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New inequalities between the inverse hyperbolic tangent and the analogue for corresponding functions

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Abstract
In this paper, we present new inequalities which reveal further relationship for both the inverse tangent function arctan(x) and the inverse hyperbolic function arctanh(x). At the same time, we give the analogue for inverse hyperbolic tangent and other corresponding functions.

Keywords: Inequalities; Inverse tangent function; Inverse hyperbolic sine function; Inverse hyperbolic tangent function; Inverse sine function

1 Introduction

Masjed-Jamei [1] obtained the following inequality for $|x| < 1$:

$$(\arctan x)^2 \leq \frac{x\ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}}. \tag{1}$$

Many similar or relative inequalities are discussed in references [2–14]. Recently, Zhu and Malesevic [15] affirmed inequality (1) for the large interval $(-\infty, \infty)$, pointed out that $\sinh^{-1}(x) = \ln(x + \sqrt{1+x^2})$, and provided the following Theorems 1–6, which (or relative results) can be also found in [11, 12].

Theorem 1 ([15]) *The inequality*

$$(\arctan x)^2 \leq \frac{x\ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} \tag{2}$$

holds for all $x \in (-\infty, \infty)$, *and the power number* 2 *is the best in* (2).

Theorem 2 ([15]) *Let* $0 < r < \infty$, $\lambda = 1$, *and* $\mu = r\ln(r + \sqrt{r^2+1})/(\sqrt{r^2+1}(\arctan r)^2)$. *Then the double inequality*

$$\lambda(\arctan x)^2 \leq \frac{x\ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} \leq \mu(\arctan x)^2 \tag{3}$$

holds for all $x \in (--r, r)$, *where* λ *and* μ *are the best constants in* (3).

Theorem 3 ([15]) *We have*

$$-\frac{1}{45}x^6 \le (\arctan x)^2 - \frac{x\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}} \le -\frac{1}{45}x^6 + \frac{4}{105}x^8, \tag{4}$$

$$\begin{aligned} -\frac{1}{45}x^6 + \frac{4}{105}x^8 - \frac{11}{225}x^{10} &\le (\arctan x)^2 - \frac{x\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}} \\ &\le -\frac{1}{45}x^6 \le +\frac{4}{105}x^8 - \frac{11}{225}x^{10} + \frac{586}{10{,}395}x^{12}. \end{aligned} \tag{5}$$

Theorem 4 ([15]) *The inequality*

$$(\operatorname{arctanh} x)^2 \le \frac{x\arcsin x}{\sqrt{1-x^2}} \tag{6}$$

holds for all $x \in (-1,1)$, *and the power number* 2 *is the best in* (6).

Theorem 5 ([15]) *Let* $0 < r < 1$, $\alpha_1 = 1$, *and* $\beta_1 = r(\arcsin r)/(\sqrt{1--r^2}(\operatorname{arctanh} r)^2)$. *Then the double inequality*

$$\alpha_1(\operatorname{arctanh} x)^2 \le \frac{x\arcsin x}{\sqrt{1-x^2}} \le \beta_1(\operatorname{arctanh} x)^2 \tag{7}$$

holds for all $x \in (-r,r)$, *where* α_1 *and* β_1 *are the best constants in* (7).

Recently, Chen and Malešević [14] proposed the following results:

$$\frac{x\operatorname{arcsinh} x}{\sqrt{1+x^2+\alpha_2x^4}} \le (\arctan x)^2 \le \frac{x\operatorname{arcsinh} x}{\sqrt{1+x^2+\beta_2x^4}}, \quad x>0, \tag{8}$$

$$\frac{x\arcsin x}{1-\alpha_3x^2} < (\operatorname{arctanh} x)^2, \quad 0<x<1, \tag{9}$$

where $\alpha_2 = \frac{2}{45}$, $\beta_2 = 0$, and $\alpha_3 = \frac{1}{2}$ are the best possible constants.

In 2020, Zhu and Malešević [13] proposed natural approximation of Masjed-Jamei's inequality and provided two-sided bounds in a polynomial form of $(\arctan x)^2 - \frac{x\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}$, which consists of explicit formulae of different degrees.

The values of μ in Theorem 2 and β_1 in Theorem 5 tend to be $+\infty$ for r tends to be $\pm\infty$ and ±1, respectively. In this paper, we obtain the following new inequalities, which improve the approximation effect of the inequalities in [15]. The main results are as follows.

Theorem 6 *The inequality*

$$(\arctan x)^2 \ge \frac{3(8+9x^2-8\sqrt{1+x^2})}{(4+11\sqrt{1+x^2})\sqrt{1+x^2}} \triangleq F(x) \tag{10}$$

holds for all $x \in (-\infty,\infty)$.

Theorem 7 *Let* $\kappa_1 = \frac{108}{11\pi^2} \approx 0.9947$ *and* $\kappa_2 = 1$. *The inequality*

$$\kappa_1(\arctan x)^2 \le F(x) \le \kappa_2(\arctan x)^2 \tag{11}$$

holds for all $x \in (-\infty, \infty)$, *where* κ_1 *and* κ_2 *are the best constants in* (11).

Theorem 8 *The inequality*

$$\frac{23}{75{,}600}x^8 \ge (\arctan x)^2 - F(x) \ge \frac{23}{75{,}600}x^8 - \frac{899}{1{,}134{,}000}x^{10} \tag{12}$$

holds for all $x \in (-\infty, \infty)$.

Theorem 9 *The inequality*

$$G_1(x) \triangleq (\operatorname{arctanh} x)^2 \le \left(\frac{-\ln(1-x^2)}{\arcsin x}\right)^2 \triangleq G_2(x) \tag{13}$$

holds for all $x \in (-1, 1)$.

Theorem 10 *Let* $\kappa_3 = 1$ *and* $\kappa_4 = \frac{16}{\pi^2} \approx 1.6211$. *The inequality*

$$\kappa_3(\operatorname{arctanh} x)^2 \le \left(\frac{-\ln(1-x^2)}{\arcsin x}\right)^2 \le \kappa_4(\operatorname{arctanh} x)^2 \tag{14}$$

holds for all $x \in (-1, 1)$, *where* κ_3 *and* κ_4 *are the best constants in* (14).

2 Proofs of Theorems 6–10

Let $\arctan x = t$, then one has that $x = \tan(t)$ and $\sqrt{1+x^2} = \sec(t)$, where $x \in (-\infty, \infty)$ and $t \in (-\pi/2, \pi/2)$. It can be verified that

$$\begin{aligned}
&(\arctan x)^2 = t^2, \\
&F(x) = -\frac{3}{4}\cos(t) - \frac{63}{16} + \frac{1125}{16(4\cos(t)+11)} = f_1(t), \\
&(\arctan x)^2 - F(x) = \bigl(t^2 - f_1(t)\bigr) = \delta_1(t), \\
&\delta_1'''(t) = \frac{(12(16\cos(t)^2 + 208\cos(t) + 1501))(\cos(t)-1)^2\sin(t)}{(4\cos(t)+11)^4}.
\end{aligned} \tag{15}$$

2.1 Proof of Theorem 6

From Eq. (15), one has that

$$\delta_1'''(t) > 0, \quad t \in (0, \pi/2), \qquad \delta_1''(0) = \delta_1'(0) = \delta_1(0) = 0, \tag{16}$$

which leads to

$$\delta_1''(t) > 0, \qquad \delta_1'(t) > 0, \quad t \in (0, \pi/2), \qquad \delta_1(t) \ge \delta_1(0) = 0, \quad t \in [0, \pi/2). \tag{17}$$

Note that $\delta_1(t) = \delta_1(-t)$, combining Eq. (15) with Eq. (17), one has that

$$\delta_1(t) \geq 0, \quad t \in (-\pi/2, \pi/2), \quad \text{and} \quad (\arctan x)^2 - F(x) \geq 0, \quad x \in (-\infty, \infty). \tag{18}$$

And we complete the proof.

2.2 Proof of Theorem 7

From Theorem 6, one has that

$$F(x) \leq \kappa_2 (\arctan x)^2.$$

Now we prove that $\kappa_1 (\arctan x)^2 \leq F(x)$. From Eq. (15), one has that

$$\begin{aligned} &\kappa_1(\arctan x)^2 - F(x) = \kappa_1 t^2 - f_1(t) = \delta_2(t), \\ &\delta_2''''(t) = -f_1''''(t) = \delta_1''''(t) > 0, \quad t \in (0, \pi/2), \\ &\delta_2''(0) = \frac{216 - 22\pi^2}{11\pi^2} \approx -0.01 < 0, \qquad \delta_2''(\pi/2) = \frac{26136 - 2250\pi^2}{1331\pi^2} \approx 0.2 > 0. \end{aligned} \tag{19}$$

From Eq. (19), there exists a unique root $t_1 \in (0, \pi/2)$ such that

$$\begin{aligned} &\delta_2''(t) < 0, \quad t \in (0, t_1), \qquad \delta_2'(0) = 0, \\ &\delta_2''(t) > 0, \quad t \in (t_1, \pi/2), \qquad \delta_2'(\pi/2) = \frac{1188 - 372\pi}{121\pi} \approx 0.05 > 0. \end{aligned} \tag{20}$$

From Eq. (19), there exists a unique root $t_2 \in (t_1, \pi/2)$ such that

$$\begin{aligned} &\delta_2'(t) < 0, \quad t \in (0, t_2), \qquad \delta_2(0) = 0, \\ &\delta_2'(t) > 0, \quad t \in (t_2, \pi/2), \qquad \delta_2(\pi/2) = 0. \end{aligned} \tag{21}$$

From Eq. (21), one has that

$$\delta_2(t) \leq 0, \quad t \in [0, t_2] \cup [t_2, \pi/2) = [0, \pi/2). \tag{22}$$

Note that $\delta_2(t) = \delta_2(-t)$, combining Eq. (19) with Eq. (22), one has that

$$\delta_2(t) \leq 0, \quad t \in (-\pi/2, \pi/2), \quad \text{and} \quad \kappa_1(\arctan x)^2 \leq F(x), \quad x \in (-\infty, \infty). \tag{23}$$

Note that

$$\lim_{x\to\infty} \frac{F(x)}{(\arctan x)^2} = \kappa_1, \qquad \lim_{x\to 0} \frac{F(x)}{(\arctan x)^2} = \kappa_2,$$

both κ_1 and κ_2 are the best constants. And the proof is completed.

2.3 Proof of Theorem 8

Let $f_2(t) = \frac{23}{75{,}600}(\tan t)^8$ and $f_3(t) = \frac{23}{75{,}600}(\tan t)^8 - \frac{899}{1{,}134{,}000}(\tan t)^{10}$. Equation (12) in Theorem 8 is equivalent to

$$\delta_3(t) = \delta_1(t) - f_2(t) \leq 0, \qquad \delta_4(t) = \delta_1(t) - f_3(t) \geq 0, \quad t \in (-\pi/2, \pi/2). \tag{24}$$

It can be verified that

$$\begin{aligned} f_2'''(t) &= \frac{23\sin(t)^5(2\cos(t)^4 - 26\cos(t)^2 + 45)}{4725(\cos t)^{11}}, \\ f_3'''(t) &= \frac{\sin(t)^5(1175\cos(t)^6 - 18871\cos(t)^4 + 50{,}261\cos(t)^2 - 29{,}667)}{28{,}350(\cos t)^{13}}. \end{aligned} \tag{25}$$

Let $\phi_1(t) = 907{,}200\cos(t)^{12} + 12{,}700{,}800\cos(t)^{11} + 97{,}807{,}500\cos(t)^{10} + 97{,}795{,}724\cos(t)^9 + 97{,}642{,}636\cos(t)^8 + 96{,}990{,}540\cos(t)^7 + 96{,}802{,}860\cos(t)^6 + 103{,}838{,}238\cos(t)^5 + 126{,}378{,}882\cos(t)^4 + 148{,}760{,}458\cos(t)^3 + 130{,}005{,}062\cos(t)^2 + 67{,}501{,}665\cos(t) + 15{,}153{,}435$ and $\phi_2(t) = 5{,}443{,}200\cos(t)^{13} + 81{,}648{,}000\cos(t)^{12} + 668{,}493{,}000\cos(t)^{11} + 1{,}255{,}037{,}200\cos(t)^{10} + 1{,}837{,}671{,}000\cos(t)^9 + 2{,}404{,}568{,}576\cos(t)^8 + 2{,}978{,}639{,}640\cos(t)^7 + 3{,}789{,}264{,}297\cos(t)^6 + 5{,}266{,}619{,}820\cos(t)^5 + 7{,}153{,}847{,}855\cos(t)^4 + 7{,}714{,}708{,}320\cos(t)^3 + 5{,}610{,}730{,}675\cos(t)^2 + 2{,}369{,}206{,}620\cos(t) + 434{,}354{,}547$. Combining Eq. (24) with Eq. (25), one has that

$$\begin{aligned} \delta_3'''(t) &= \frac{\sin(t)(\cos(t) - 1)^3}{(4\cos(t) + 11)^4(\cos t)^{11}}\phi_1(t) < 0, \quad \forall t \in (0, \pi/2), \\ \delta_4'''(t) &= \frac{\sin(t)(\cos(t) - 1)^4}{28{,}350(4\cos(t) + 11)^4(\cos t)^{13}}\phi_2(t) > 0, \quad \forall t \in (0, \pi/2), \\ \delta_3''(0) &= 0, \qquad \delta_4''(0) = 0. \end{aligned} \tag{26}$$

From Eq. (25), one has that

$$\delta_3''(t) < 0, \qquad \delta_4''(t) > 0, \quad \forall t \in (0, \pi/2), \qquad \delta_3'(0) = 0, \qquad \delta_4'(0) = 0. \tag{27}$$

From Eq. (27), one obtains that

$$\delta_3'(t) < 0, \qquad \delta_4'(t) > 0, \quad \forall t \in (0, \pi/2), \qquad \delta_3(0) = 0, \qquad \delta_4(0) = 0, \tag{28}$$

which leads to

$$\delta_3(t) \leq 0, \qquad \delta_4(t) \geq 0, \quad \forall t \in [0, \pi/2). \tag{29}$$

Note that $\delta_i(t) = \delta_i(-t)$, $i = 3, 4$, combining with Eq. (29), both Eq. (24) and Theorem 8 are proved.

2.4 Proof of Theorem 9

Let $\arcsin(x) = s$, then one has that $x = \sin(s)$, where $x \in (-1,1)$, $s \in (-\pi/2, \pi/2)$. It can be verified that

$$\begin{aligned} &(\operatorname{arctanh} x) = \frac{1}{2}\ln\left(\frac{1+\sin(s)}{1-\sin(s)}\right) > 0, \\ &\left(\frac{-\ln(1-x^2)}{\arcsin x}\right) = \frac{-\ln(1-(\sin s)^2)}{s} > 0, \quad s \in (0,\pi/2). \end{aligned} \tag{30}$$

Let

$$\begin{aligned} &(\operatorname{arctanh} x) - \left(\frac{-\ln(1-x^2)}{\arcsin x}\right) = \frac{1}{2}\ln\left(\frac{1+\sin(s)}{1-\sin(s)}\right) - \frac{-\ln(1-(\sin s)^2)}{s} = \delta_5(s), \\ &\delta_6(s) = \delta_5'(s)\cdot s^2, \qquad \phi_3(s) = -2 + \sin(s)s + 2\cos(s). \end{aligned} \tag{31}$$

It can be verified that

$$\phi_3''(s) = -\sin(s)s < 0, \quad s \in (0,\pi/2), \qquad \phi_3'(0) = \phi_3(0) = 0,$$

which leads to

$$\phi_3(s) \le 0, \qquad \delta_6'(s) = \frac{s}{(\cos s)^2}\phi_3(s) \le 0, \qquad \delta_6(0) = 0, \quad s \in [0,\pi/2). \tag{32}$$

Combining Eq. (31) with Eq. (32), one obtains that

$$\delta_6(s) \le 0, \qquad \delta_5'(s) \le 0, \qquad \delta_5(0) = 0, \quad s \in [0,\pi/2). \tag{33}$$

Combining Eq. (31) with Eq. (33), we have that

$$\delta_5(s) \le 0, \quad s \in [0,\pi/2), \qquad 0 \le (\operatorname{arctanh} x)^2 \le \left(\frac{-\ln(1-x^2)}{\arcsin x}\right)^2, \quad x \in [0,1). \tag{34}$$

Note that $G_i(-x) = G_i(x)$, $i = 1,2$, combining with Eq. (34), we have proved both Eq. (13) and Theorem 9.

2.5 Proof of Theorem 10

Directly from Theorem 9, we have proved the left-hand side in Eq. (14) in Theorem 10.

$$\kappa_3(\operatorname{arctanh} x)^2 \le \left(\frac{-\ln(1-x^2)}{\arcsin x}\right)^2.$$

Now, we will prove the right-hand side of Eq. (14). Combining with Eq. (30), let

$$\begin{aligned} &\frac{4}{\pi}(\operatorname{arctanh} x) - \left(\frac{-\ln(1-x^2)}{\arcsin x}\right) = \frac{4}{2\pi}\ln\left(\frac{1+\sin(s)}{1-\sin(s)}\right) - \frac{-\ln(1-(\sin s)^2)}{s} \triangleq \delta_7(s), \\ &\delta_8(s) = \delta_7'(s)\cdot s^2, \qquad \phi_4(s) = \frac{2(2\sin(s)s + 4\cos(s) - \pi)}{\pi}. \end{aligned} \tag{35}$$

It can be verified that

$$\phi_4''(s) = \frac{-4\sin(s)s}{\pi} < 0, \quad s \in (0,\pi/2), \qquad \phi_4'(0) = \phi_4(\pi/2) = 0,$$

which leads to

$$\phi_4(s) \geq 0, \qquad \delta_8'(s) = \frac{s}{(\cos s)^2}\phi_4(s) \geq 0, \qquad \delta_8(0) = 0, \quad s \in [0,\pi/2). \tag{36}$$

Combining Eq. (35) with Eq. (36), one obtains that

$$\delta_8(s) \geq 0, \qquad \delta_7'(s) \geq 0, \qquad \delta_7(0) = 0, \quad s \in [0,\pi/2). \tag{37}$$

Combining Eq. (35) with Eq. (37), we have that

$$\begin{aligned} &\delta_7(s) \geq 0, \quad s \in [0,\pi/2), \\ &0 \leq \left(\frac{-\ln(1-x^2)}{\arcsin x}\right)^2 \leq \left(\frac{4}{\pi}\operatorname{arctanh} x\right)^2, \quad x \in [0,1). \end{aligned} \tag{38}$$

Note that $G_i(-x) = G_i(x)$, $i = 1,2$, combining with Eq. (38), one obtains that

$$\left(\frac{-\ln(1-x^2)}{\arcsin x}\right)^2 \leq \kappa_2(\operatorname{arctanh} x)^2, \quad x \in (-1,1). \tag{39}$$

Combining Theorem 9 with Eq. (39), we have completed the proofs of both Eq. (14) and Theorem 10.

3 Discussions and conclusions

The values of μ in Theorem 2 and β_1 in Theorem 5 tend to be $+\infty$ for r tends to be $\pm\infty$ and ± 1, respectively, while the values of κ_i in Theorems 7 and 10 are constant. The error plots of the bounds from Eq. (2) and Eq. (6) in [15], from Eq. (8) and Eq. (9) in [14], and from Eq. (6) and Eq. (13) are plotted in Fig. 1. It shows that the results of Eq. (11) and

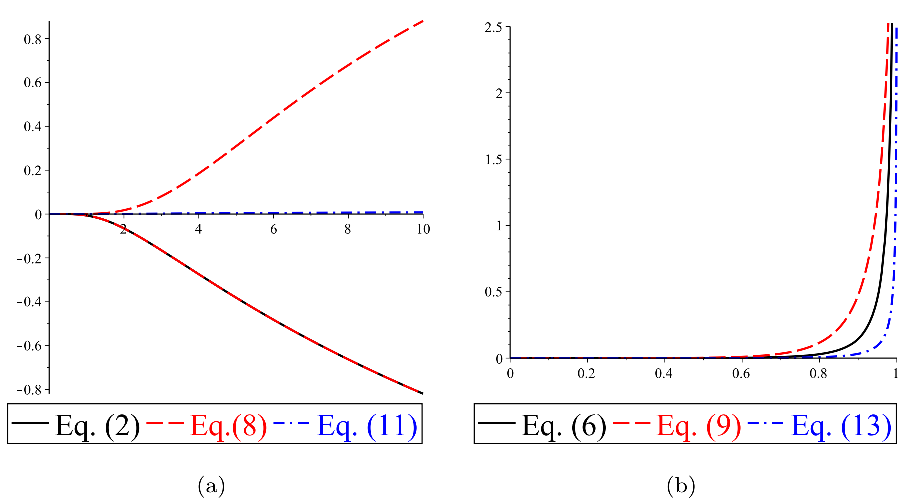

Figure 1 Error plots of bounds from (**a**) Eq. (2), (8), and (11); and (**b**) Eq. (6), (9), and (13)

Eq. (13) in this paper achieve better approximation effect than those of Eq. (2), Eq. (6), Eq. (8), and Eq. (9).

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