1}{2n}\sum_{s=1}^{n}\biggl(&(a+b)^{-2q}{}_2F_1\left(2q,1;2;\frac{a-b}{a+b}\right)\\
&- \frac{(a+b)^{-2q}}{s+1}{}_2F_1\left(2q,s+1;s+2;\frac{a-b}{a+b}\right)\biggr),
\end{aligned} \tag{2.6}$$

$$C_3 = \frac{1}{2n}\sum_{s=1}^{n}\left((2a)^{-2q}{}_2F_1\left(2q,1;2;\frac{a-b}{2a}\right) - \frac{(2a)^{-2q}}{s+1}{}_2F_1\left(2q,1;s+2;\frac{a-b}{2a}\right)\right), \tag{2.7}$$

$$C_4 = \frac{1}{2n}\sum_{s=1}^{n}\left(2(2a)^{-2q}{}_2F_1\left(2q,1;2;\frac{a-b}{2a}\right) - \frac{(2a)^{-2q}}{s+1}{}_2F_1\left(2q,s+1;s+2;\frac{a-b}{2a}\right)\right). \tag{2.8}$$

Proof It follows from Lemma 2.5 and Hölder's integral inequality together with the n-polynomial harmonic convexity of $|f'|^q$ that

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2} + \frac{(2-\lambda-\mu)}{2}f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx\right| \\
&\quad\leq \frac{ab(b-a)}{4}\left[\int_0^1 \left|\frac{4(1-\lambda-t)}{((1-t)a+(1+t)b)^2}\right|\left|f'\left(\frac{2ab}{((1-t)a+(1+t)b)}\right)\right| dt \right.\\
&\qquad\left. + \int_0^1 \left|\frac{4(\mu-t)}{((2-t)a+tb)^2}\right|\left|f'\left(\frac{2ab}{(2-t)a+tb)}\right)\right| dt\right] \\
&\quad\leq \frac{ab(b-a)}{4}\left\{\left(\int_0^1 4(1-\lambda-t)^p\,dt\right)^{\frac{1}{p}}\right. \\
&\qquad\times\left[\int_0^1 \frac{1}{A_a^{2q}}\left(\frac{1}{2n}\sum_{s=1}^{n}\left[2-(1-t)^s\right]\left|f'(a)\right|^q + \frac{1}{2n}\sum_{s=1}^{n}\left[1-t^s\right]\left|f'(b)\right|^q\right) dt\right]^{\frac{1}{q}} \\
&\qquad + \left(\int_0^1 4(\mu-t)^p \mathrm{d}t\right)^{\frac{1}{p}} \\
&\qquad\left.\times\left[\int_0^1 \frac{1}{A_b^{2q}}\left(\frac{1}{2n}\sum_{s=1}^{n}\left[1-(1-t)^s\right]\left|f'(a)\right|^q + \frac{1}{2n}\sum_{s=1}^{n}\left[2-t^s\right]\left|f'(b)\right|^q\right) \mathrm{d}t\right]^{\frac{1}{q}}\right\} \\
&\quad= \frac{ab(b-a)}{4}\left\{\left(\psi_1^{\frac{1}{p}}\left(C_1\left|f'(a)\right|^q + C_2\left|f'(b)\right|\right)^{\frac{1}{q}}\right)\right. \\
&\qquad\left. + \left(\psi_2^{\frac{1}{p}}\left(C_3\left|f'(a)\right|^q + C_4\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right)\right\}.
\end{aligned}$$

This completes the proof. □

From Theorem 2.6 we get the following Corollaries 2.7 and 2.8 immediately.

Corollary 2.7 *Let $\lambda = \mu$. Then Theorem 2.6 leads to the conclusion that*

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\lambda f(b)}{2} + (1-\lambda)f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx\right| \\
&\quad\leq \frac{ab(b-a)}{4}\psi_1^{\frac{1}{p}}\left[\left\{C_1\left|f'(a)\right|^q + C_2\left|f'(b)\right|^q\right\}^{\frac{1}{q}} + \left\{C_3\left|f'(a)\right|^q + C_4\left|f'(b)\right|^q\right\}^{\frac{1}{q}}\right],
\end{aligned}$$

where ψ_1, C_1, C_2, C_3, and C_4 are given by (2.3), (2.5), (2.6), (2.7), and (2.8), respectively.

Corollary 2.8 *Let $\lambda = \mu = 1/2$ and $\lambda = \mu = 1/3$. Then Theorem 2.6 leads to*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{4}+\frac{1}{2}f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\,dx\right|\\&\quad\le\frac{ab(b-a)}{4}\left(\frac{8^{\frac{1}{p}}}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}}\right)\left[\left\{C_1\left|f'(a)\right|^q+C_2\left|f'(b)\right|^q\right\}^{\frac{1}{q}}\right.\\&\qquad\left.+\left\{C_3\left|f'(a)\right|^q+C_4\left|f'(b)\right|^q\right\}^{\frac{1}{q}}\right]\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{6}+\frac{2}{3}f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\,dx\right|\\&\quad\le\frac{ab(b-a)}{4}\left(4\left(\frac{(\frac{2}{3})^{p+1}+(\frac{1}{3})^{p+1}}{(p+1)}\right)\right)^{\frac{1}{p}}\left[\left\{C_1\left|f'(a)\right|^q+C_2\left|f'(b)\right|^q\right\}^{\frac{1}{q}}\right.\\&\qquad\left.+\left\{C_3\left|f'(a)\right|^q+C_4\left|f'(b)\right|^q\right\}^{\frac{1}{q}}\right],\end{aligned}$$

where C_1, C_2, C_3, and C_4 are given by (2.5), (2.6), (2.7), and (2.8), respectively.

Theorem 2.9 *Let $p,q>1$ with $1/p+1/q=1$, $\lambda,\mu\in[0,1]$, and $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function such that $f'\in L[a,b]$ and $|f'|^q$ is an n-polynomial harmonically convex function. Then*

$$\begin{aligned}&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\,dx\right|\\&\quad\le\frac{ab(b-a)}{4}\left[\frac{4}{(a+b)^2}\left({}_2F_1\left(2p,1;2;\frac{a-b}{a+b}\right)\right)^{\frac{1}{p}}\left(C_5\left|f'(a)\right|^q+C_6\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.\\&\qquad\left.+\frac{1}{a^2}\left({}_2F_1\left(2p,1;2;\frac{a-b}{2a}\right)\right)^{\frac{1}{p}}\left(C_7\left|f'(a)\right|^q+C_8\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right],\end{aligned}\tag{2.9}$$

where

$$\begin{aligned}C_5&=\frac{1}{2n}\sum_{s=1}^n\left(2\frac{(1-\lambda)^{q+1}+\lambda^{q+1}}{q+1}-\int_0^1|1-\lambda-t|^q(1-t)^s\,dt\right),\\C_6&=\frac{1}{2n}\sum_{s=1}^n\left(\frac{(1-\lambda)^{q+1}+\lambda^{q+1}}{q+1}-\int_0^1|1-\lambda-t|^qt^s\,dt\right),\\C_7&=\frac{1}{2n}\sum_{s=1}^n\left(\frac{(1-\mu)^{q+1}+\mu^{q+1}}{q+1}-\int_0^1|\mu-t|^q(1-t)^s\,dt\right),\\C_8&=\frac{1}{2n}\sum_{s=1}^n\left(2\frac{(1-\mu)^{q+1}+\mu^{q+1}}{q+1}-\int_0^1|\mu-t|^qt^s\,dt\right).\end{aligned}$$

Proof Using Lemma 2.5, Hölder's integral inequality, and the n-polynomial harmonic convexity of $|f'|^q$, we have

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{(2-\lambda-\mu)}{2}f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\,dx\right| \\
&\leq \frac{ab(b-a)}{4}\left[\int_0^1\left|\frac{4(1-\lambda-t)}{((1-t)a+(1+t)b)^2}\right|\left|f'\left(\frac{2ab}{((1-t)a+(1+t)b)}\right)\right|dt\right. \\
&\quad\left.+\int_0^1\left|\frac{4(\mu-t)}{((2-t)a+tb)^2}\right|\left|f'\left(\frac{2ab}{(2-t)a+tb)}\right)\right|dt\right] \\
&\leq \frac{ab(b-a)}{4}\left\{4\left(\int_0^1\frac{1}{A_a^{2p}}\,dt\right)^{\frac{1}{p}}\right. \\
&\quad\times\left[\int_0^1|1-\lambda-t|^q\left(\frac{1}{2n}\sum_{s=1}^{n}\bigl[2-(1-t)^s\bigr]\bigl|f'(a)\bigr|^q+\frac{1}{2n}\sum_{s=1}^{n}\bigl[1-t^s\bigr]\bigl|f'(b)\bigr|^q\right)dt\right]^{\frac{1}{q}} \\
&\quad+4\left(\int_0^1\frac{1}{A_b^{2p}}\,dt\right)^{\frac{1}{p}} \\
&\quad\left.\times\left[\int_0^1|\mu-t|^q\mathrm{d}t\left(\frac{1}{2n}\sum_{s=1}^{n}\bigl[1-(1-t)^s\bigr]\bigl|f'(a)\bigr|^q+\frac{1}{2n}\sum_{s=1}^{n}\bigl[2-t^s\bigr]\bigl|f'(b)\bigr|^q\right)\right]^{\frac{1}{q}}\right\} \\
&=\frac{ab(b-a)}{4}\left[\frac{4}{(a+b)^2}\left({}_2F_1\left(2p,1;2;\frac{a-b}{a+b}\right)\right)^{\frac{1}{p}}\bigl(C_5\bigl|f'(a)\bigr|^q+C_6\bigl|f'(b)\bigr|^q\bigr)^{\frac{1}{q}}\right. \\
&\quad\left.+\frac{1}{a^2}\left({}_2F_1\left(2p,1;2;\frac{a-b}{2a}\right)\right)^{\frac{1}{p}}\bigl(C_7\bigl|f'(a)\bigr|^q+C_8\bigl|f'(b)\bigr|^q\bigr)^{\frac{1}{q}}\right].
\end{aligned}$$

This completes the proof. □

Remark 2.10 Let $n = 1$. Then inequality (2.9) reduces to the following inequality for harmonically convex function:

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\,dx\right| \\
&\leq \frac{ab(b-a)}{4}\left[\frac{4}{(a+b)^2}\left({}_2F_1\left(2p,1;2;\frac{a-b}{a+b}\right)\right)^{\frac{1}{p}}\bigl(C_5^*\bigl|f'(a)\bigr|^q+C_6^*\bigl|f'(b)\bigr|^q\bigr)^{\frac{1}{q}}\right. \\
&\quad\left.+\frac{1}{a^2}\left({}_2F_1\left(2p,1;2;\frac{a-b}{2a}\right)\right)^{\frac{1}{p}}\bigl(C_7^*\bigl|f'(a)\bigr|^q+C_8^*\bigl|f'(b)\bigr|^q\bigr)^{\frac{1}{q}}\right],
\end{aligned}$$

where

$$\begin{aligned}
C_5^* &= \frac{1}{2}\int_0^1|1-\lambda-t|^q(1+t)\,dt \\
&= \frac{1}{2}\left(\frac{(1-\lambda)^{q+1}+(\lambda)^{q+1}}{q+1}+\frac{(1-\lambda)^{q+2}+(q+2-\lambda)(\lambda)^{q+1}}{(q+1)(q+2)}\right) \\
&= \frac{1}{2(q+1)(q+2)}\bigl((q+3-\lambda)(1-\lambda)^{q+1}+(2q+4-\lambda)\lambda^{q+1}\bigr),
\end{aligned}$$

$$\begin{aligned}
C_6^* &= \frac{1}{2}\int_0^1 |1-\lambda-t|^q(1-t)\,dt \\
&= \frac{1}{2}\left(\frac{(1-\lambda)^{q+1}+(\lambda)^{q+1}}{q+1} - \frac{(1-\lambda)^{q+2}+(q+2-\lambda)(\lambda)^{q+1}}{(q+1)(q+2)}\right) \\
&= \frac{1}{2(q+1)(q+2)}\big((q+1+\lambda)(1-\lambda)^{q+1}+\lambda^{q+2}\big), \\
C_7^* &= \frac{1}{2}\sum_{s=1}^{n}\int_0^1 |\mu-t|^q t\,dt = \frac{1}{2(q+1)(q+2)}\big((\mu)^{q+2}+(q+1+\mu)(1-\mu)^{q+1}\big), \\
C_8^* &= \frac{1}{2}\sum_{s=1}^{n}\int_0^1 |\mu-t|^q(2-t)\,dt \\
&= \frac{1}{2}\left(2\left(\frac{(\mu)^{q+1}+(1-\mu)^{q+1}}{q+1}\right) - \frac{(\mu)^{q+2}+(q+1+\mu)(1-\mu)^{q+1}}{(q+1)(q+2)}\right) \\
&= \frac{1}{2(q+1)(q+2)}\big((q+3-\mu)(1-\mu)^{q+1}+(2q+4-\mu)\mu^{q+1}\big).
\end{aligned}$$

Remark 2.11 If $q=1$ and $n=1$, then Theorem 2.9 reduces to

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx\right| \\
&\quad\le \frac{ab(b-a)}{4}\left[\frac{4}{(a+b)^2}\left({}_2F_1\left(2p,1;2;\frac{a-b}{a+b}\right)\right)^{\frac{1}{p}}\left(C_5^{**}\left|f'(a)\right| + C_6^{**}\left|f'(b)\right|\right)\right. \\
&\quad\quad \left. + \frac{1}{a^2}\left({}_2F_1\left(2p,1;2;\frac{a-b}{2a}\right)\right)^{\frac{1}{p}}\left(C_7^{**}\left|f'(a)\right| + C_8^{**}\left|f'(b)\right|\right)\right],
\end{aligned}$$

where

$$\begin{aligned}
C_5^{**} &= \frac{1}{12}\big[(4-\lambda)(1-\lambda)^2+(6-\lambda)\lambda^2\big], \\
C_6^{**} &= \frac{1}{12}\big[(2+\lambda)(1-\lambda)^2+\lambda^3\big], \\
C_7^{**} &= \frac{1}{12}\big[(2+\mu)(1-\mu)^2+\mu^3\big], \\
C_8^{**} &= \frac{1}{12}\big[(4-\mu)(1-\mu)^2+(6-\mu)\mu^2\big].
\end{aligned}$$

Remark 2.12 If $q=1$, $n=1$, and $\lambda=\mu$, then inequality (2.9) becomes

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\lambda f(b)}{2} + (1-\lambda) f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx\right| \\
&\quad\le \frac{ab(b-a)}{4}\left[\frac{4}{(a+b)^2}\left({}_2F_1\left(2p,1;2;\frac{a-b}{a+b}\right)\right)^{\frac{1}{p}}\left(C_5^{**}\left|f'(a)\right| + C_6^{**}\left|f'(b)\right|\right)\right. \\
&\quad\quad \left. + \frac{1}{a^2}\left({}_2F_1\left(2p,1;2;\frac{a-b}{2a}\right)\right)^{\frac{1}{p}}\left(C_6^{**}\left|f'(a)\right| + C_5^{**}\left|f'(b)\right|\right)\right].
\end{aligned}$$

Remark 2.13 Let $q = 1$, $n = 1$, and $\lambda = \mu = 1/2$. Then Theorem 2.9 leads to the conclusion that

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{4}+\frac{1}{2}f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\,dx\right| \\
&\quad\leq\frac{ab(b-a)}{384}\left[\frac{4}{(a+b)^2}\left({}_2F_1\left(2p,1;2;\frac{a-b}{a+b}\right)\right)^{\frac{1}{p}}\left(18\left|f'(a)\right|+6\left|f'(b)\right|\right)\right. \\
&\qquad\left.+\frac{1}{a^2}\left({}_2F_1\left(2p,1;2;\frac{a-b}{2a}\right)\right)^{\frac{1}{p}}\left(6\left|f'(a)\right|+18\left|f'(b)\right|\right)\right].
\end{aligned}$$

3 Conclusion

In this paper, we have introduced a new class of harmonically convex functions, which are called n-polynomial harmonically convex functions, derived several new versions of the Hermite–Hadamard inequality using the class of n-polynomial harmonically convex functions and a new integral identity for the differentiable function. We have also discussed some special cases of the obtained results which show that our results are the generalizations and extensions of some previously known results for the harmonically convex functions. Our ideas and approach may lead to a lot of follow-up research.

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Authors' contributions

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Author details

[1]Department of Mathematics, Government College University, Faisalabad, Pakistan. [2]Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan. [3]Department of Mathematics, Huzhou University, Huzhou, China. [4]Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, Changsha University of Science & Technology, Changsha, China.

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