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New Hermite–Jensen–Mercer-type inequalities via k -fractional integrals

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Abstract

In the article, we establish several novel Hermite–Jensen–Mercer-type inequalities for convex functions in the framework of the k -fractional conformable integrals by use of our new approaches. Our obtained results are the generalizations, improvements, and extensions of some previously known results, and our ideas and methods may lead to a lot of follow-up research.

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1 Introduction

Convex function [1–20] is an important concept that has come to the fore among many other function classes with its many features and areas of use. Giving the definition as an inequality containing linear combinations has helped in using convex functions for classical inequalities. Jensen inequality [21, 22] is one of these inequalities for convex functions, which can be stated as follows.

Let $(\mu_1, \mu_2, \dots, \mu_n) \in [0, 1]^n$ with $\sum_{i=1}^n \mu_i = 1$ and τ be a convex function on the interval $[\theta, \vartheta]$. Then the inequality

$$\tau\left(\sum_{i=1}^n \mu_i x_i\right) \leq \left(\sum_{i=1}^n \mu_i \tau(x_i)\right) \quad (1.1)$$

holds for all $x_i \in [\theta, \vartheta]$ ($i = 1, 2, \dots, n$).

Another important inequality for convex functions is the Hermite–Hadamard inequality [23, 24], which has been proved by numerous ways and has many generalizations and extensions [25–29]. This inequality can generate bounds on the average value of convex functions to reveal its functionality with applications to numerical analysis and error calculation formulas such as trapezoidal and midpoint quadrature formulas. Now, we recall the Hermite–Hadamard inequality as follows.

Let $\tau : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function. Then the Hermite–Hadamard inequality

$$\tau\left(\frac{\theta + \vartheta}{2}\right) \leq \frac{1}{\vartheta - \theta} \int_{\theta}^{\vartheta} \tau(\lambda) d\lambda \leq \frac{\tau(\theta) + \tau(\vartheta)}{2}$$

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holds for all $\theta, \vartheta \in J$ with $\theta \neq \vartheta$. If τ is a concave function on J , then the above inequality is reversed.

There are many interesting studies in the literature for the Jensen inequality, for example, the Jensen–Mercer inequality is a new variant of the Jensen inequality given by Mercer in [30]. Later, Matković et al. [31] generalized the Jensen–Mercer inequality to operators and gave its many applications. Recently, the Jensen–Mercer inequality has been the subject of intensive research.

The following Theorem 1.1 for convex functions can be found in [32].

Theorem 1.1 ([32]) *Let τ be a convex function defined on $[\theta, \vartheta]$. Then the inequality*

$$\tau\left(\theta + \vartheta - \sum_{i=1}^n \mu_i x_i\right) \leq \tau(\theta) + \tau(\vartheta) - \sum_{i=1}^n \mu_i \tau(x_i) \tag{1.2}$$

holds for all $x_i \in [\theta, \vartheta]$ and $\mu_i \in [0, 1]$ with $\sum_{i=1}^n \mu_i = 1$.

Next, we recall the definitions of the Euler Gamma $\Gamma(\cdot)$ and Beta $B(\cdot, \cdot)$ functions, which will be used in the article:

$$\Gamma(\theta) = \int_0^\infty e^{-\lambda} \lambda^{\theta-1} d\lambda, \quad B(r, s) = \int_0^1 \lambda^{r-1} (1 - \lambda)^{s-1} d\lambda.$$

The concept of fractional order derivative and integral [33–40] that will shed light on some unknown points about differential equations and solutions of some fractional order differential equations, which proved to be useful for their solution, is a novelty in applied sciences as well as in mathematics. New derivatives and integrals contribute to the solution of differential equations that are expressed and solved in classical analysis, as well as using fractional order derivatives and integrals. In addition, it has increased its contribution to the literature with applications in areas such as engineering, biostatistics, and mathematical biology. Fractional derivative and integral operators not only differ from each other in terms of singularity, locality, and kernels, but also brought innovations to fractional analysis in terms of their usage areas and spaces. The new integral operators put forward by the researchers working in the field of fractional analysis led to new approaches, results, and methods in applied mathematics, engineering, and many other fields, and they have found the expected response in inequality theory. Many new integral inequalities and bounds to known inequalities have been found by using new integral operators. The new trends, improvements, and advances on fractional calculus and real world applications can be found in the literature [41–60]. Now let us remember some integral operators that are well known to be useful in fractional analysis.

Definition 1.2 ([61]) *Let $\alpha > 0$, $0 \leq \theta < \vartheta$, and $\tau \in [\theta, \vartheta]$. Then the Riemann–Liouville integrals $J_{\theta^+}^\alpha \tau$ and $J_{\vartheta^-}^\alpha \tau$ of order α are defined by*

$$(J_{\theta^+}^\alpha) \tau(y) = \frac{1}{\Gamma(\alpha)} \int_\theta^y (y - \lambda)^{\alpha-1} \tau(\lambda) d\lambda \quad (y > \theta) \tag{1.3}$$

and

$$({}^{\alpha}J_{\vartheta-}^{\alpha})\tau(y) = \frac{1}{\Gamma(\alpha)} \int_y^{\vartheta} (\lambda - y)^{\alpha-1} \tau(\lambda) d\lambda \quad (y < \vartheta), \tag{1.4}$$

respectively, where $(J_{\theta+}^0)\tau(y) = (J_{\vartheta-}^0)\tau(y) = \tau(y)$.

In [62], Jarad et al. defined the new fractional integral operators as follows:

$${}^{\beta}J_{\theta}^{\alpha}\tau(y) = \frac{1}{\Gamma(\beta)} \int_{\theta}^y \left(\frac{(y - \theta)^{\alpha} - (\lambda - \theta)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{\tau(\lambda)}{(\lambda - \theta)^{1-\alpha}} d\lambda \tag{1.5}$$

and

$${}^{\beta}J_{\vartheta}^{\alpha}\tau(y) = \frac{1}{\Gamma(\beta)} \int_y^{\vartheta} \left(\frac{(\vartheta - y)^{\alpha} - (\vartheta - \lambda)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{\tau(\lambda)}{(\vartheta - \lambda)^{1-\alpha}} d\lambda. \tag{1.6}$$

Remark 1.3 From (1.5) and (1.6) we clearly see that

- (i) If $\theta = 0$ and $\alpha = 1$, then (1.5) reduces to the Riemann–Liouville operator given in (1.3).
- (ii) If $\theta = 0$ and $\alpha \rightarrow 0$, then the new conformable fractional integral coincides with the generalized fractional integral (see [63]).
- (iii) Furthermore, (1.6) becomes the Riemann–Liouville operator if we set $\vartheta = 0$ and $\alpha = 1$. It also corresponds the Hadamard fractional integral [63] once we take $\vartheta = 0$ and $\alpha \rightarrow 0$ in the generalized fractional integral.

The generalized k -fractional conformable integrals [64] are defined by

$${}^{\beta}J_{\theta+}^{\alpha}\tau(y) = \frac{1}{k\Gamma_k(\beta)} \int_{\theta}^y \left(\frac{(y - \theta)^{\alpha} - (\lambda - \theta)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}-1} \frac{\tau(\lambda)}{(\lambda - \theta)^{1-\alpha}} d\lambda \tag{1.7}$$

and

$${}^{\beta}J_{\vartheta-}^{\alpha}\tau(y) = \frac{1}{k\Gamma_k(\beta)} \int_y^{\vartheta} \left(\frac{(\vartheta - y)^{\alpha} - (\vartheta - \lambda)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}-1} \frac{\tau(\lambda)}{(\vartheta - \lambda)^{1-\alpha}} d\lambda. \tag{1.8}$$

If $k > 0$, then the k -Gamma function Γ_k is defined as

$$\Gamma_k(\alpha) = \lim_{m \rightarrow \infty} \frac{m!k^m(mk)^{\frac{\alpha}{k}-1}}{(\alpha)_{m,k}}. \tag{1.9}$$

If $\text{Re}(\alpha) > 0$, then the k -Gamma function in integral form is defined as

$$\Gamma_k(\alpha) = \int_0^{\infty} e^{-\frac{\mu}{k}} \mu^{\alpha-1} d\mu \tag{1.10}$$

with $\alpha\Gamma_k(\alpha) = \Gamma_k(\alpha + k)$.

The main purpose of the article is to reveal new and more general Hermite–Jensen–Mercer-type inequalities for convex functions with the help of k -fractional integral op-

erator. For this purpose, Hölder inequality and its variants have been used in addition to various analysis processes. With the special versions of the main findings, many inequalities in the literature were obtained and the importance of the results was emphasized.

2 New Hermite–Jensen–Mercer type inequalities

Theorem 2.1 *Let $\alpha, \beta > 0$ and $\tau : [\theta, \vartheta] \rightarrow R$ be a convex mapping. Then the inequality*

$$\begin{aligned} \tau\left(\theta + \vartheta - \frac{x+y}{2}\right) &\leq \frac{2^{\alpha\frac{\beta}{k}-1} \alpha^{\frac{\beta}{k}} \Gamma_k(\beta+k)}{(y-x)^{\alpha\frac{\beta}{k}}} \\ &\quad \times \left\{ {}_k J_{(\theta+\vartheta-\frac{x+y}{2})+}^{\beta\alpha} \tau(\theta + \vartheta - x) + {}_k J_{(\theta+\vartheta-\frac{x+y}{2})-}^{\beta\alpha} \tau(\theta + \vartheta - y) \right\} \\ &\leq \tau(\theta) + \tau(\vartheta) - \left(\frac{\tau(x) + \tau(y)}{2}\right) \end{aligned} \tag{2.1}$$

holds for all $x, y \in [\theta, \vartheta]$.

Proof Since τ is convex, to prove the first inequality, we write

$$\begin{aligned} \tau\left(\theta + \vartheta - \frac{x_1+y_1}{2}\right) &= \tau\left(\frac{\theta + \vartheta - x_1 + \theta + \vartheta - y_1}{2}\right) \\ &\leq \frac{\tau(\theta + \vartheta - x_1) + \tau(\theta + \vartheta - y_1)}{2} \end{aligned}$$

for all $x_1, y_1 \in [\theta, \vartheta]$.

Let $x_1 = \frac{\lambda}{2}x + \frac{2-\lambda}{2}y$ and $y_1 = \frac{2-\lambda}{2}x + \frac{\lambda}{2}y$. Then for $x, y \in [\theta, \vartheta]$ and $\lambda \in [0, 1]$, we have

$$\begin{aligned} 2\tau\left(\theta + \vartheta - \frac{x+y}{2}\right) &\leq \tau\left(\theta + \vartheta - \left(\frac{\lambda}{2}x + \frac{2-\lambda}{2}y\right)\right) \\ &\quad + \tau\left(\theta + \vartheta - \left(\frac{2-\lambda}{2}x + \frac{\lambda}{2}y\right)\right). \end{aligned} \tag{2.2}$$

Multiplying both sides of (2.2) by $\left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1} (1-\lambda)^{\alpha-1}$ and integrating the obtained inequality with respect to λ over $[0, 1]$, and then combining the resulting inequality with the definition of the integral operator gives

$$\begin{aligned} &2\tau\left(\theta + \vartheta - \frac{x+y}{2}\right) \int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1} (1-\lambda)^{\alpha-1} d\lambda \\ &\leq \int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1} (1-\lambda)^{\alpha-1} \\ &\quad \times \left(\tau\left(\theta + \vartheta - \left(\frac{\lambda}{2}x + \frac{2-\lambda}{2}y\right)\right) + \tau\left(\theta + \vartheta - \left(\frac{2-\lambda}{2}x + \frac{\lambda}{2}y\right)\right)\right) d\lambda \\ &= \int_{\theta+\vartheta-y}^{\theta+\vartheta-\frac{x+y}{2}} \left(\frac{1-\left(\frac{(\theta+\vartheta-\frac{x+y}{2})-\lambda_1}{\frac{y-x}{2}}\right)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1} \left(\frac{(\theta + \vartheta - \frac{x+y}{2}) - \lambda_1}{\frac{y-x}{2}}\right)^{\alpha-1} \tau(\lambda_1) \frac{2}{y-x} d\lambda_1 \end{aligned}$$

$$\begin{aligned}
 & + \int_{\theta+\vartheta-x}^{\theta+\vartheta-\frac{x+y}{2}} \left(\frac{1 - \left(\frac{\lambda_2 - (\theta+\vartheta-\frac{x+y}{2})}{\frac{y-x}{2}} \right)^\alpha}{\alpha} \right)^{\frac{\beta}{k}-1} \left(\frac{\lambda_2 - (\theta+\vartheta-\frac{x+y}{2})}{\frac{y-x}{2}} \right)^{\alpha-1} \tau(\lambda_2) \frac{2}{y-x} d\lambda_2 \\
 & = \left(\frac{2}{y-x} \right)^{\alpha \frac{\beta}{k}} \left\{ \Gamma_k(\beta) {}_k J_{(\theta+\vartheta-\frac{x+y}{2})^+}^\alpha \tau(\theta+\vartheta-y) + \Gamma_k(\beta) {}_k J_{(\theta+\vartheta-\frac{x+y}{2})^-}^\alpha \tau(\theta+\vartheta-x) \right\}.
 \end{aligned}$$

Note that

$$\int_0^1 \left(\frac{1 - (1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}-1} (1-\lambda)^{\alpha-1} d\lambda = \frac{1}{\frac{\beta}{k} \alpha^{\frac{\beta}{k}}}. \tag{2.3}$$

Therefore,

$$\begin{aligned}
 & 2\tau\left(\theta+\vartheta-\frac{x+y}{2}\right) \frac{1}{\frac{\beta}{k} \alpha^{\frac{\beta}{k}}} \\
 & \leq \left(\frac{2}{y-x} \right)^{\alpha \frac{\beta}{k}} \left\{ \Gamma_k(\beta) {}_k J_{(\theta+\vartheta-\frac{x+y}{2})^+}^\alpha \tau(\theta+\vartheta-y) + \Gamma_k(\beta) {}_k J_{(\theta+\vartheta-\frac{x+y}{2})^-}^\alpha \tau(\theta+\vartheta-x) \right\}.
 \end{aligned}$$

This completes the proof of the first inequality of (2.1).

To prove the second inequality, by a similar discussion, making use of the convexity of τ , for $\lambda \in [0, 1]$, we have

$$\tau\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right) \leq \tau(\theta)+\tau(\vartheta)-\left(\frac{\lambda}{2}\tau(x)+\frac{2-\lambda}{2}\tau(y)\right) \tag{2.4}$$

and

$$\tau\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right) \leq \tau(\theta)+\tau(\vartheta)-\left(\frac{2-\lambda}{2}\tau(x)+\frac{\lambda}{2}\tau(y)\right). \tag{2.5}$$

Adding (2.4) and (2.5) leads to

$$\begin{aligned}
 & \tau\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right) + \tau\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right) \\
 & \leq 2[\tau(\theta)+\tau(\vartheta)] - [\tau(x)+\tau(y)].
 \end{aligned} \tag{2.6}$$

Multiplying (2.6) by $\left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}(1-\lambda)^{\alpha-1}$ and integrating the obtained inequality with respect to λ over $[0, 1]$ gives

$$\begin{aligned}
 & \int_0^1 \left(\frac{1 - (1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}-1} (1-\lambda)^{\alpha-1} \\
 & \quad \times \left\{ \tau\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right) + \tau\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right) \right\} d\lambda \\
 & \leq \{2[\tau(\theta)+\tau(\vartheta)] - [\tau(x)+\tau(y)]\} \int_0^1 \left(\frac{1 - (1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}-1} (1-\lambda)^{\alpha-1} d\lambda
 \end{aligned}$$

$$\begin{aligned} & \times \left(\frac{2}{y-x}\right)^{\alpha \frac{\beta}{k}} \left\{ \Gamma_k(\beta) {}_k J_{(\theta+\vartheta-\frac{x+y}{2})^+}^{\alpha} \tau(\theta+\vartheta-y) + \Gamma_k(\beta) {}_k J_{(\theta+\vartheta-\frac{x+y}{2})^-}^{\alpha} \tau(\theta+\vartheta-x) \right\} \\ & \leq \frac{1}{\frac{\beta}{k} \alpha^{\frac{\beta}{k}}} \left\{ 2[\tau(\theta) + \tau(\vartheta)] - [\tau(x) + \tau(y)] \right\}, \end{aligned}$$

which completes the proof of the desired inequality. □

Remark 2.2 From Theorem 2.1, we clearly see that:

- (i) If we take $k = 1, x = \theta,$ and $y = \vartheta$ in Theorem 2.1, then we get Theorem 2.1 of [65].
- (ii) If we take $\alpha = k = 1, x = \theta,$ and $y = \vartheta$ in Theorem 2.1, then we get Theorem 2 of [66].

Theorem 2.3 *Let $\alpha, \beta > 0$ and $\tau : [\theta, \vartheta] \rightarrow R$ be a convex function. Then the inequalities*

$$\begin{aligned} \tau\left(\theta + \vartheta - \frac{x+y}{2}\right) & \leq \tau(\theta) + \tau(\vartheta) - \frac{\alpha^{\frac{\beta}{k}} \Gamma_k(\beta+k)}{2(y-x)^{\alpha \frac{\beta}{k}}} \left\{ {}_k J_{x^+}^{\alpha} \tau(y) + {}_k J_{y^-}^{\alpha} \tau(x) \right\} \\ & \leq \tau(\theta) + \tau(\vartheta) - \tau\left(\frac{x+y}{2}\right) \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \tau\left(\theta + \vartheta - \frac{x+y}{2}\right) & \leq \frac{\alpha^{\frac{\beta}{k}} \Gamma_k(\beta+k)}{2(y-x)^{\alpha \frac{\beta}{k}}} \left\{ {}_k J_{(\theta+\vartheta-x)^+}^{\alpha} \tau(\theta+\vartheta-y) + {}_k J_{(\theta+\vartheta-y)^-}^{\alpha} \tau(\theta+\vartheta-x) \right\} \\ & \leq \frac{\tau(\theta+\vartheta-x) + \tau(\theta+\vartheta-y)}{2} \leq \tau(\theta) + \tau(\vartheta) - \frac{\tau(x) + \tau(y)}{2} \end{aligned} \tag{2.8}$$

hold for all $x, y \in [\theta, \vartheta]$.

Proof It follows from the Jensen–Mercer inequality that

$$\tau\left(\theta + \vartheta - \frac{x_1 + y_1}{2}\right) \leq \tau(\theta) + \tau(\vartheta) - \frac{\tau(x_1) + \tau(y_1)}{2} \tag{2.9}$$

for all $x_1, y_1 \in [\theta, \vartheta]$.

By changing the variables $x_1 = \lambda x + (1 - \lambda)y$ and $y_1 = (1 - \lambda)x + \lambda y$ for $x, y \in [\theta, \vartheta]$ and $\lambda \in [0, 1]$ in (2.9), we get

$$\tau\left(\theta + \vartheta - \frac{x+y}{2}\right) \leq \tau(\theta) + \tau(\vartheta) - \frac{\tau(\lambda x + (1 - \lambda)y) + \tau((1 - \lambda)x + \lambda y)}{2}. \tag{2.10}$$

Multiplying (2.10) by $\left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1} (1-\lambda)^{\alpha-1}$ and integrating the obtained inequality with respect to λ over $[0, 1]$ leads to the conclusion that

$$\begin{aligned} & \tau\left(\theta + \vartheta - \frac{x+y}{2}\right) \int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} (1-\lambda)^{\alpha-1} d\lambda \\ & \leq \int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} (1-\lambda)^{\alpha-1} \\ & \quad \times \left\{ \tau(\theta) + \tau(\vartheta) - \frac{\tau(\lambda x + (1 - \lambda)y) + \tau((1 - \lambda)x + \lambda y)}{2} \right\} d\lambda, \end{aligned}$$

that is,

$$\tau\left(\theta + \vartheta - \frac{x+y}{2}\right) \leq \tau(\theta) + \tau(\vartheta) - \frac{\alpha^{\frac{\beta}{k}} \Gamma_k(\beta+k)}{2(y-x)^{\alpha \frac{\beta}{k}}} \left\{ {}^{\beta} J_{x^+}^{\alpha} \tau(y) + {}^{\beta} J_{y^-}^{\alpha} \tau(x) \right\}, \tag{2.11}$$

which completes the proof of the first inequality of (2.7).

To prove the second inequality of (2.7), from the convexity of τ , for $\lambda \in [0, 1]$ we obtain

$$\begin{aligned} \tau\left(\frac{x+y}{2}\right) &= \tau\left(\frac{\lambda x + (1-\lambda)y + (1-\lambda)x + \lambda y}{2}\right) \\ &\leq \frac{\tau(\lambda x + (1-\lambda)y) + \tau((1-\lambda)x + \lambda y)}{2}. \end{aligned} \tag{2.12}$$

Multiplying (2.12) by $\left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1} (1-\lambda)^{\alpha-1}$ and then by using integration with respect to λ over $[0, 1]$, we have

$$\begin{aligned} \tau\left(\frac{x+y}{2}\right) &\int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} (1-\lambda)^{\alpha-1} d\lambda \\ &\leq \int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} (1-\lambda)^{\alpha-1} \left\{ \frac{\tau(\lambda x + (1-\lambda)y) + \tau((1-\lambda)x + \lambda y)}{2} \right\} d\lambda, \end{aligned}$$

that is,

$$\begin{aligned} \tau\left(\frac{x+y}{2}\right) &\leq \frac{\alpha^{\frac{\beta}{k}} \Gamma_k(\beta+k)}{2(y-x)^{\alpha \frac{\beta}{k}}} \left\{ {}^{\beta} J_{x^+}^{\alpha} \tau(y) + {}^{\beta} J_{y^-}^{\alpha} \tau(x) \right\}, \\ -\tau\left(\frac{x+y}{2}\right) &\geq -\frac{\alpha^{\frac{\beta}{k}} \Gamma_k(\beta+k)}{2(y-x)^{\alpha \frac{\beta}{k}}} \left\{ {}^{\beta} J_{x^+}^{\alpha} \tau(y) + {}^{\beta} J_{y^-}^{\alpha} \tau(x) \right\}. \end{aligned} \tag{2.13}$$

Adding $\tau(\theta) + \tau(\vartheta)$ to both sides of (2.13), we obtain

$$\tau(\theta) + \tau(\vartheta) - \tau\left(\frac{x+y}{2}\right) \geq \tau(\theta) + \tau(\vartheta) - \frac{\alpha^{\frac{\beta}{k}} \Gamma_k(\beta+k)}{2(y-x)^{\alpha \frac{\beta}{k}}} \left\{ {}^{\beta} J_{x^+}^{\alpha} \tau(y) + {}^{\beta} J_{y^-}^{\alpha} \tau(x) \right\}. \tag{2.14}$$

Combining (2.11) and (2.14), we get (2.7). To prove inequality (2.8), we use the convexity of τ to get

$$\begin{aligned} \tau\left(\theta + \vartheta - \frac{x_1+y_1}{2}\right) &= \tau\left(\frac{\theta + \vartheta - x_1 + \theta + \vartheta - y_1}{2}\right) \\ &\leq \frac{\tau(\theta + \vartheta - x_1) + \tau(\theta + \vartheta - y_1)}{2} \end{aligned} \tag{2.15}$$

for all $x_1, y_1 \in [\theta, \vartheta]$.

Let $x_1 = \lambda x + (1-\lambda)y$ and $y_1 = (1-\lambda)x + \lambda y$. Then (2.15) leads to

$$\begin{aligned} \tau\left(\theta + \vartheta - \frac{x+y}{2}\right) &\leq \left\{ \frac{\tau(\theta + \vartheta - (\lambda x + (1-\lambda)y)) + \tau(\theta + \vartheta - ((1-\lambda)x + \lambda y))}{2} \right\}. \end{aligned} \tag{2.16}$$

Multiplying (2.16) by $\left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}(1-\lambda)^{\alpha-1}$ and then by integrating the resulting inequality with respect to λ over $[0, 1]$, we have

$$\begin{aligned} & \tau\left(\theta + \vartheta - \frac{x+y}{2}\right) \int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} (1-\lambda)^{\alpha-1} d\lambda \\ & \leq \int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} (1-\lambda)^{\alpha-1} \\ & \quad \times \left\{ \frac{\tau(\theta + \vartheta - (\lambda x + (1-\lambda)y)) + \tau(\theta + \vartheta - ((1-\lambda)x + \lambda y))}{2} \right\} d\lambda, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} & \tau\left(\theta + \vartheta - \frac{x+y}{2}\right) \\ & \leq \frac{\alpha^{\frac{\beta}{k}} \Gamma_k(\beta+k)}{2(y-x)^{\alpha \frac{\beta}{k}}} \left\{ {}_k J_{(\theta+\vartheta-x)^+}^\beta \tau(\theta + \vartheta - y) + {}_k J_{(\theta+\vartheta-y)^-}^\beta \tau(\theta + \vartheta - x) \right\}. \end{aligned} \tag{2.17}$$

It follows from the convexity of τ that

$$\tau(\lambda(\theta + \vartheta - x) + (1-\lambda)(\theta + \vartheta - y)) \leq \lambda\tau(\theta + \vartheta - x) + (1-\lambda)\tau(\theta + \vartheta - y)$$

and

$$\tau((1-\lambda)(\theta + \vartheta - x) + \lambda(\theta + \vartheta - y)) \leq (1-\lambda)\tau(\theta + \vartheta - x) + \lambda\tau(\theta + \vartheta - y).$$

Adding the above two inequalities and using the Jensen–Mercer inequality gives

$$\begin{aligned} & \tau(\lambda(\theta + \vartheta - x) + (1-\lambda)(\theta + \vartheta - y)) + \tau((1-\lambda)(\theta + \vartheta - x) + \lambda(\theta + \vartheta - y)) \\ & \leq \tau(\theta + \vartheta - x) + \tau(\theta + \vartheta - y) \leq 2(\tau(\theta) + \tau(\vartheta)) - (\tau(x) + \tau(y)). \end{aligned} \tag{2.18}$$

Multiplying (2.18) by $\left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^{\frac{\beta}{k}-1}(1-\lambda)^{\alpha-1}$ and then by using integration with respect to λ over $[0, 1]$, we have

$$\begin{aligned} & \int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} (1-\lambda)^{\alpha-1} \\ & \quad \times \left\{ \tau(\lambda(\theta + \vartheta - x) + (1-\lambda)(\theta + \vartheta - y)) + \tau((1-\lambda)(\theta + \vartheta - x) + \lambda(\theta + \vartheta - y)) \right\} d\lambda \\ & \leq 2(\tau(\theta) + \tau(\vartheta)) - (\tau(x) + \tau(y)) \int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^{\frac{\beta}{k}} (1-\lambda)^{\alpha-1} d\lambda, \end{aligned}$$

that is,

$$\begin{aligned} & \frac{\alpha^{\frac{\beta}{k}} \Gamma_k(\beta+k)}{2(y-x)^{\alpha \frac{\beta}{k}}} \left\{ {}_k J_{(\theta+\vartheta-x)^+}^\beta \tau(\theta + \vartheta - y) + {}_k J_{(\theta+\vartheta-y)^-}^\beta \tau(\theta + \vartheta - x) \right\} \\ & \leq (\tau(\theta) + \tau(\vartheta)) - \left(\frac{\tau(x) + \tau(y)}{2} \right). \end{aligned} \tag{2.19}$$

Combining (2.17) and (2.19) leads to (2.8). □

Remark 2.4 Let $\alpha = \beta = k = 1$. Then Theorem 2.3 leads to the conclusion that

$$\begin{aligned} \text{(i)} \quad \tau\left(\theta + \vartheta - \frac{x+y}{2}\right) &\leq \tau(\theta) + \tau(\vartheta) - \int_0^1 \tau(\lambda x + (1-\lambda)y) \, d\lambda \\ &\leq \tau(\theta) + \tau(\vartheta) - \tau\left(\frac{x+y}{2}\right) \end{aligned}$$

and

$$\begin{aligned} \text{(ii)} \quad \tau\left(\theta + \vartheta - \frac{x+y}{2}\right) &\leq \frac{1}{y-x} \int_x^y \tau(\theta + \vartheta - \lambda) \, d\lambda \\ &\leq \tau(\theta) + \tau(\vartheta) - \frac{\tau(x) + \tau(y)}{2}, \end{aligned}$$

which was also proved in Theorem 2.1 of [67].

Lemma 2.5 *Let $\alpha, \beta > 0, \theta < \vartheta$ and $\tau : [\theta, \vartheta] \rightarrow R$ be a differentiable mapping such that $\tau' \in L[\theta, \vartheta]$. Then the inequality*

$$\begin{aligned} &\frac{2^{\alpha \frac{\beta}{k}-1} \alpha^{\frac{\beta}{k}} \Gamma_k(\beta+k)}{(y-x)^{\alpha \frac{\beta}{k}}} \left\{ {}^{\beta} J_{(\theta+\vartheta-\frac{x+y}{2})^+}^{\alpha} \tau(\theta + \vartheta - x) + {}^{\beta} J_{(\theta+\vartheta-\frac{x+y}{2})^-}^{\alpha} \tau(\theta + \vartheta - y) \right\} \\ &\quad - \tau\left(\theta + \vartheta - \frac{x+y}{2}\right) \\ &= \frac{(y-x)\alpha^{\frac{\beta}{k}}}{4} \int_0^1 \left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}} \\ &\quad \times \left\{ \tau'\left(\theta + \vartheta - \left(\frac{2-\lambda}{2}x + \frac{\lambda}{2}y\right)\right) - \tau'\left(\theta + \vartheta - \left(\frac{\lambda}{2}x + \frac{2-\lambda}{2}y\right)\right) \right\} d\lambda \end{aligned} \tag{2.20}$$

holds for all $x, y \in [\theta, \vartheta]$.

Proof Let

$$I = \frac{y-x}{4} \alpha^{\frac{\beta}{k}} \{I_1 - I_2\}, \tag{2.21}$$

where

$$I_1 = \int_0^1 \left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}} \tau'\left(\theta + \vartheta - \left(\frac{2-\lambda}{2}x + \frac{\lambda}{2}y\right)\right) d\lambda$$

and

$$I_2 = \int_0^1 \left(\frac{1-(1-\lambda)^{\alpha}}{\alpha}\right)^{\frac{\beta}{k}} \tau'\left(\theta + \vartheta - \left(\frac{\lambda}{2}x + \frac{2-\lambda}{2}y\right)\right) d\lambda.$$

Then integrating by parts, we get

$$\begin{aligned}
 &= -\frac{2}{\alpha^{\frac{\beta}{k}}(y-x)}\tau\left(\theta+\vartheta-\frac{x+y}{2}\right) \\
 &\quad +\frac{2\alpha^{\frac{\beta}{k}}}{\alpha^{\frac{\beta}{k}}(y-x)}\int_0^1(1-(1-\lambda)^\alpha)^{\frac{\beta}{k}-1}(1-\lambda)^{\alpha-1}\tau\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right)d\lambda \\
 &= -\frac{2}{\alpha^{\frac{\beta}{k}}(y-x)}\tau\left(\theta+\vartheta-\frac{x+y}{2}\right) \\
 &\quad +\frac{2\alpha^{\frac{\beta}{k}}}{\alpha^{\frac{\beta}{k}-1}(y-x)^{\alpha^{\frac{\beta}{k}+1}}}\int_{\theta+\vartheta-\frac{x+y}{2}}^{\theta+\vartheta-x}\left(\left(\frac{y-x}{2}\right)^\alpha-\left(\lambda_1-\left(\theta+\vartheta-\frac{x+y}{2}\right)\right)^\alpha\right)^{\frac{\beta}{k}-1} \\
 &\quad \times\frac{\tau(\lambda_1)}{(\lambda_1-(\theta+\vartheta-\frac{x+y}{2}))^{1-\alpha}}d\lambda_1 \\
 &= -\frac{2}{\alpha^{\frac{\beta}{k}}(y-x)}\tau\left(\theta+\vartheta-\frac{x+y}{2}\right) \\
 &\quad +\left(\frac{2}{y-x}\right)^{\alpha^{\frac{\beta}{k}+1}}\frac{\Gamma_k(\beta+k)\beta J_{k(\theta+\vartheta-\frac{x+y}{2})}^\alpha}{\alpha^{\frac{\beta}{k}-1}}\tau(\theta+\vartheta-x). \tag{2.22}
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 I_2 &= \int_0^1\left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\tau'\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right)d\lambda \\
 &= \frac{2}{\alpha^{\frac{\beta}{k}}(y-x)}\tau\left(\theta+\vartheta-\frac{x+y}{2}\right) \\
 &\quad -\left(\frac{2}{y-x}\right)^{\alpha^{\frac{\beta}{k}+1}}\frac{\Gamma_k(\beta+k)\beta J_{k(\theta+\vartheta-\frac{x+y}{2})}^\alpha}{\alpha^{\frac{\beta}{k}-1}}\tau(\theta+\vartheta-y). \tag{2.23}
 \end{aligned}$$

Therefore, inequality (2.20) follows from (2.21)–(2.23). □

Remark 2.6 Lemma 2.5 leads to the conclusion that:

- (i) If we take $k = 1, x = \theta,$ and $y = \vartheta,$ then we can get Lemma 3.1 of [65].
- (ii) If we take $\alpha = k = 1, x = \theta,$ and $y = \vartheta,$ then Lemma 2.5 reduces to Lemma 1.1 of [68].

Lemma 2.7 *Let $\alpha, \beta > 0, \theta < \vartheta$ and $\tau : [\theta, \vartheta] \rightarrow R$ be a differentiable mapping such that $\tau' \in L[\theta, \vartheta].$ Then the identity*

$$\begin{aligned}
 &\frac{\tau(\theta+\vartheta-x)+\tau(\theta+\vartheta-y)}{2}-\frac{\alpha^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^\alpha} \\
 &\quad \times\left\{\beta J_{k(\theta+\vartheta-y)}^\alpha\tau(\theta+\vartheta-x)+\beta J_{k(\theta+\vartheta-x)}^\alpha\tau(\theta+\vartheta-y)\right\} \\
 &= \frac{(y-x)\alpha^{\frac{\beta}{k}}}{2}\int_0^1\left[\left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^{\frac{\beta}{k}}-\left(\frac{1-\lambda^\alpha}{\alpha}\right)^{\frac{\beta}{k}}\right] \\
 &\quad \times\tau'(\theta+\vartheta-(\lambda x+(1-\lambda)y))d\lambda \tag{2.24}
 \end{aligned}$$

holds for all $x, y \in [\theta, \vartheta].$

Proof Let

$$\begin{aligned}
 I &= \frac{(y-x)\alpha^{\frac{\beta}{k}}}{2} \int_0^1 \left[\left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} - \left(\frac{1-\lambda^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right] \tau'(\theta + \vartheta - (\lambda x + (1-\lambda)y)) \, d\lambda. \\
 &= \frac{(y-x)\alpha^{\frac{\beta}{k}}}{2} \{I_1 - I_2\}.
 \end{aligned} \tag{2.25}$$

Then we clearly see that

$$\begin{aligned}
 I_1 &= \int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \tau'(\theta + \vartheta - (\lambda x + (1-\lambda)y)) \, d\lambda \\
 &= \frac{1}{\alpha^{\frac{\beta}{k}}} \frac{\tau(\theta + \vartheta - x)}{y-x} - \frac{\frac{\beta}{k}}{y-x} \int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}-1} \\
 &\quad \times (1-\lambda)^{\alpha-1} \tau(\theta + \vartheta - (\lambda x + (1-\lambda)y)) \, d\lambda \\
 &= \frac{1}{\alpha^{\frac{\beta}{k}}} \frac{\tau(\theta + \vartheta - x)}{y-x} - \frac{\Gamma_k(\beta + k)}{(y-x)^{\alpha\frac{\beta}{k}+1}} \{ {}^\beta J_{(\theta+\vartheta-x)^+}^\alpha \tau(\theta + \vartheta - y) \}
 \end{aligned} \tag{2.26}$$

and

$$\begin{aligned}
 I_2 &= \int_0^1 \left(\frac{1-\lambda^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \tau'(\theta + \vartheta - (\lambda x + (1-\lambda)y)) \, d\lambda \\
 &= -\frac{1}{\alpha^{\frac{\beta}{k}}} \frac{\tau(\theta + \vartheta - y)}{y-x} + \frac{\frac{\beta}{k}}{y-x} \int_0^1 \left(\frac{1-\lambda^\alpha}{\alpha} \right)^{\frac{\beta}{k}-1} \lambda^{\alpha-1} \tau(\theta + \vartheta - (\lambda x + (1-\lambda)y)) \, d\lambda \\
 &= -\frac{1}{\alpha^{\frac{\beta}{k}}} \frac{\tau(\theta + \vartheta - y)}{y-x} + \frac{\Gamma_k(\beta + k)}{(y-x)^{\alpha\frac{\beta}{k}+1}} \{ {}^\beta J_{(\theta+\vartheta-y)^-}^\alpha \tau(\theta + \vartheta - x) \}.
 \end{aligned} \tag{2.27}$$

Therefore, identity (2.24) follows from (2.25)–(2.27). □

Corollary 2.8 *If we take $\alpha = \beta = k = 1$, then Lemma 2.7 leads to the equality*

$$\begin{aligned}
 &\frac{\tau(\theta + \vartheta - x) + \tau(\theta + \vartheta - y)}{2} - \frac{1}{y-x} \int_{\theta+\vartheta-y}^{\theta+\vartheta-x} \tau(\lambda) \, d\lambda \\
 &= \frac{y-x}{2} \int_0^1 (2\lambda - 1) \tau'(\theta + \vartheta - (\lambda x + (1-\lambda)y)) \, d\lambda.
 \end{aligned} \tag{2.28}$$

Remark 2.9 If we take $x = \theta$ and $y = \vartheta$ in Corollary 2.8, then equality (2.28) becomes the equality

$$\frac{\tau(\theta) + \tau(\vartheta)}{2} - \frac{1}{\vartheta - \theta} \int_\theta^\vartheta \tau(\lambda) \, d\lambda = \frac{\vartheta - \theta}{2} \int_0^1 (2\lambda - 1) \tau'((1-\lambda)\theta + \lambda\vartheta) \, d\lambda,$$

which was proved in Lemma 2.1 of [69].

Theorem 2.10 *Let $\alpha, \beta > 0, \theta < \vartheta$ and $\tau : [\theta, \vartheta] \rightarrow R$ be a differentiable mapping such that $\tau' \in L[\theta, \vartheta]$ and $|\tau'|$ is a convex mapping on $[\theta, \vartheta]$. Then the inequality*

$$\begin{aligned} & \left| \frac{2^{\alpha} \frac{\beta}{k} - 1}{(y-x)^{\alpha \frac{\beta}{k}}} \Gamma_k(\beta+k) \left\{ {}_k^{\beta} J_{(\theta+\vartheta-\frac{x+y}{2})^+}^{\alpha} \tau(\theta+\vartheta-x) + {}_k^{\beta} J_{(\theta+\vartheta-\frac{x+y}{2})^-}^{\alpha} \tau(\theta+\vartheta-y) \right\} \right. \\ & \quad \left. - \tau\left(\theta+\vartheta-\frac{x+y}{2}\right) \right| \\ & \leq \frac{y-x}{4} \alpha^{\frac{\beta}{k}} \left[(|\tau'(\theta)| + |\tau'(\vartheta)|) \left(\frac{1}{\alpha^{\frac{\beta}{k}+1}} B\left(\frac{\beta}{k}+1, \frac{1}{\alpha}\right) \right) \right. \\ & \quad \left. - \left\{ |\tau'(x)| \left(\frac{1}{2\alpha^{\frac{\beta}{k}+1}} \left(B\left(\frac{\beta}{k}+1, \frac{2}{\alpha}\right) + B\left(\frac{\beta}{k}+1, \frac{1}{\alpha}\right) \right) \right) \right\} \right. \\ & \quad \left. + \left\{ |\tau'(y)| \left(\frac{1}{2\alpha^{\frac{\beta}{k}+1}} \left(B\left(\frac{\beta}{k}+1, \frac{2}{\alpha}\right) - B\left(\frac{\beta}{k}+1, \frac{1}{\alpha}\right) \right) \right) \right\} \right. \\ & \quad \left. + (|\tau'(\theta)| + |\tau'(\vartheta)|) \left(\frac{1}{\alpha^{\frac{\beta}{k}+1}} B\left(\frac{\beta}{k}+1, \frac{1}{\alpha}\right) \right) \right. \\ & \quad \left. - \left\{ |\tau'(x)| \left(\frac{1}{2\alpha^{\frac{\beta}{k}+1}} \left(B\left(\frac{\beta}{k}+1, \frac{1}{\alpha}\right) - B\left(\frac{\beta}{k}+1, \frac{2}{\alpha}\right) \right) \right) \right\} \right. \\ & \quad \left. + \left\{ |\tau'(y)| \left(\frac{1}{2\alpha^{\frac{\beta}{k}+1}} \left(B\left(\frac{\beta}{k}+1, \frac{2}{\alpha}\right) + B\left(\frac{\beta}{k}+1, \frac{1}{\alpha}\right) \right) \right) \right\} \right], \tag{2.29} \end{aligned}$$

holds for all $x, y \in [\theta, \vartheta]$.

Proof It follows from Lemma 2.5 and Jensen–Mercer inequality using the convexity of $|\tau'|$ that

$$\begin{aligned} & \left| \frac{2^{\alpha} \frac{\beta}{k} - 1}{(y-x)^{\alpha \frac{\beta}{k}}} \Gamma_k(\beta+k) \left\{ {}_k^{\beta} J_{(\theta+\vartheta-\frac{x+y}{2})^+}^{\alpha} \tau(\theta+\vartheta-x) + {}_k^{\beta} J_{(\theta+\vartheta-\frac{x+y}{2})^-}^{\alpha} \tau(\theta+\vartheta-y) \right\} \right. \\ & \quad \left. - \tau\left(\theta+\vartheta-\frac{x+y}{2}\right) \right| \\ & \leq \frac{y-x}{4} \alpha^{\frac{\beta}{k}} \left\{ \int_0^1 \left(\frac{1-(1-\lambda)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}} \left| \tau'\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right) \right| d\lambda \right. \\ & \quad \left. + \int_0^1 \left(\frac{1-(1-\lambda)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}} \left| \tau'\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right) \right| d\lambda \right\} \\ & \leq \frac{y-x}{4} \alpha^{\frac{\beta}{k}} \left\{ \int_0^1 \left(\frac{1-(1-\lambda)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}} \left\{ |\tau'(\theta)| + |\tau'(\vartheta)| - \left(\frac{2-\lambda}{2} |\tau'(x)| + \frac{\lambda}{2} |\tau'(y)| \right) \right\} d\lambda \right. \\ & \quad \left. + \int_0^1 \left(\frac{1-(1-\lambda)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}} \left\{ |\tau'(\theta)| + |\tau'(\vartheta)| - \left(\frac{\lambda}{2} |\tau'(x)| + \frac{2-\lambda}{2} |\tau'(y)| \right) \right\} d\lambda \right\} \\ & \leq \frac{y-x}{4} \alpha^{\frac{\beta}{k}} \times \left\{ (|\tau'(\theta)| + |\tau'(\vartheta)|) \int_0^1 \left(\frac{1-(1-\lambda)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}} d\lambda \right. \\ & \quad \left. - \left(|\tau'(x)| \int_0^1 \left(\frac{1-(1-\lambda)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}} \frac{2-\lambda}{2} d\lambda + |\tau'(y)| \int_0^1 \left(\frac{1-(1-\lambda)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}} \frac{\lambda}{2} d\lambda \right) \right\} \end{aligned}$$

$$\begin{aligned}
 &+ \left\{ (|\tau'(\theta)| + |\tau'(\vartheta)|) \int_0^1 \left(\frac{1 - (1 - \lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\lambda \right. \\
 &\left. - \left(|\tau'(x)| \int_0^1 \left(\frac{1 - (1 - \lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \lambda \frac{d\lambda}{2} + |\tau'(y)| \int_0^1 \left(\frac{1 - (1 - \lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \frac{2 - \lambda}{2} d\lambda \right) \right\}.
 \end{aligned}$$

Therefore, inequality (2.29) can be derived after some simple calculations. □

Remark 2.11 From Theorem 2.10 we clearly see that:

- (i) If we take $k = 1$, $x = \theta$, and $y = \vartheta$ in Theorem 2.10, then we get Theorem 3.1 of [65].
- (ii) If we take $\alpha = k = 1$, $x = \theta$, and $y = \vartheta$ in Theorem 2.10, then we obtain Theorem 5 of [68] in the case of $q = 1$.

Theorem 2.12 Let $q > 1$, $\alpha, \beta > 0$, $\theta < \vartheta$ and $\tau : [\theta, \vartheta] \rightarrow R$ be a differentiable mapping such that $\tau' \in L[\theta, \vartheta]$ and $|\tau'|^q$ is a convex mapping on $[\theta, \vartheta]$. Then the inequality

$$\begin{aligned}
 &\left| \frac{2^{\alpha \frac{\beta}{k} - 1} \alpha^{\frac{\beta}{k}} \Gamma_k(\beta + k)}{(y - x)^{\alpha \frac{\beta}{k}}} \left\{ {}_k^{\beta} J_{(\theta + \vartheta - \frac{x+y}{2})^+}^{\alpha} \tau(\theta + \vartheta - x) + {}_k^{\beta} J_{(\theta + \vartheta - \frac{x+y}{2})^-}^{\alpha} \tau(\theta + \vartheta - y) \right\} \right. \\
 &\quad \left. - \tau \left(\theta + \vartheta - \frac{x + y}{2} \right) \right| \\
 &\leq \frac{y - x}{4} \alpha^{\frac{\beta}{k}} \left(\frac{1}{\alpha^{\frac{\beta}{k} + 1}} B \left(\frac{\beta}{k} + 1, \frac{1}{\alpha} \right) \right)^{1 - \frac{1}{q}} \\
 &\quad \times \left\{ (|\tau'(\theta)|^q + |\tau'(\vartheta)|^q) \left(\left(\frac{1}{\alpha^{\frac{\beta}{k} + 1}} B \left(\frac{\beta}{k} + 1, \frac{1}{\alpha} \right) \right) \right) \right. \\
 &\quad \left. - \left\{ |\tau'(x)|^q \left(\frac{1}{2\alpha^{\frac{\beta}{k} + 1}} \left(B \left(\frac{\beta}{k} + 1, \frac{1}{\alpha} \right) - B \left(\frac{\beta}{k} + 1, \frac{2}{\alpha} \right) \right) \right) \right. \right. \\
 &\quad \left. \left. + |\tau'(y)|^q \left(\frac{1}{2\alpha^{\frac{\beta}{k} + 1}} \left(B \left(\frac{\beta}{k} + 1, \frac{2}{\alpha} \right) + B \left(\frac{\beta}{k} + 1, \frac{1}{\alpha} \right) \right) \right) \right\} \right. \\
 &\quad \left. + (|\tau'(\theta)|^q + |\tau'(\vartheta)|^q) \left(\left(\frac{1}{\alpha^{\frac{\beta}{k} + 1}} B \left(\frac{\beta}{k} + 1, \frac{1}{\alpha} \right) \right) \right) \right. \\
 &\quad \left. - \left\{ |\tau'(x)|^q \left(\frac{1}{2\alpha^{\frac{\beta}{k} + 1}} \left(B \left(\frac{\beta}{k} + 1, \frac{2}{\alpha} \right) - B \left(\frac{\beta}{k} + 1, \frac{1}{\alpha} \right) \right) \right) \right. \right. \\
 &\quad \left. \left. + |\tau'(y)|^q \left(\frac{1}{2\alpha^{\frac{\beta}{k} + 1}} \left(B \left(\frac{\beta}{k} + 1, \frac{1}{\alpha} \right) + B \left(\frac{\beta}{k} + 1, \frac{2}{\alpha} \right) \right) \right) \right\} \right\}^{\frac{1}{q}} \tag{2.30}
 \end{aligned}$$

holds for all $x, y \in [\theta, \vartheta]$.

Proof It follows from Lemma 2.5, Jensen–Mercer inequality, power-mean inequality, and the convexity of function $|\tau'|^q$ that

$$\begin{aligned}
 &\left| \frac{2^{\alpha \frac{\beta}{k} - 1} \alpha^{\frac{\beta}{k}} \Gamma_k(\beta + k)}{(y - x)^{\alpha \frac{\beta}{k}}} \left\{ {}_k^{\beta} J_{(\theta + \vartheta - \frac{x+y}{2})^+}^{\alpha} \tau(\theta + \vartheta - x) + {}_k^{\beta} J_{(\theta + \vartheta - \frac{x+y}{2})^-}^{\alpha} \tau(\theta + \vartheta - y) \right\} \right. \\
 &\quad \left. - \tau \left(\theta + \vartheta - \frac{x + y}{2} \right) \right|
 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{y-x}{4} \alpha^{\frac{\beta}{k}} \left\{ \left(\int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\lambda \right)^{1-\frac{1}{q}} \right. \\
 &\quad \times \left(\int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \left| \tau' \left(\theta + \vartheta - \left(\frac{\lambda}{2}x + \frac{2-\lambda}{2}y \right) \right) \right|^q d\lambda \right)^{\frac{1}{q}} \\
 &\quad + \left(\int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\lambda \right)^{1-\frac{1}{q}} \\
 &\quad \times \left. \left(\int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \left| \tau' \left(\theta + \vartheta - \left(\frac{2-\lambda}{2}x + \frac{\lambda}{2}y \right) \right) \right|^q d\lambda \right)^{\frac{1}{q}} \right\} \\
 &\leq \frac{y-x}{4} \alpha^{\frac{\beta}{k}} \left\{ \left(\int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\lambda \right)^{1-\frac{1}{q}} \left(\int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right. \right. \\
 &\quad \times \left. \left. \left(|\tau'(\theta)|^q + |\tau'(\vartheta)|^q - \left(\frac{\lambda}{2}|\tau'(x)|^q + \frac{2-\lambda}{2}|\tau'(y)|^q \right) \right) d\lambda \right)^{\frac{1}{q}} \right. \\
 &\quad + \left(\int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\lambda \right)^{1-\frac{1}{q}} \left(\int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right. \\
 &\quad \times \left. \left. \left(|\tau'(\theta)|^q + |\tau'(\vartheta)|^q - \left(\frac{2-\lambda}{2}|\tau'(x)|^q + \frac{\lambda}{2}|\tau'(y)|^q \right) \right) d\lambda \right)^{\frac{1}{q}} \right\}. \tag{2.31}
 \end{aligned}$$

Making simple simplifications, we get (2.30) from (2.31). □

Remark 2.13 Theorem 2.12 leads to the conclusion that:

- (i) If we take $k = 1$, $x = \theta$, and $y = \vartheta$ in Theorem 2.12, then we get Theorem 3.2 of [65].
- (ii) Let $\alpha = k = 1$, $x = \theta$, and $y = \vartheta$, then Theorem 2.12 reduces to Theorem 5 of [68].

Theorem 2.14 *Let $\alpha, \beta > 0$, $p, q > 1$ with $1/p + 1/q = 1$, $\theta < \vartheta$ and $\tau : [\theta, \vartheta] \rightarrow R$ be a differentiable mapping such that $\tau' \in L[\theta, \vartheta]$ and $|\tau'|^q$ is a convex mapping on $[\theta, \vartheta]$. Then one has*

$$\begin{aligned}
 &\left| \frac{2^\alpha \beta^{-1} \alpha^{\frac{\beta}{k}} \Gamma_k(\beta + k)}{(y-x)^{\alpha \frac{\beta}{k}}} \left\{ {}^\beta J_{(\theta+\vartheta-\frac{x+y}{2})^+}^\alpha \tau(\theta + \vartheta - x) + {}^\beta J_{(\theta+\vartheta-\frac{x+y}{2})^-}^\alpha \tau(\theta + \vartheta - y) \right\} \right. \\
 &\quad \left. - \tau \left(\theta + \vartheta - \frac{x+y}{2} \right) \right| \\
 &\leq \frac{y-x}{4} \alpha^{\frac{\beta}{k}} \left(\frac{1}{\alpha^{\frac{\beta}{k} p + 1}} B \left(\frac{\beta}{k} p + 1, \frac{1}{\alpha} \right) \right)^{\frac{1}{p}} \\
 &\quad \times \left\{ \left(|\tau'(\theta)|^q + |\tau'(\vartheta)|^q - \left(\frac{3|\tau'(x)|^q + |\tau'(y)|^q}{4} \right) \right)^{\frac{1}{q}} \right. \\
 &\quad \left. + \left(|\tau'(\theta)|^q + |\tau'(\vartheta)|^q - \left(\frac{|\tau'(x)|^q + 3|\tau'(y)|^q}{4} \right) \right)^{\frac{1}{q}} \right\} \tag{2.32}
 \end{aligned}$$

for all $x, y \in [\theta, \vartheta]$.

Proof By using Lemma 2.5, and the Jensen–Mercer and Hölder integral inequalities, we obtain

$$\begin{aligned} & \left| \frac{2^{\alpha \frac{\beta}{k}-1} \alpha^{\frac{\beta}{k}} \Gamma_k(\beta+k)}{(y-x)^{\alpha \frac{\beta}{k}}} \left\{ {}^{\beta} J_{(\theta+\vartheta-\frac{x+y}{2})^+}^{\alpha} \tau(\theta+\vartheta-x) + {}^{\beta} J_{(\theta+\vartheta-\frac{x+y}{2})^-}^{\alpha} \tau(\theta+\vartheta-y) \right\} \right. \\ & \quad \left. - \tau\left(\theta+\vartheta-\frac{x+y}{2}\right) \right| \\ & \leq \frac{y-x}{4} \alpha^{\frac{\beta}{k}} \left(\int_0^1 \left(\frac{1-(1-\lambda)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k} p} d\lambda \right)^{\frac{1}{p}} \\ & \quad \times \left\{ \left(\int_0^1 \left| \tau'\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right) \right|^q d\lambda \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_0^1 \left| \tau'\left(\theta+\vartheta-\left(\frac{\lambda}{2}x+\frac{2-\lambda}{2}y\right)\right) \right|^q d\lambda \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

It follows from the convexity of $|\tau'|^q$ that

$$\begin{aligned} & \left| \tau'\left(\theta+\vartheta-\left(\frac{2-\lambda}{2}x+\frac{\lambda}{2}y\right)\right) \right|^q \\ & \leq |\tau'(\theta)|^q + |\tau'(\vartheta)|^q - \left(\frac{2-\lambda}{2} |\tau'(x)|^q + \frac{\lambda}{2} |\tau'(y)|^q \right) \\ & \leq \frac{y-x}{4} \alpha^{\frac{\beta}{k}} \left(\int_0^1 \left(\frac{1-(1-\lambda)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k} p} d\lambda \right)^{\frac{1}{p}} \\ & \quad \times \left\{ \left(\int_0^1 \left(|\tau'(\theta)|^q + |\tau'(\vartheta)|^q - \left(\frac{2-\lambda}{2} |\tau'(x)|^q + \frac{\lambda}{2} |\tau'(y)|^q \right) \right) d\lambda \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_0^1 \left(|\tau'(\theta)|^q + |\tau'(\vartheta)|^q - \left(\frac{\lambda}{2} |\tau'(x)|^q + \frac{2-\lambda}{2} |\tau'(y)|^q \right) \right) d\lambda \right)^{\frac{1}{q}} \right\} \\ & \leq \frac{y-x}{4} \alpha^{\frac{\beta}{k}} \left(\frac{1}{\alpha^{\frac{\beta}{k} p+1}} B\left(\frac{\beta}{k} p+1, \frac{1}{\alpha}\right) \right)^{\frac{1}{p}} \\ & \quad \times \left\{ \left(|\tau'(\theta)|^q + |\tau'(\vartheta)|^q - \left(\frac{3|\tau'(x)|^q + |\tau'(y)|^q}{4} \right) \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(|\tau'(\theta)|^q + |\tau'(\vartheta)|^q - \left(\frac{|\tau'(x)|^q + 3|\tau'(y)|^q}{4} \right) \right)^{\frac{1}{q}} \right\}, \end{aligned}$$

which completes the proof. □

Corollary 2.15 *Let $\alpha = k = 1$. Then Theorem 2.14 leads to*

$$\begin{aligned} & \left| \frac{1}{y-x} \int_{\theta+\vartheta-y}^{\theta+\vartheta-x} \tau(\lambda) d\lambda - \tau\left(\theta+\vartheta-\frac{x+y}{2}\right) \right| \\ & \leq \frac{1}{2^{\frac{1}{p}}} \times \left\{ \left(|\tau'(\theta)|^q + |\tau'(\vartheta)|^q - \left(\frac{3|\tau'(x)|^q + |\tau'(y)|^q}{4} \right) \right)^{\frac{1}{q}} \right. \end{aligned}$$

$$+ \left(|\tau'(\theta)|^q + |\tau'(\vartheta)|^q - \left(\frac{|\tau'(x)|^q + 3|\tau'(y)|^q}{4} \right) \right)^{\frac{1}{q}} \Big\}.$$

Theorem 2.16 *Let $\alpha, \beta > 0, p, q > 1$ with $1/p + 1/q = 1, \theta < \vartheta$ and $\tau : [\theta, \vartheta] \rightarrow R$ be a differentiable mapping such that $\tau' \in L[\theta, \vartheta]$ and $|\tau'|^q$ is a convex mapping on $[\theta, \vartheta]$. Then the inequality*

$$\begin{aligned} & \left| \frac{2^{\alpha \frac{\beta}{k} - 1} \alpha^{\frac{\beta}{k}} \Gamma_k(\beta + k)}{(y-x)^{\alpha \frac{\beta}{k}}} \left\{ {}_k J_{(\theta+\vartheta-\frac{x+y}{2})^+}^{\beta} \tau(\theta + \vartheta - x) + {}_k J_{(\theta+\vartheta-\frac{x+y}{2})^-}^{\beta} \tau(\theta + \vartheta - y) \right\} \right. \\ & \quad \left. - \tau\left(\theta + \vartheta - \frac{x+y}{2}\right) \right| \\ & \leq \frac{(y-x)\alpha^{\frac{\beta}{k}}}{4} \left\{ (|\tau'(\theta)|^q + |\tau'(\vartheta)|^q) \left(\frac{B(\frac{\beta}{k} + 1, \frac{1}{\alpha})}{\alpha^{\frac{\beta}{k} + 1}} \right) \right. \\ & \quad - \left(\left(\frac{B(\frac{\beta}{k} + 1, \frac{2}{\alpha}) + B(\frac{\beta}{k} + 1, \frac{1}{\alpha})}{2\alpha^{\frac{\beta}{k} + 1}} \right) |\tau'(x)|^q \right. \\ & \quad \left. + \left(\frac{B(\frac{\beta}{k} + 1, \frac{1}{\alpha}) - B(\frac{\beta}{k} + 1, \frac{2}{\alpha})}{2\alpha^{\frac{\beta}{k} + 1}} \right) |\tau'(y)|^q \right)^{\frac{1}{q}} \\ & \quad + (|\tau'(\theta)|^q + |\tau'(\vartheta)|^q) \left(\frac{B(\frac{\beta}{k} + 1, \frac{1}{\alpha})}{\alpha^{\frac{\beta}{k} + 1}} \right) - \left(\left(\frac{B(\frac{\beta}{k} + 1, \frac{1}{\alpha}) - B(\frac{\beta}{k} + 1, \frac{2}{\alpha})}{2\alpha^{\frac{\beta}{k} + 1}} \right) |\tau'(x)|^q \right. \\ & \quad \left. + \left(\frac{B(\frac{\beta}{k} + 1, \frac{2}{\alpha}) + B(\frac{\beta}{k} + 1, \frac{1}{\alpha})}{2\alpha^{\frac{\beta}{k} + 1}} \right) |\tau'(y)|^q \right)^{\frac{1}{q}} \Big\} \tag{2.33} \end{aligned}$$

holds for all $x, y \in [\theta, \vartheta]$.

Proof It follows from Lemma 2.5, Jensen–Mercer inequality, convexity of $|\tau'|^q$, and Hölder integral inequality that

$$\begin{aligned} & \left| \frac{2^{\alpha \frac{\beta}{k} - 1} \alpha^{\frac{\beta}{k}} \Gamma_k(\beta + k)}{(y-x)^{\alpha \frac{\beta}{k}}} \left\{ {}_k J_{(\theta+\vartheta-\frac{x+y}{2})^+}^{\beta} \tau(\theta + \vartheta - x) + {}_k J_{(\theta+\vartheta-\frac{x+y}{2})^-}^{\beta} \tau(\theta + \vartheta - y) \right\} \right. \\ & \quad \left. - \tau\left(a + b - \frac{x+y}{2}\right) \right| \\ & \leq \frac{(y-x)\alpha^{\frac{\beta}{k}}}{4} \left\{ \left(\int_0^1 1 d\lambda \right)^{\frac{1}{p}} \right. \\ & \quad \times \left(\int_0^1 \left(\frac{1 - (1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \left| \tau'\left(\theta + \vartheta - \left(\frac{2-\lambda}{2}x + \frac{\lambda}{2}y\right)\right) \right|^q d\lambda \right)^{\frac{1}{q}} \\ & \quad + \left(\int_0^1 1 d\lambda \right)^{\frac{1}{p}} \times \left(\int_0^1 \left(\frac{1 - (1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \left| \tau'\left(\theta + \vartheta - \left(\frac{\lambda}{2}x + \frac{2-\lambda}{2}y\right)\right) \right|^q d\lambda \right)^{\frac{1}{q}} \Big\} \\ & \leq \frac{(y-x)\alpha^{\frac{\beta}{k}}}{4} \left\{ \left(\int_0^1 \left(\frac{1 - (1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right. \right. \end{aligned}$$

$$\begin{aligned} & \times \left[|\tau'(\theta)|^q + |\tau'(\vartheta)|^q - \left(\frac{2-\lambda}{2} |\tau'(x)|^q + \frac{\lambda}{2} |\tau'(y)|^q \right) \right]^{\frac{1}{q}} \\ & + \left(\int_0^1 \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \left[|\tau'(\theta)|^q + |\tau'(\vartheta)|^q - \left(\frac{\lambda}{2} |\tau'(x)|^q + \frac{2-\lambda}{2} |\tau'(y)|^q \right) \right]^{\frac{1}{q}} \right)^{\frac{1}{q}}. \end{aligned}$$

By making necessary changes, we get (2.33). □

Theorem 2.17 *Let $\theta < \vartheta$ and $\tau : [\theta, \vartheta] \rightarrow R$ be a differentiable mapping such that $\tau' \in L[\theta, \vartheta]$ and $|\tau'|$ is a convex mapping on $[\theta, \vartheta]$. Then one has*

$$\begin{aligned} & \left| \frac{\tau(\theta + \vartheta - x) + \tau(\theta + \vartheta - y)}{2} - \frac{\alpha^{\frac{\beta}{k}} \Gamma_k(\beta + k)}{2(y-x)^{\alpha \frac{\beta}{k}}} \right. \\ & \quad \left. \times \left\{ {}_k^{\beta} J_{(\theta+\vartheta-y)^+}^{\alpha} \tau(\theta + \vartheta - x) + {}_k^{\beta} J_{(\theta+\vartheta-x)^-}^{\alpha} \tau(\theta + \vartheta - y) \right\} \right| \\ & \leq \frac{(y-x)\alpha^{\frac{\beta}{k}}}{2} \left[\left\{ (|\tau'(\theta)| + |\tau'(\vartheta)|) \frac{B(\frac{1}{\alpha}, \frac{\beta}{k} + 1)}{\alpha^{\frac{\beta}{k}+1}} \right. \right. \\ & \quad - \frac{|\tau'(x)|}{\alpha^{\frac{\beta}{k}+1}} \left\{ B_{\frac{1}{2\alpha}} \left(\frac{1}{\alpha}, \frac{\beta}{k} + 1 \right) + B \left(\frac{2}{\alpha}, \frac{\beta}{k} + 1 \right) - B \left(\frac{1}{\alpha}, \frac{\beta}{k} + 1 \right) \right\} \right. \\ & \quad - \frac{|\tau'(y)|}{\alpha^{\frac{\beta}{k}+1}} \left\{ B_{\frac{1}{2\alpha}} \left(\frac{1}{\alpha}, \frac{\beta}{k} + 1 \right) - B \left(\frac{2}{\alpha}, \frac{\beta}{k} + 1 \right) \right\} \left. \right\} \\ & \quad + \left\{ (|\tau'(\theta)| + |\tau'(\vartheta)|) \frac{B(\frac{1}{\alpha}, \frac{\beta}{k} + 1)}{\alpha^{\frac{\beta}{k}+1}} \right. \\ & \quad - \frac{|\tau'(x)|}{\alpha^{\frac{\beta}{k}+1}} \left\{ B_{\frac{1}{2\alpha}} \left(\frac{1}{\alpha}, \frac{\beta}{k} + 1 \right) - B \left(\frac{2}{\alpha}, \frac{\beta}{k} + 1 \right) \right\} \right. \\ & \quad \left. \left. - \frac{|\tau'(y)|}{\alpha^{\frac{\beta}{k}+1}} \left\{ B_{\frac{1}{2\alpha}} \left(\frac{1}{\alpha}, \frac{\beta}{k} + 1 \right) + B \left(\frac{2}{\alpha}, \frac{\beta}{k} + 1 \right) - B \left(\frac{1}{\alpha}, \frac{\beta}{k} + 1 \right) \right\} \right\} \right] \tag{2.34} \end{aligned}$$

for all $x, y \in [\theta, \vartheta]$.

Proof By using Lemma 2.7 and similar arguments as in the the proofs the previous theorems, we get

$$\begin{aligned} & \left| \frac{\tau(\theta + \vartheta - x) + \tau(\theta + \vartheta - y)}{2} - \frac{\alpha^{\frac{\beta}{k}} \Gamma_k(\beta + k)}{2(y-x)^{\alpha \frac{\beta}{k}}} \right. \\ & \quad \left. \times \left\{ {}_k^{\beta} J_{(\theta+\vartheta-y)^+}^{\alpha} \tau(\theta + \vartheta - x) + {}_k^{\beta} J_{(\theta+\vartheta-x)^-}^{\alpha} \tau(\theta + \vartheta - y) \right\} \right| \\ & \leq \frac{(y-x)\alpha^{\frac{\beta}{k}}}{2} \int_0^1 \left| \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} - \left(\frac{1-\lambda^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right| |\tau'(\theta + \vartheta - (\lambda x + (1-\lambda)y))| d\lambda \\ & \leq \frac{(y-x)\alpha^{\frac{\beta}{k}}}{2} \int_0^1 \left| \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} - \left(\frac{1-\lambda^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right| \\ & \quad \times \{ |\tau'(\theta)| + |\tau'(\vartheta)| - (\lambda |\tau'(x)| + (1-\lambda) |\tau'(y)|) \} d\lambda \\ & \leq \frac{(y-x)\alpha^{\frac{\beta}{k}}}{2} \left[\int_0^{\frac{1}{2}} \left[\left(\frac{1-\lambda^\alpha}{\alpha} \right)^{\frac{\beta}{k}} - \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right] \right. \end{aligned}$$

$$\begin{aligned} & \times \left\{ |\tau'(\theta)| + |\tau'(\vartheta)| - (\lambda|\tau'(x)| + (1-\lambda)|\tau'(y)|) \right\} d\lambda \\ & + \int_{\frac{1}{2}}^1 \left[\left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} - \left(\frac{1-\lambda^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right] \\ & \times \left\{ |\tau'(\theta)| + |\tau'(\vartheta)| - (\lambda|\tau'(x)| + (1-\lambda)|\tau'(y)|) \right\} d\lambda \Big]. \end{aligned}$$

This completes the proof. □

3 New inequalities via improved Hölder inequality

Theorem 3.1 *Let $\alpha, \beta > 0, p, q > 1$ with $1/p + 1/q = 1, \theta < \vartheta$ and $\tau : [\theta, \vartheta] \rightarrow R$ be a differentiable mapping such that $\tau' \in L[\theta, \vartheta]$ and $|\tau'|^q$ is a convex mapping on $[\theta, \vartheta]$. Then one has*

$$\begin{aligned} & \left| \frac{2^{\alpha \frac{\beta}{k}-1} \alpha^{\frac{\beta}{k}} \Gamma_k(\beta + k)}{(y-x)^{\alpha \frac{\beta}{k}}} \left\{ {}_k^{\beta} J_{(\theta+\vartheta-\frac{x+y}{2})^+}^{\alpha} \tau(\theta + \vartheta - x) + {}_k^{\beta} J_{(\theta+\vartheta-\frac{x+y}{2})^-}^{\alpha} \tau(\theta + \vartheta - y) \right\} \right. \\ & \quad \left. - \tau\left(\theta + \vartheta - \frac{x+y}{2}\right) \right| \\ & \leq \frac{(y-x)\alpha^{\frac{\beta}{k}}}{2} \left[\left\{ \left(\frac{B(\frac{2}{\alpha}, \frac{\beta}{k}p + 1)}{\alpha^{\frac{\beta}{k}p+1}} \right)^{\frac{1}{p}} \right. \right. \\ & \quad \times \left(\frac{|\tau'(\theta)|^q + |\tau'(\vartheta)|^q}{2} - \left(\frac{5}{12}|\tau'(x)|^q + \frac{1}{12}|\tau'(y)|^q \right) \right)^{\frac{1}{q}} \\ & \quad + \left(\frac{B(\frac{1}{\alpha}, \frac{\beta}{k}p + 1) - B(\frac{2}{\alpha}, \frac{\beta}{k}p + 1)}{\alpha^{\frac{\beta}{k}p+1}} \right)^{\frac{1}{p}} \\ & \quad \times \left. \left(\frac{|\tau'(\theta)|^q + |\tau'(\vartheta)|^q}{2} - \left(\frac{1}{3}|\tau'(x)|^q + \frac{1}{6}|\tau'(y)|^q \right) \right)^{\frac{1}{q}} \right\} \\ & \quad + \left\{ \left(\frac{B(\frac{2}{\alpha}, \frac{\beta}{k}p + 1)}{\alpha^{\frac{\beta}{k}p+1}} \right)^{\frac{1}{p}} \left(\frac{|\tau'(\theta)|^q + |\tau'(\vartheta)|^q}{2} - \left(\frac{1}{12}|\tau'(x)|^q + \frac{5}{12}|\tau'(y)|^q \right) \right) \right)^{\frac{1}{q}} \\ & \quad + \left(\frac{B(\frac{1}{\alpha}, \frac{\beta}{k}p + 1) - B(\frac{2}{\alpha}, \frac{\beta}{k}p + 1)}{\alpha^{\frac{\beta}{k}p+1}} \right)^{\frac{1}{p}} \\ & \quad \times \left. \left(\frac{|\tau'(\theta)|^q + |\tau'(\vartheta)|^q}{2} - \left(\frac{1}{6}|\tau'(x)|^q + \frac{1}{3}|\tau'(y)|^q \right) \right)^{\frac{1}{q}} \right\} \Big] \tag{3.1} \end{aligned}$$

for all $x, y \in [\theta, \vartheta]$.

Proof It follows from Lemma 2.5, Jensen–Mercer inequality, the convexity of $|\tau'|^q$, and Hölder–İşcan integral inequality given in Theorem 1.4 of [70] that

$$\begin{aligned} & \left| \frac{2^{\alpha \frac{\beta}{k}-1} \alpha^{\frac{\beta}{k}} \Gamma_k(\beta + k)}{(y-x)^{\alpha \frac{\beta}{k}}} \left\{ {}_k^{\beta} J_{(\theta+\vartheta-\frac{x+y}{2})^+}^{\alpha} \tau(\theta + \vartheta - x) + {}_k^{\beta} J_{(\theta+\vartheta-\frac{x+y}{2})^-}^{\alpha} \tau(\theta + \vartheta - y) \right\} \right. \\ & \quad \left. - \tau\left(a + b - \frac{x+y}{2}\right) \right| \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{(y-x)\alpha^{\frac{\beta}{k}}}{4} \left[\left\{ \left(\int_0^1 (1-\lambda) \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{kp}} d\lambda \right)^{\frac{1}{p}} \right. \right. \\
 &\quad \times \left(\int_0^1 (1-\lambda) \left| \tau' \left(\theta + \vartheta - \left(\frac{2-\lambda}{2}x + \frac{\lambda}{2}y \right) \right) \right|^q d\lambda \right)^{\frac{1}{q}} \\
 &\quad \times \left(\int_0^1 \lambda \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{kp}} d\lambda \right)^{\frac{1}{p}} \left(\int_0^1 \lambda \left| \tau' \left(\theta + \vartheta - \left(\frac{2-\lambda}{2}x + \frac{\lambda}{2}y \right) \right) \right|^q d\lambda \right)^{\frac{1}{q}} \Big\} \\
 &\quad + \left\{ \left(\int_0^1 (1-\lambda) \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{kp}} d\lambda \right)^{\frac{1}{p}} \right. \\
 &\quad \times \left(\int_0^1 (1-\lambda) \left| \tau' \left(\theta + \vartheta - \left(\frac{\lambda}{2}x + \frac{2-\lambda}{2}y \right) \right) \right|^q d\lambda \right)^{\frac{1}{q}} \\
 &\quad \times \left(\int_0^1 \lambda \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{kp}} d\lambda \right)^{\frac{1}{p}} \\
 &\quad \times \left. \left(\int_0^1 \lambda \left| \tau' \left(\theta + \vartheta - \left(\frac{\lambda}{2}x + \frac{2-\lambda}{2}y \right) \right) \right|^q d\lambda \right)^{\frac{1}{q}} \right\} \Big] \\
 &\leq \frac{(y-x)\alpha^{\frac{\beta}{k}}}{4} \left[\left\{ \left(\int_0^1 (1-\lambda) \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{kp}} d\lambda \right)^{\frac{1}{p}} \right. \right. \\
 &\quad \times \left(\int_0^1 (1-\lambda) \left[|\tau'(\theta)|^q + |\tau'(\vartheta)|^q - \left(\frac{2-\lambda}{2}|\tau'(x)|^q + \frac{\lambda}{2}|\tau'(y)|^q \right) \right] d\lambda \right)^{\frac{1}{q}} \\
 &\quad \times \left(\int_0^1 \lambda \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{kp}} d\lambda \right)^{\frac{1}{p}} \\
 &\quad \times \left. \left(\int_0^1 \lambda \left[|\tau'(\theta)|^q + |\tau'(\vartheta)|^q - \left(\frac{2-\lambda}{2}|\tau'(x)|^q + \frac{\lambda}{2}|\tau'(y)|^q \right) \right] d\lambda \right)^{\frac{1}{q}} \right\} \\
 &\quad + \left\{ \left(\int_0^1 (1-\lambda) \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{kp}} d\lambda \right)^{\frac{1}{p}} \right. \\
 &\quad \times \left(\int_0^1 (1-\lambda) \left[|\tau'(\theta)|^q + |\tau'(\vartheta)|^q - \left(\frac{\lambda}{2}|\tau'(x)|^q + \frac{2-\lambda}{2}|\tau'(y)|^q \right) \right] d\lambda \right)^{\frac{1}{q}} \\
 &\quad \times \left(\int_0^1 \lambda \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{kp}} d\lambda \right)^{\frac{1}{p}} \\
 &\quad \times \left. \left(\int_0^1 \lambda \left[|\tau'(\theta)|^q + |\tau'(\vartheta)|^q - \left(\frac{\lambda}{2}|\tau'(x)|^q + \frac{2-\lambda}{2}|\tau'(y)|^q \right) \right] d\lambda \right)^{\frac{1}{q}} \right\} \Big].
 \end{aligned}$$

By making use of some computations, one can get the required result. □

Theorem 3.2 *Let $\alpha, \beta > 0, p, q > 1$ with $1/p + 1/q = 1, \theta < \vartheta$ and $\tau : [\theta, \vartheta] \rightarrow R$ be a differentiable mapping such that $\tau' \in L[\theta, \vartheta]$ and $|\tau'|^q$ is a convex mapping on $[\theta, \vartheta]$. Then the inequality*

$$\begin{aligned}
 &\left| \frac{2^\alpha \frac{\beta}{k} - 1 \alpha^{\frac{\beta}{k}} \Gamma_k(\beta + k)}{(y-x)\alpha^{\frac{\beta}{k}}} \left\{ k^{\beta} J_{(\theta+\vartheta-\frac{x+y}{2})^+}^{\alpha} \tau(\theta + \vartheta - x) + k^{\beta} J_{(\theta+\vartheta-\frac{x+y}{2})^-}^{\alpha} \tau(\theta + \vartheta - y) \right\} \right. \\
 &\quad \left. - \tau \left(\theta + \vartheta - \frac{x+y}{2} \right) \right|
 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{(y-x)\alpha^{\frac{\beta}{k}}}{4} \left[\left\{ \left(\frac{B(\frac{2}{\alpha}, \frac{\beta}{k} + 1)}{\alpha^{\frac{\beta}{k+1}}} \right)^{1-\frac{1}{q}} \right. \right. \\
 &\quad \times \left((|\tau'(\theta)|^q + |\tau'(\vartheta)|^q) \left(\frac{B(\frac{2}{\alpha}, \frac{\beta}{k} + 1)}{\alpha^{\frac{\beta}{k+1}}} \right) \right. \\
 &\quad - \left. \left(\frac{1}{2\alpha^{\frac{\beta}{k+1}}} |\tau'(x)|^q \left(B\left(\frac{2}{\alpha}, \frac{\beta}{k} + 1\right) + B\left(\frac{3}{\alpha}, \frac{\beta}{k} + 1\right) \right) \right. \right. \\
 &\quad \left. \left. + \frac{1}{2\alpha^{\frac{\beta}{k+1}}} |\tau'(y)|^q \left(B\left(\frac{2}{\alpha}, \frac{\beta}{k} + 1\right) - B\left(\frac{3}{\alpha}, \frac{\beta}{k} + 1\right) \right) \right) \right) \right]^{\frac{1}{q}} \\
 &\quad + \left(\frac{B(\frac{1}{\alpha}, \frac{\beta}{k} + 1) - B(\frac{2}{\alpha}, \frac{\beta}{k} + 1)}{\alpha^{\frac{\beta}{k+1}}} \right)^{1-\frac{1}{q}} \\
 &\quad \times \left((|\tau'(\theta)|^q + |\tau'(\vartheta)|^q) \left(\frac{B(\frac{1}{\alpha}, \frac{\beta}{k} + 1) - B(\frac{2}{\alpha}, \frac{\beta}{k} + 1)}{\alpha^{\frac{\beta}{k+1}}} \right) \right. \\
 &\quad - \left. \left(\frac{1}{2\alpha^{\frac{\beta}{k+1}}} |\tau'(x)|^q \left(B\left(\frac{1}{\alpha}, \frac{\beta}{k} + 1\right) - B\left(\frac{3}{\alpha}, \frac{\beta}{k} + 1\right) \right) \right) \right. \\
 &\quad \left. \left. + \frac{1}{2\alpha^{\frac{\beta}{k+1}}} |\tau'(y)|^q \left(B\left(\frac{1}{\alpha}, \frac{\beta}{k} + 1\right) - 2B\left(\frac{2}{\alpha}, \frac{\beta}{k} + 1\right) + B\left(\frac{3}{\alpha}, \frac{\beta}{k} + 1\right) \right) \right) \right)^{\frac{1}{q}} \Big\} \\
 &\quad + \left\{ \left(\frac{B(\frac{2}{\alpha}, \frac{\beta}{k} + 1)}{\alpha^{\frac{\beta}{k+1}}} \right)^{1-\frac{1}{q}} \left((|\tau'(\theta)|^q + |\tau'(\vartheta)|^q) \left(\frac{B(\frac{2}{\alpha}, \frac{\beta}{k} + 1)}{\alpha^{\frac{\beta}{k+1}}} \right) \right. \right. \\
 &\quad - \left. \left(\frac{1}{2\alpha^{\frac{\beta}{k+1}}} |\tau'(x)|^q \left(B\left(\frac{2}{\alpha}, \frac{\beta}{k} + 1\right) - B\left(\frac{3}{\alpha}, \frac{\beta}{k} + 1\right) \right) \right) \right. \\
 &\quad \left. \left. + \frac{1}{2\alpha^{\frac{\beta}{k+1}}} |\tau'(y)|^q \left(B\left(\frac{2}{\alpha}, \frac{\beta}{k} + 1\right) + B\left(\frac{3}{\alpha}, \frac{\beta}{k} + 1\right) \right) \right) \right) \right)^{\frac{1}{q}} \\
 &\quad + \left(\frac{B(\frac{1}{\alpha}, \frac{\beta}{k} + 1) - B(\frac{2}{\alpha}, \frac{\beta}{k} + 1)}{\alpha^{\frac{\beta}{k+1}}} \right)^{1-\frac{1}{q}} \\
 &\quad \times \left((|\tau'(\theta)|^q + |\tau'(\vartheta)|^q) \left(\frac{B(\frac{1}{\alpha}, \frac{\beta}{k} + 1) - B(\frac{2}{\alpha}, \frac{\beta}{k} + 1)}{\alpha^{\frac{\beta}{k+1}}} \right) \right. \\
 &\quad - \left. \left(\frac{1}{2\alpha^{\frac{\beta}{k+1}}} |\tau'(x)|^q \left(B\left(\frac{1}{\alpha}, \frac{\beta}{k} + 1\right) - 2B\left(\frac{2}{\alpha}, \frac{\beta}{k} + 1\right) + B\left(\frac{3}{\alpha}, \frac{\beta}{k} + 1\right) \right) \right) \right. \\
 &\quad \left. \left. + \frac{1}{2\alpha^{\frac{\beta}{k+1}}} |\tau'(y)|^q \left(B\left(\frac{1}{\alpha}, \frac{\beta}{k} + 1\right) - B\left(\frac{3}{\alpha}, \frac{\beta}{k} + 1\right) \right) \right) \right)^{\frac{1}{q}} \Big], \tag{3.2}
 \end{aligned}$$

holds for all $x, y \in [\theta, \vartheta]$.

Proof It follows from Lemma 2.5, Jensen–Mercer inequality, the convexity of $|\tau'|^q$, and the improved power-mean integral inequality given in Theorem 1.5 of [70] that

$$\begin{aligned}
 &\left| \frac{2\alpha^{\frac{\beta}{k}-1} \alpha^{\frac{\beta}{k}} \Gamma_k(\beta+k)}{(y-x)\alpha^{\frac{\beta}{k}}} \left\{ {}_k^{\beta} J_{(\theta+\vartheta-\frac{x+y}{2})^+}^{\alpha} \tau(\theta+\vartheta-x) + {}_k^{\beta} J_{(\theta+\vartheta-\frac{x+y}{2})^-}^{\alpha} \tau(\theta+\vartheta-y) \right\} \right. \\
 &\quad \left. - \tau\left(\theta+\vartheta-\frac{x+y}{2}\right) \right|
 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{(y-x)\alpha^{\frac{\beta}{k}}}{4} \left[\left\{ \left(\int_0^1 (1-\lambda) \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\lambda \right)^{1-\frac{1}{q}} \right. \right. \\
 &\quad \times \left(\int_0^1 (1-\lambda) \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \left| \tau' \left(\theta + \vartheta - \left(\frac{2-\lambda}{2}x + \frac{\lambda}{2}y \right) \right) \right|^q d\lambda \right)^{\frac{1}{q}} \\
 &\quad + \left(\int_0^1 \lambda \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\lambda \right)^{1-\frac{1}{q}} \\
 &\quad \times \left. \left(\int_0^1 \lambda \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \left| \tau' \left(\theta + \vartheta - \left(\frac{2-\lambda}{2}x + \frac{\lambda}{2}y \right) \right) \right|^q d\lambda \right)^{\frac{1}{q}} \right\} \\
 &\quad + \left\{ \left(\int_0^1 (1-\lambda) \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\lambda \right)^{1-\frac{1}{q}} \right. \\
 &\quad \times \left(\int_0^1 (1-\lambda) \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \left| \tau' \left(\theta + \vartheta - \left(\frac{\lambda}{2}x + \frac{2-\lambda}{2}y \right) \right) \right|^q d\lambda \right)^{\frac{1}{q}} \\
 &\quad + \left(\int_0^1 \lambda \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\lambda \right)^{1-\frac{1}{q}} \\
 &\quad \times \left. \left(\int_0^1 \lambda \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \left| \tau' \left(\theta + \vartheta - \left(\frac{\lambda}{2}x + \frac{2-\lambda}{2}y \right) \right) \right|^q d\lambda \right)^{\frac{1}{q}} \right\} \\
 &\leq \frac{(y-x)\alpha^{\frac{\beta}{k}}}{4} \left[\left\{ \left(\int_0^1 (1-\lambda) \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\lambda \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-\lambda) \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right. \right. \right. \\
 &\quad \times \left. \left[|\tau'(\theta)|^q + |\tau'(\vartheta)|^q - \left(\frac{2-\lambda}{2}|\tau'(x)|^q + \frac{\lambda}{2}|\tau'(y)|^q \right) \right] d\lambda \right)^{\frac{1}{q}} \\
 &\quad + \left(\int_0^1 \lambda \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\lambda \right)^{1-\frac{1}{q}} \left(\int_0^1 \lambda \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right. \\
 &\quad \times \left. \left[|\tau'(\theta)|^q + |\tau'(\vartheta)|^q - \left(\frac{2-\lambda}{2}|\tau'(x)|^q + \frac{\lambda}{2}|\tau'(y)|^q \right) \right] d\lambda \right)^{\frac{1}{q}} \Big\} \\
 &\quad + \left\{ \left(\int_0^1 (1-\lambda) \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\lambda \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-\lambda) \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right. \right. \\
 &\quad \times \left. \left[|\tau'(\theta)|^q + |\tau'(\vartheta)|^q - \left(\frac{\lambda}{2}|\tau'(x)|^q + \frac{2-\lambda}{2}|\tau'(y)|^q \right) \right] d\lambda \right)^{\frac{1}{q}} \\
 &\quad + \left(\int_0^1 \lambda \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\lambda \right)^{1-\frac{1}{q}} \left(\int_0^1 \lambda \left(\frac{1-(1-\lambda)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right. \\
 &\quad \times \left. \left[|\tau'(\theta)|^q + |\tau'(\vartheta)|^q - \left(\frac{\lambda}{2}|\tau'(x)|^q + \frac{2-\lambda}{2}|\tau'(y)|^q \right) \right] d\lambda \right)^{\frac{1}{q}} \Big\} \Big].
 \end{aligned}$$

By computing the above integrals, one can obtain the required result. □

4 Conclusions

The Hermite–Kadamard inequality is one of the most important inequalities for convex functions and in the theory of inequalities, while the Hermite–Jensen–Mercer inequality is a variant of the Hermite–Kadamard inequality which has attracted the attention of many researchers in recently years due to its many applications in pure and applied mathemat-

ics, as well as in physics. Therefore, it is important to further generalize and improve the Hermite–Jensen–Mercer inequality. In the article, we have found new methods to generalize the Hermite–Jensen–Mercer inequality to the fractional integrals, established several novel Hermite–Jensen–Mercer-type inequalities for convex functions in the framework of the k -fractional conformable integrals, generalized and improved many previously known results in the literature. The ideas and techniques we put forward are likely to open new research directions in this field and lead to a large number of follow-up studies.

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