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NEW INEQUALITIES FOR POLYNOMIALS

BY

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ABSTRACT. Using a recently developed method to determine bound-preserving convolution operators in the unit disk, we derive various refinements and generalizations of the well-known inequalities of S. Bernstein and M. Riesz for polynomials. Many of these results take into account the size of one or more of the coefficients of the polynomial in question. Other results of similar nature are obtained from a new interpolation formula.

1. Introduction. Let $\mathscr{P}_n$ be the class of polynomials $P(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ of degree at most n. We write

$$\|P\| := \max_{|z|=1}|P(z)|, \quad M_P(R) := \max_{|z|=R}|P(z)|.$$

According to a well-known result of S. Bernstein (for references see [**18**]),

$$\|P'\| \leqslant n\|P\|. \tag{1.1}$$

It is also well known (see [**14**, p. 346 or **11**, vol. 1, Problem III 269, p. 137]) that

$$M_P(R) \leqslant R^n\|P\| \quad \text{for } R > 1. \tag{1.2}$$

In both (1.1) and (1.2), equality holds only when $P(z)$ is a constant multiple of z^n, i.e., if and only if all coefficients a_ν, except a_n, are zero. Thus, we should be able to say more if any of them is known to be different from zero.

Here we obtain inequalities similar to the above which take into account the coefficients of P. They are established by a fairly uniform procedure, and most of them constitute refinements of (1.1) or (1.2). Later we obtain other refinements of the above inequalities. We wish to draw the attention of the reader to Theorem 8 in particular.

First of all we investigate the dependence of $\|P'\|$ and $M_P(R)$ on $\|P\|$ and $|a_0|$. In fact, we consider $\|P(Rz) - P(z)\|$ rather than $\|P'\|$ and prove:

THEOREM 1. *Let $P \in \mathscr{P}_n$. Then for all $R > 1$,*

$$\|P(Rz) - P(z)\| + \psi_n(R)|P(0)| \leqslant (R^n - 1)\|P\|, \tag{1.3}$$

where

$$\psi_n(R) := \frac{(R-1)(R^{n-1} + R^{n-2})\{R^{n+1} + R^n - (n+1)R + (n-1)\}}{R^{n+1} + R^n - (n-1)R + (n-3)}$$

if $n \geqslant 2$, and $\psi_1(R) := R - 1$. The coefficient of $|P(0)|$ is the best possible for each R.

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Dividing both sides of (1.3) by $R - 1$ and letting R tend to 1, we obtain

COROLLARY 1. *For $P \in \mathscr{P}_n$ we have*

$$\|P'\| + \varepsilon_n |P(0)| \leqslant n\|P\|, \tag{1.4}$$

where $\varepsilon_n = 2n/(n+2)$ if $n \geqslant 2$, whereas $\varepsilon_1 = 1$. The coefficient of $|P(0)|$ is the best possible for each n.

The corresponding refinement of (1.2) is contained in

THEOREM 2. *Let $P \in \mathscr{P}_n$, $n \geqslant 2$. Then for all $R > 1$,*

$$M_P(R) + (R^n - R^{n-2})|P(0)| \leqslant R^n \|P\|. \tag{1.5}$$

The coefficient of $|P(0)|$ is the best possible for each R.

Before stating any of the other results, we wish to describe

2. The method of proof. Given two analytic functions

$$f(z) := \sum_{\nu=0}^{\infty} a_\nu z^\nu, \quad g(z) := \sum_{\nu=0}^{\infty} b_\nu z^\nu \qquad (|z| < 1),$$

the function

$$(f * g)(z) := \sum_{\nu=0}^{\infty} a_\nu b_\nu z^\nu \qquad (|z| < 1)$$

is said to be their Hadamard product.

Let us denote by $\mathscr{B}_n$ the subclass of $\mathscr{P}_n$ consisting of those polynomials Q for which

$$\|Q * P\| \leqslant \|P\| \quad \text{for all } P \in \mathscr{P}_n. \tag{2.1}$$

In order to prove our inequalities we divide both sides by the coefficient of $\|P\|$ and express the resulting quantity on the left as $\|Q * P\|$. After that we must show that $Q \in \mathscr{B}_n$. We have a fairly straightforward method to do that if $Q(0) \neq 0$. In order to describe it we find it convenient to introduce *the subclass $\mathscr{B}_n^0$ of $\mathscr{B}_n$ consisting of those polynomials Q in $\mathscr{B}_n$ for which* $Q(0) = 1$. The class $\mathscr{B}_n^0$ is closely related to *the class $\mathscr{R}$ of all analytic functions f in $|z| < 1$ such that* $f(0) = 1$ *and* $\operatorname{Re} f(z) > 1/2$ *for* $|z| < 1$. To be precise, we have [**19**, **17**, p. 124]:

LEMMA 1. *A polynomial Q belongs to $\mathscr{B}_n^0$ if and only if there exists $f \in \mathscr{R}$ such that $f(z) - Q(z) = o(z^n)$, $z \to 0$.*

This leads us to the following characterization of polynomials in $\mathscr{B}_n^0$, which will be used on several occasions:

LEMMA 2. *The polynomial $Q(z) := \sum_{k=0}^{n} a_k z^k$, where $a_0 = 1$, belongs to $\mathscr{B}_n^0$ if and only if the matrix*

$$M(a_0, a_1, \ldots, a_n) := \begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ \bar{a}_1 & a_0 & \cdots & a_{n-1} \\ \cdot & \cdot & \cdots & \cdot \\ \bar{a}_n & \bar{a}_{n-1} & \cdots & a_0 \end{bmatrix}$$

is positive semidefinite.

PROOF. If $Q \in \mathscr{B}_n^0$ then, by Lemma 1, there exists $f \in \mathscr{R}$ such that $f(z) - Q(z) = o(z^n)$. Using the Carathéodory-Toeplitz Theorem [**21**, p. 157], we see that $M(a_0, a_1, \ldots, a_n)$ must be positive semidefinite. Conversely, if $M(a_0, a_1, \ldots, a_n)$ is positive semidefinite then, again using the Carathéodory-Toeplitz Theorem, we obtain that Q extends to a function $f \in \mathscr{R}$, whence $f(z) - Q(z) = o(z^n)$, $z \to 0$, and Lemma 1 shows that $Q \in \mathscr{B}_n^0$.

In order to study the definiteness of the matrix $M(1, a_1, \ldots, a_n)$ associated with the polynomial $Q(z) := 1 + \sum_{k=1}^n a_k z^k$, we use the following well-known result [**4**, vol. 1, p. 337] from linear algebra:

LEMMA 3. *The hermitian matrix*

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \qquad a_{ij} = \bar{a}_{ji},$$

is positive definite if and only if the corresponding leading principal minors

$$D_k = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \cdot & \cdot & \cdots & \cdot \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix}, \qquad k = 1, 2, \ldots, n,$$

are all positive.

We make one further observation. For that, let us associate with $P \in \mathscr{P}_n$ the polynomial $\tilde{P}(z) := z^n \overline{P(1/\bar{z})}$; thus $\tilde{P}$ depends on the class $\mathscr{P}_n$ and not just on P. Observe that

$$Q \in \mathscr{B}_n \Leftrightarrow \tilde{Q} \in \mathscr{B}_n, \tag{2.2}$$

which is obvious but very useful.

3. Specific details. For sake of brevity we denote the matrix

$$\begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ b_1 & & & \\ \vdots & & & a_1 \\ b_n & & b_1 & a_0 \end{bmatrix}$$

by

$$M\left(a_0, \begin{matrix} a_1, \ldots, a_n \\ \\ b_1, \ldots, b_n \end{matrix} \right);$$

the one obtained by deleting its i_1th, $\ldots$, i_mth rows and j_1th, $\ldots$, j_mth columns, respectively, will be written as

$$M_{i_1, \ldots, i_m; j_1, \ldots, j_m}\left(a_0, \begin{matrix} a_1, \ldots, a_n \\ \\ b_1, \ldots, b_n \end{matrix} \right).$$

The notations for the corresponding determinants will be

$$D\begin{pmatrix} & a_1,\ldots,a_n \\ a_0, & \\ & b_1,\ldots,b_n \end{pmatrix}$$

and

$$D_{i_1,\ldots,i_m;j_1,\ldots,j_m}\begin{pmatrix} & a_1,\ldots,a_n \\ a_0, & \\ & b_1,\ldots,b_n \end{pmatrix},$$

respectively.

PROOF OF THEOREM 1. The case $n = 1$ is trivial so let $n \geqslant 2$. First we note that

$$\frac{1}{R^n-1}\{\|P(Rz)-P(z)\|+\psi_n(R)|P(0)|\}$$
$$=\sup_{(R^n-1)|\alpha|<\psi_n(R)}\frac{1}{R^n-1}\|P(Rz)-P(z)+(R^n-1)\bar{\alpha}P(0)\|.$$

Next we observe that $(R^n-1)^{-1}\{P(Rz)-P(z)+(R^n-1)\bar{\alpha}P(0)\}$ is the Hadamard product of P and the polynomial

$$Q_\alpha(z):=\frac{1}{R^n-1}\{(R^n-1)z^n+(R^{n-1}-1)z^{n-1}+\cdots$$
$$+(R-1)z+(R^n-1)\bar{\alpha}\}.$$

Inequality (1.3) will therefore be established provided we show that $Q_\alpha \in \mathscr{B}_n$ if $(R^n-1)|\alpha| < \psi_n(R)$. For the sharpness of the result we need only prove that for every $\sigma > \psi_n(R)$ there exists a number α, with $(R^n-1)|\alpha| = \sigma$, such that $Q_\alpha \notin \mathscr{B}_n$.

According to (2.2) we may just as well find the set of alphas for which

$$Q_\alpha(z)=1+\frac{1}{R^n-1}\{(R^{n-1}-1)z+\cdots+(R-1)z^{n-1}+(R^n-1)\alpha z^n\}$$

is in $\mathscr{B}_n^0$. In accordance with Lemma 2 we have to study the definiteness of the hermitian matrix

$$M_1(\alpha,n):=M\begin{pmatrix} & R^{n-1}-1,R^{n-2}-1,\ldots,R-1,(R^n-1)\alpha \\ R^n-1, & \\ & R^{n-1}-1,R^{n-2}-1,\ldots,R-1,(R^n-1)\bar{\alpha} \end{pmatrix}.$$

Now, via Lemma 3 we are led to the problem of determining the values of α for which the determinant $\det(M_1(\alpha,n))$ of $M_1(\alpha,n)$ and its other leading principal minors are all positive.

We note that

$$\det(M_1(\alpha,n))=C_{1,n}(R)+(-1)^n2(R^n-1)B_{1,n}(R)\operatorname{Re}\alpha \tag{3.1}$$
$$-(R^n-1)^2A_{1,n}(R)|\alpha|^2,$$

where

$$A_{1,n}(R) := D\left(R^n - 1, \begin{matrix} R^{n-1}-1, R^{n-2}-1, \dots, R^2-1 \\ R^{n-1}-1, R^{n-2}-1, \dots, R^2-1 \end{matrix}\right),$$

$$B_{1,n}(R) := D\left(R^{n-1} - 1, \begin{matrix} R^{n}-1, R^{n-1}-1, \dots, R^3-1, R^2-1 \\ R^{n-2}-1, R^{n-3}-1, \dots, R-1, 0 \end{matrix}\right),$$

and

$$C_{1,n}(R) := D\left(R^n - 1, \begin{matrix} R^{n-1}-1, R^{n-2}-1, \dots, R-1, 0 \\ R^{n-1}-1, R^{n-2}-1, \dots, R-1, 0 \end{matrix}\right).$$

This is explained as follows: clearly, $\det(M_1(\alpha, n))$ is of the form

$$C_{1,n}(R) + b(R^n - 1)(\alpha + \bar{\alpha}) - (R^n - 1)^2 A_{1,n}(R)|\alpha|^2.$$

In order to determine the value of b, we may expand $\det(M_1(\alpha, n))$ by its first row. The term in α will come only from the last element in the row. Now, expanding the corresponding cofactor by its first column, the conclusion becomes transparent.

In order to evaluate $A_{1,n}(R)$, we may perform the following operations one after the other:

(a) subtract the $(i+1)$th row from the ith row, $i = 1, 2, \dots, n-2$;

(b) factor out $R - 1$ from each of the first $n - 2$ rows;

(c) from the ith row subtract $1/R$ times the $(i+1)$th row, $i = 1, 2, \dots, n-3$;

(d) from the jth column subtract $1/R$ times the $(j+1)$th column, $j = 1, 2, \dots, n - 2$;

(e) add the first column to the second, the (new) second column to the third, and so on. Thus (for $n \geqslant 3$) we obtain

$$A_{1,n}(R) = (R-1)^{n-2}\begin{vmatrix} R^{n-1}+R^{n-2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & R^{n-1}+R^{n-2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & R^{n-1}+R^{n-2} & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & R^{n-1}+R^{n-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & R^{n-1}+R^{n-2} & -R^{n-1} \\ \left(\frac{1}{R}-1\right) & 2\left(\frac{1}{R}-1\right) & 3\left(\frac{1}{R}-1\right) & \cdots & (n-3)\left(\frac{1}{R}-1\right) & (n-2)\left(\frac{1}{R}-1\right) & R^n-1 \end{vmatrix}$$

$$= (R-1)^{n-2}(R^{n-1}+R^{n-2})^{n-3}\begin{vmatrix} R^{n-1}+R^{n-2} & -R^{n-1} \\ (n-2)\left(\frac{1}{R}-1\right) & R^n-1 \end{vmatrix}$$

$$= (R-1)^{n-2}(R^{n-1}+R^{n-2})^{n-3}R^{n-2}\{R^{n+1}+R^n-(n-1)R+(n-3)\}.$$

It is seen directly that the same formula holds for $n = 2$ as well.

As for $B_{1,n}(R)$, we perform the following operations:

(a) subtract the $(i+1)$th row from the ith row, $i = 1, 2, \ldots, n-1$;

(b) factor out $R - 1$ from each of the first $n - 1$ rows;

(c) from the ith row subtract $1/R$ times the $(i-1)$th row, $i = 2, 3, \ldots, n-1$, and obtain (for $n \geqslant 2$)

$$B_{1,n}(R) = (R-1)^{n-1} \begin{vmatrix} R^{n-2} & R^{n-1} & -R^{n-1} & \cdots & -R^4 & -R^3 & -R^2 \\ 0 & 0 & R^{n-1}+R^{n-2} & \cdots & R^3-R^5 & R^2-R^4 & R-R^3 \\ 0 & 0 & 0 & \cdots & R^4-R^6 & R^3-R^5 & R^2-R^4 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & 0 & R^{n-1}+R^{n-2} & R^{n-3}-R^{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & R^{n-1}+R^{n-2} \\ 0 & R-1 & R^2-1 & \cdots & R^{n-3}-1 & R^{n-2}-1 & R^{n-1}-1 \end{vmatrix}$$

$$= (R-1)^{n-1}R^{n-2} \begin{vmatrix} 0 & R^{n-1}+R^{n-2} & \cdots & R^3-R^5 & R^2-R^4 & R-R^3 \\ 0 & 0 & \cdots & R^4-R^6 & R^3-R^5 & R^2-R^4 \\ \vdots & & & & & \\ 0 & 0 & \cdots & 0 & R^{n-1}+R^{n-2} & R^{n-3}-R^{n-1} \\ 0 & 0 & \cdots & 0 & 0 & R^{n-1}+R^{n-2} \\ R-1 & R^2-1 & \cdots & R^{n-3}-1 & R^{n-2}-1 & R^{n-1}-1 \end{vmatrix}$$

$$= (-1)^n R^{n-2}(R-1)^n (R^{n-1}+R^{n-2})^{n-2}.$$

Finally, in order to evaluate $C_{1,n}(R)$, we may perform the same sequence of operations (with $n-1$ replaced by $n+1$) as for $A_{1,n}(R)$. It turns out that (for $n \geqslant 2$)

$$C_{1,n}(R) = (R-1)^n (R^{n-1}+R^{n-2})^{n-1} R^{n-2}\{R^{n+1}+R^n-(n+1)R+n-1\}.$$

Hence, if $n \geqslant 2$, then $\det(M_1(\alpha, n)) > 0$ provided that

$$\begin{aligned} f_1(\alpha) := {} & \frac{\det(M_1(\alpha, n))}{(R-1)^{n-2}(R^{n-1}+R^{n-2})^{n-3}R^{n-2}} \\ = {} & (R-1)^2(R^{n-1}+R^{n-2})^2\{R^{n+1}+R^n-(n+1)R+(n-1)\} \\ & +2(R-1)^2(R^{n-1}+R^{n-2})(R^n-1)\operatorname{Re}\alpha \\ & -(R^n-1)^2\{R^{n+1}+R^n-(n-1)R+(n-3)\}|\alpha|^2 \end{aligned}$$

is positive, and definitely if

$$\begin{aligned} & (R-1)^2(R^{n-1}+R^{n-2})^2\{R^{n+1}+R^n-(n+1)R+(n-1)\} \\ & \quad -2(R-1)^2(R^{n-1}+R^{n-2})(R^n-1)|\alpha| \\ & \quad -(R^n-1)^2\{R^{n+1}+R^n-(n-1)R+(n-3)\}|\alpha|^2 > 0. \end{aligned}$$

Thus $\det(M_1(\alpha, n)) > 0$ if

$$(R^n - 1)|\alpha| < \frac{(R-1)(R^{n-1}+R^{n-2})\{R^{n+1}+R^n-(n+1)R+(n-1)\}}{R^{n+1}+R^n-(n-1)R+(n-3)} =: \psi_n(R).$$

Further, we observe that for $R > 1$, all the leading principal minors

$$m_{1,1} = R^n - 1,$$

$$m_{1,k} := D\left(R^n - 1, \begin{matrix} R^{n-1}-1,\ldots,R^{n-k+1}-1 \\ R^{n-1}-1,\ldots,R^{n-k+1}-1 \end{matrix}\right), \qquad k = 2,3,\ldots,n,$$

of $M_1(\alpha, n)$ are positive. In fact, proceeding as in the case of $A_{1,n}(R)$, it is seen that $m_{1,k}$ is equal to

$$R^{n-2}(R-1)^k(R^{n-1}+R^{n-2})^{k-2}\left(R^n + 2\sum_{\nu=0}^{n-1} R^\nu - k\right) > 0.$$

By Lemmas 2 and 3 it follows that if $(R^n - 1)|\alpha| < \psi_n(R)$, then

$$\{(R^n-1)+(R^{n-1}-1)z+\cdots+(R-1)z^{n-1}+(R^n-1)\alpha z^n\}/(R^n-1) \in \mathscr{B}_n^0.$$

This, in conjunction with (2.2), implies that (1.3) holds for all $R > 1$ and all $P \in \mathscr{P}_n$.

On the other hand, we note that for every $\sigma > \psi_n(R)$ there corresponds a complex number α_σ, with $|\alpha_\sigma| = \sigma$, such that $f_1(\alpha_\sigma) < 0$, so $\det(M_1(\alpha_\sigma, n)) < 0$. From Lemma 3 it follows that

$$\{(R^n-1)+(R^{n-1}-1)z+\cdots+(R-1)z^{n-1}+(R^n-1)\alpha_\sigma z^n\}/(R^n-1) \notin \mathscr{B}_n^0,$$

i.e. for each given $R > 1$ *there exists* a polynomial $P \in \mathscr{P}_n$ such that

$$\|P(Rz) - P(z) + \alpha_\sigma P(0)\| > (R^n - 1)\|P\|,$$

and a fortiori

$$\|P(Rz) - P(z)\| + \sigma|P(0)| > (R^n - 1)\|P\|.$$

With this, Theorem 1 is proved for $n \geqslant 2$.

Proof of Theorem 2. Let $R > 1$ and note that

$$\frac{1}{R^n}\{M_P(R) + (R^n - R^{n-2})|P(0)|\} = \left\|\left\{\sum_{k=0}^{n} R^{k-n}z^k + (1 - R^{-2})e^{i\gamma}\right\} * P(z)\right\|$$

for some $\gamma \in \mathbf{R}$. In view of (2.2) and Lemma 2, it is therefore enough to prove that

$$q(z;\alpha) := \sum_{k=0}^{n-1}\frac{1}{R^k}z^k + \left(\frac{1}{R^n} + \alpha\right)z^n \in \mathscr{B}_n^0 \Leftrightarrow |\alpha| \leqslant 1 - \frac{1}{R^2}.$$

Accordingly we study the definiteness of

$$M_2(\alpha, n) := M\begin{pmatrix} & \frac{1}{R}, \frac{1}{R^2}, \dots, \frac{1}{R^{n-1}}, \frac{1}{R^n} + \alpha \\ 1, & \\ & \frac{1}{R}, \frac{1}{R^2}, \dots, \frac{1}{R^{n-1}}, \frac{1}{R^n} + \bar{\alpha} \end{pmatrix}. \tag{3.2}$$

Let $\alpha_1 := \alpha + 1/R^n$. Then, as in (3.1), the determinant of $M_2(\alpha, n)$ can be developed in the form

$$\det(M_2(\alpha, n)) = C_{2,n}(R) + (-1)^n 2B_{2,n}(R)\operatorname{Re}(\alpha_1) - A_{2,n}(R)|\alpha_1|^2,$$

where

$$A_{2,n}(R) := D\begin{pmatrix} & \frac{1}{R}, \frac{1}{R^2}, \dots, \frac{1}{R^{n-2}} \\ 1, & \\ & \frac{1}{R}, \frac{1}{R^2}, \dots, \frac{1}{R^{n-2}} \end{pmatrix},$$

$$B_{2,n}(R) := D\begin{pmatrix} & 1, \frac{1}{R}, \dots, \frac{1}{R^{n-3}}, \frac{1}{R^{n-2}} \\ \frac{1}{R}, & \\ & \frac{1}{R^2}, \frac{1}{R^3}, \dots, \frac{1}{R^{n-1}}, 0 \end{pmatrix},$$

and

$$C_{2,n}(R) := D\begin{pmatrix} & \frac{1}{R}, \frac{1}{R^2}, \dots, \frac{1}{R^{n-1}}, 0 \\ 1, & \\ & \frac{1}{R}, \frac{1}{R^2}, \dots, \frac{1}{R^{n-1}}, 0 \end{pmatrix}.$$

It turns out that for $n \geqslant 2$,

$$A_{2,n}(R) = \left(1 - \frac{1}{R^2}\right)^{n-2}, \quad B_{2,n}(R) = (-1)^n \frac{1}{R^n}\left(1 - \frac{1}{R^2}\right)^{n-2},$$

and

$$C_{2,n}(R) = \left(1 - \frac{1}{R^2}\right)^n - \frac{1}{R^{2n}}\left(1 - \frac{1}{R^2}\right)^{n-2}.$$

Hence,

$$\det(M_2(\alpha, n)) = \left(1 - \frac{1}{R^2}\right)^{n-2}\left\{\left(1 - \frac{1}{R^2}\right)^2 - |\alpha|^2\right\}, \tag{3.3}$$

so that $\det(M_2(\alpha, n)) > 0$ if and only if $|\alpha| < 1 - 1/R^2$. Further, we observe that the leading principal minors of $M_2(\alpha, n)$, namely

$$(3.4)\quad m_{2,1} := 1, m_{2,k} := D\begin{pmatrix} & \frac{1}{R},\dots,\frac{1}{R^{k-1}} \\ 1, & \\ & \frac{1}{R},\dots,\frac{1}{R^{k-1}} \end{pmatrix} = \left(1 - \frac{1}{R^2}\right)^{k-1}, \qquad 2 \leqslant k \leqslant n,$$

are all positive for $R > 1$. Thus we have proved that $q(z; \alpha) \in \mathscr{B}_n^0$ if $|\alpha| < 1 - 1/R^2$ and *does not* belong to $\mathscr{B}_n^0$ if $|\alpha| > 1 - 1/R^2$. Hence, Theorem 2 holds.

4. Dependence on the other coefficients. As we have mentioned earlier there is strict inequality in (1.1) as well as in (1.2) if any of the coefficients a_ν $(0 \leqslant \nu \leqslant n-1)$ are different from zero. The dependence of $\|P'\|$ on $|a_1|$ is given in Theorem 3, whereas Theorem 4 contains the corresponding refinement of (1.2).

THEOREM 3. *For $P \in \mathscr{P}_n$,*

$$(4.1)\qquad \|P'\| + c_n|P'(0)| \leqslant n\|P\|,$$

where $c_1 = 0$, $c_2 = \sqrt{2} - 1$, $c_3 = 1/\sqrt{2}$, whereas for $n \geqslant 4$, c_n is the unique root of the equation

$$(4.2)\qquad 16n - 8(3n+2)x^2 - 16x^3 + (n+4)x^4 = 0$$

lying in $(0,1)$. The coefficient of $|P'(0)|$ is the best possible for each n.

PROOF. It is clear that

$$\|P'\| + c_n|P'(0)| = \sup_{|\alpha| < c_n} \left\| \left\{ \sum_{k=2}^{n} kz^k + (1+\bar{\alpha})z \right\} * P(z) \right\|.$$

So again using (2.2) and Lemmas 2 and 3, we are led to study the definiteness of the hermitian matrix $\xi(1+\alpha, 1+\bar{\alpha})$, where

$$\xi(x, y) := M\begin{pmatrix} & n-1, n-2, \dots, 2, x, 0 \\ n, & \\ & n-1, n-2, \dots, 2, y, 0 \end{pmatrix}.$$

The leading principal minors of $\xi(1+\alpha, 1+\bar{\alpha})$ of order $n-1$ or less, namely

$$m_{3,1} = n,$$

$$m_{3,k} := D\begin{pmatrix} & n-1, \dots, n-k+1 \\ n, & \\ & n-1, \dots, n-k+1 \end{pmatrix} = 2^{k-2}(2n-k+1), \qquad 2 \leqslant k \leqslant n-1,$$

are, of course, all positive.

In order to study the leading principal minor of order n we consider the determinant

$$G(x, y) := D\begin{pmatrix} & n-1, n-2, \dots, 2, x \\ n, & \\ & n-1, n-2, \dots, 2, y \end{pmatrix}.$$

Now according to a well-known formula for the derivative of a determinant (see [**2**, pp. 25–26]), we have

$$\frac{\partial G}{\partial x} = (-1)^{n+1} D\begin{pmatrix} n-1, & \begin{matrix} n, n-1, \dots, 4, 3 \\ n-2, n-3, \dots, 2, y \end{matrix} \end{pmatrix}, \quad \frac{\partial^2 G}{\partial x^2} = 0,$$

$$\frac{\partial^2 G}{\partial x \partial y} = -D\begin{pmatrix} n, & \begin{matrix} n-1, n-2, \dots, 4, 3 \\ n-1, n-2, \dots, 4, 3 \end{matrix} \end{pmatrix}, \quad \text{etc.}$$

Evaluating the determinants involved, we see that

$$G(1,1) = (n+1)2^{n-2}, \quad \frac{\partial G}{\partial x}(1,1) = \frac{\partial G}{\partial y}(1,1) = 2^{n-3},$$

$$\frac{\partial^2 G}{\partial x^2}(1,1) = \frac{\partial^2 G}{\partial y^2}(1,1) = 0, \quad \frac{\partial^2 G}{\partial x \partial y}(1,1) = -(n+3)2^{n-4},$$

whereas all the higher-order partial derivatives vanish. Thus,

$$G(x, y) = (n+1)2^{n-2} + 2^{n-3}(x-1) + 2^{n-3}(y-1) \\ -(n+3)2^{n-4}(x-1)(y-1),$$

and, in particular,

$$G(1+\alpha, 1+\bar{\alpha}) = (n+1)2^{n-2} + 2^{n-2}\operatorname{Re}\alpha - (n+3)2^{n-4}|\alpha|^2.$$

Hence the nth principal minor of $\xi(1+\alpha, 1+\bar{\alpha})$ is positive if

$$(4.3) \qquad |\alpha| < 2(n+1)/(n+3) =: c_n^*.$$

Next, let us denote the determinant of $\xi(x, y)$ by $X(x, y)$, so that $X(y, x) = X(x, y)$. Again using the formula for the derivative of a determinant, we obtain

$$\frac{\partial X}{\partial x} = (-1)^{n+1} D_{1;n}\begin{pmatrix} n, & \begin{matrix} n-1, n-2, \dots, 3, 2, x, 0 \\ n-1, n-2, \dots, 3, 2, y, 0 \end{matrix} \end{pmatrix}$$

$$+ (-1)^{n+1} D_{2;n+1}\begin{pmatrix} n, & \begin{matrix} n-1, n-2, \dots, 3, 2, x, 0 \\ n-1, n-2, \dots, 3, 2, y, 0 \end{matrix} \end{pmatrix},$$

$$\frac{\partial^2 X}{\partial x^2} = 2D\begin{pmatrix} n-2, & \begin{matrix} n-1, n, n-1, \dots, 6, 5, 4 \\ n-3, n-4, n-5, \dots, 2, y, 0 \end{matrix} \end{pmatrix},$$

$$\frac{\partial^2 X}{\partial x \partial y} = -d_1(x, y) - d_2(x, y) - d_3(x, y) - d_4(x, y),$$

where

$$d_1(x, y) := D_{n-1,n+1;n-1,n+1}\begin{pmatrix} n, & n-1, n-2, \ldots, 2, x, 0 \\ & n-1, n-2, \ldots, 2, y, 0 \end{pmatrix},$$

$$d_2(x, y) := D_{1,n+1;2,n}\begin{pmatrix} n, & n-1, n-2, \ldots, 2, x, 0 \\ & n-1, n-2, \ldots, 2, y, 0 \end{pmatrix},$$

$$d_3(x, y) := D_{2,n;1,n+1}\begin{pmatrix} n, & n-1, n-2, \ldots, 2, x, 0 \\ & n-1, n-2, \ldots, 2, y, 0 \end{pmatrix},$$

and

$$d_4(x, y) := D_{2,n+1;2,n+1}\begin{pmatrix} n, & n-1, n-2, \ldots, 2, x, 0 \\ & n-1, n-2, \ldots, 2, y, 0 \end{pmatrix}.$$

Further,

$$\frac{\partial^3 X}{\partial x^3} = \frac{\partial^4 X}{\partial x^4} = \frac{\partial^4 X}{\partial x^3 \partial y} = 0,$$

whereas for $n \geqslant 4$,

$$\begin{aligned}\frac{\partial^3 X}{\partial x^2 \partial y} = {} & 2(-1)^{n+1} D_{1,n-1,n+1;n-1,n,n+1}\begin{pmatrix} n, & n-1, n-2, \ldots, 2, x, 0 \\ & n-1, n-2, \ldots, 2, y, 0 \end{pmatrix} \\ & + 2(-1)^{n+1} D_{1,2,n+1;2,n,n+1}\begin{pmatrix} n, & n-1, n-2, \ldots, 2, x, 0 \\ & n-1, n-2, \ldots, 2, y, 0 \end{pmatrix},\end{aligned}$$

and

$$\frac{\partial^4 X}{\partial x^2 \partial y^2} = 4D\begin{pmatrix} n, & n-1, \ldots, 5, 4 \\ & n-1, \ldots, 5, 4 \end{pmatrix} = (n+4)2^{n-3}.$$

It is not difficult to see that $X(1,1) = n2^{n-1}$ and the two determinants involved in $(\partial X/\partial x)(1,1)$ are separately equal to zero. Also, $(\partial^2 X/\partial x^2)(1,1)$ is easily seen to be zero. Besides, $d_1(1,1) = d_4(1,1) = (n+1)2^{n-2}$ and $d_2(1,1) = d_3(1,1) = n2^{n-3}$, so that $(\partial^2 X/\partial x \partial y)(1,1) = -(3n+2)2^{n-2}$. Since $(\partial^3 X/\partial x^2 \partial y)(1,1)$ turns out to be equal to -2^{n-1}, and the partial derivatives of order higher than four are all zero, we obtain

$$\begin{aligned}X(x, y) = {} & n2^{n-1} - (3n+2)2^{n-2}(x-1)(y-1) - 2^{n-2}(x-1)^2(y-1) \\ & -2^{n-2}(x-1)(y-1)^2 + (n+4)2^{n-5}(x-1)^2(y-1)^2,\end{aligned}$$

so

$$X(1+\alpha, 1+\bar{\alpha}) = n2^{n-1} - (3n+2)2^{n-2}|\alpha|^2 - 2^{n-1}|\alpha|^2 \operatorname{Re}\alpha + (n+4)2^{n-5}|\alpha|^4.$$

For $n \geqslant 4$ we are thus led to equation (4.2). Its smallest positive root being c_n by hypothesis, it follows that $X(1+\alpha, 1+\bar{\alpha}) > 0$ for $0 \leqslant |\alpha| < c_n$. By Descartes' rule of signs, (4.2) cannot have more than two positive roots. Since the expression on its left side is positive for $x = 0$, negative for $x = 1$, and positive for all large values of x, it has just one root in $(0,1)$, which we call c_n, and another in $(1,\infty)$. Referring to (4.3) we see that $c_n < c_n^*$, so for $n \geqslant 4$ the determinant of $\xi(1+\alpha, 1+\bar{\alpha})$, as well as the determinants of the other leading principal minors, is positive if $|\alpha| < c_n$. Hence, by Lemma 3, $\{n + (n-1)z + \cdots + 2z^{n-2} + (1+\alpha)z^{n-1}\}/n \in \mathscr{B}_n^0$ if $|\alpha| < c_n$, which proves (4.1) for $n \geqslant 4$. Besides, referring to equation (4.2) we note that to every $\tau > c_n$ there corresponds a complex number α_τ, with $|\alpha_\tau| = \tau$, such that $X(1+\alpha_\tau, 1+\bar{\alpha}_\tau) < 0$, so, again by Lemma 3,

$$\{n + (n-1)z + \cdots + 2z^{n-2} + (1+\alpha_\tau)z^{n-1}\}/n \notin \mathscr{B}_n^0.$$

In other words, for each given $\tau > c_n$ there exists a polynomial $P \in \mathscr{P}_n$ such that

$$\|P'\| + \tau|P'(0)| \geqslant \|P'(z) + \alpha_\tau P'(0)\| > n\|P\|.$$

With this, Theorem 3 is proved for $n \geqslant 4$.

The example $P(z) := z$ shows that c_1 must be zero. In the case $n = 2$ we have to study the matrix

$$M\begin{pmatrix} & 1+\alpha, 0 \\ 2, & \\ & 1+\bar{\alpha}, 0 \end{pmatrix},$$

and it is readily seen that (4.1) does hold with $c_2 = \sqrt{2} - 1$. In order to prove (4.1) in the case $n = 3$, we need to consider the matrix

$$\mathscr{M}(\alpha,\gamma) := M\begin{pmatrix} & 2, 1+|\alpha|e^{i\gamma}, 0 \\ 3, & \\ & 2, 1+|\alpha|e^{-i\gamma}, 0 \end{pmatrix}.$$

The leading principal minor of order 3 of $\mathscr{M}(\alpha,\gamma)$ is easily seen to be positive for $|\alpha| < 4/3$, whereas $\det(\mathscr{M}(\alpha,\gamma))$, being equal to

$$12 - 22|\alpha|^2 + 2|\alpha|^2\cos 2\gamma + 4|\alpha|^3\cos\gamma + |\alpha|^4,$$

is positive for *all* $\gamma \in \mathbf{R}$, provided that $|\alpha| < 1/\sqrt{2}$. Since the determinant is negative for some $\gamma \in \mathbf{R}$ if $|\alpha| > 1/\sqrt{2}$ it follows that (4 1) holds with

$$c_3 = \min(4/3, 1/\sqrt{2}) = 1/\sqrt{2}.$$

THEOREM 4. *Let $R \geqslant 1$. If we denote by $\varphi_n(R)$ the best possible constant such that*

$$M_P(R) + \varphi_n(R)|P'(0)| \leqslant R^n\|P\| \quad \text{for all } P \in \mathscr{P}_n, \tag{4.4}$$

then

$$\varphi_1(R) = 0, \quad \varphi_2(R) = R\left(\sqrt{(R^2+1)/2} - 1\right),$$

$$\varphi_3(R) = (R^2 - R)\left(\sqrt{R^2+R+1} - 1\right),$$

whereas, for $n \geqslant 4$,

$$\varphi_n(R) = (R^{n-1} - R^{n-3})\left(R^2 + 2 - 2\sqrt{R^2+1}\right)^{1/2}.$$

PROOF. Let $R > 1$ and note that

$$\frac{1}{R^n}\{M_P(R) + \varphi_n(R)|P'(0)|\}$$

$$= \left\|\left\{\sum_{k=2}^{n} \frac{1}{R^{n-k}} z^k + \left(\frac{1}{R^{n-1}} + \frac{1}{R^n}\varphi_n(R)\right)e^{i\gamma}z + \frac{1}{R^n}\right\} * P(z)\right\|$$

for some $\gamma \in \mathbf{R}$. For reasons which are now very familar, we set

$$\zeta(x, y) := M\begin{pmatrix} & \frac{1}{R}, \frac{1}{R^2}, \ldots, \frac{1}{R^{n-2}}, \frac{1}{R^{n-1}} + x, \frac{1}{R^n} \\ 1, & \\ & \frac{1}{R}, \frac{1}{R^2}, \ldots, \frac{1}{R^{n-2}}, \frac{1}{R^{n-1}} + y, \frac{1}{R^n} \end{pmatrix}$$

and study the definiteness of $\zeta(\alpha, \bar{\alpha})$.

Let us denote the determinant of $\zeta(x, y)$ by $Z(x, y)$, so that $Z(y, x) = Z(x, y)$. Using the formula for the derivative of a determinant, we obtain for $n \geqslant 4$ (in the case $n = 4$ some of the determinants need to be interpreted in an obvious way):

$$\frac{\partial Z}{\partial x} = (-1)^{n+1}(D_{1,1}(x, y) + D_{1,2}(x, y)),$$

where

$$D_{1,1}(x, y) := D_{1;n}\begin{pmatrix} & \frac{1}{R}, \frac{1}{R^2}, \ldots, \frac{1}{R^{n-2}}, \frac{1}{R^{n-1}} + x, \frac{1}{R^n} \\ 1, & \\ & \frac{1}{R}, \frac{1}{R^2}, \ldots, \frac{1}{R^{n-2}}, \frac{1}{R^{n-1}} + y, \frac{1}{R^n} \end{pmatrix},$$

$$D_{1,2}(x, y) := D_{2;n+1}\begin{pmatrix} & \frac{1}{R}, \frac{1}{R^2}, \ldots, \frac{1}{R^{n-2}}, \frac{1}{R^{n-1}} + x, \frac{1}{R^n} \\ 1, & \\ & \frac{1}{R}, \frac{1}{R^2}, \ldots, \frac{1}{R^{n-2}}, \frac{1}{R^{n-1}} + y, \frac{1}{R^n} \end{pmatrix};$$

$$\frac{\partial^2 Z}{\partial x^2} = 2D_2(x, y) := 2D_{1,2;n,n+1}\begin{pmatrix} & \frac{1}{R}, \frac{1}{R^2}, \ldots, \frac{1}{R^{n-2}}, \frac{1}{R^{n-1}} + x, \frac{1}{R^n} \\ 1, & \\ & \frac{1}{R}, \frac{1}{R^2}, \ldots, \frac{1}{R^{n-2}}, \frac{1}{R^{n-1}} + y, \frac{1}{R^n} \end{pmatrix},$$

$$\frac{\partial^2 Z}{\partial x \partial y} = -(D_{2,1}(x, y) + D_{2,2}(x, y) + D_{2,3}(x, y) + D_{2,4}(x, y)),$$

where

$$D_{2,1}(x, y) := D_{n-1,n+1;n-1,n+1}\begin{pmatrix} & \frac{1}{R}, \frac{1}{R^2}, \ldots, \frac{1}{R^{n-2}}, \frac{1}{R^{n-1}} + x, \frac{1}{R^n} \\ 1, & \\ & \frac{1}{R}, \frac{1}{R^2}, \ldots, \frac{1}{R^{n-2}}, \frac{1}{R^{n-1}} + y, \frac{1}{R^n} \end{pmatrix},$$

$$D_{2,2}(x,y) := D_{1,n+1;2,n}\begin{pmatrix} 1, & \begin{matrix} \frac{1}{R}, \frac{1}{R^2}, \ldots, \frac{1}{R^{n-2}}, \frac{1}{R^{n-1}} + x, \frac{1}{R^n} \\ \\ \frac{1}{R}, \frac{1}{R^2}, \ldots, \frac{1}{R^{n-2}}, \frac{1}{R^{n-1}} + y, \frac{1}{R^n} \end{matrix} \end{pmatrix},$$

$$D_{2,3}(x,y) := D_{2,n;1,n+1}\begin{pmatrix} 1, & \begin{matrix} \frac{1}{R}, \frac{1}{R^2}, \ldots, \frac{1}{R^{n-2}}, \frac{1}{R^{n-1}} + x, \frac{1}{R^n} \\ \\ \frac{1}{R}, \frac{1}{R^2}, \ldots, \frac{1}{R^{n-2}}, \frac{1}{R^{n-1}} + y, \frac{1}{R^n} \end{matrix} \end{pmatrix},$$

$$D_{2,4}(x,y) := D_{2,n+1;2,n+1}\begin{pmatrix} 1, & \begin{matrix} \frac{1}{R}, \frac{1}{R^2}, \ldots, \frac{1}{R^{n-2}}, \frac{1}{R^{n-1}} + x, \frac{1}{R^n} \\ \\ \frac{1}{R}, \frac{1}{R^2}, \ldots, \frac{1}{R^{n-2}}, \frac{1}{R^{n-1}} + y, \frac{1}{R^n} \end{matrix} \end{pmatrix};$$

$$\frac{\partial^3 Z}{\partial x^3} = \frac{\partial^4 Z}{\partial x^4} = \frac{\partial^4 Z}{\partial x^3 \partial y} = 0,$$

$$\frac{\partial^3 Z}{\partial x^2 \partial y} = 2(-1)^{n+1}(D_{3,1}(x,y) + D_{3,2}(x,y)),$$

where

$$D_{3,1}(x,y) := D_{1,n-1,n+1;n-1,n,n+1}\begin{pmatrix} 1, & \begin{matrix} \frac{1}{R}, \frac{1}{R^2}, \ldots, \frac{1}{R^{n-2}}, \frac{1}{R^{n-1}} + x, \frac{1}{R^n} \\ \\ \frac{1}{R}, \frac{1}{R^2}, \ldots, \frac{1}{R^{n-2}}, \frac{1}{R^{n-1}} + y, \frac{1}{R^n} \end{matrix} \end{pmatrix},$$

$$D_{3,2}(x,y) := D_{1,2,n+1;2,n,n+1}\begin{pmatrix} 1, & \begin{matrix} \frac{1}{R}, \frac{1}{R^2}, \ldots, \frac{1}{R^{n-2}}, \frac{1}{R^{n-1}} + x, \frac{1}{R^n} \\ \\ \frac{1}{R}, \frac{1}{R^2}, \ldots, \frac{1}{R^{n-2}}, \frac{1}{R^{n-1}} + y, \frac{1}{R^n} \end{matrix} \end{pmatrix},$$

and

$$\frac{\partial^4 Z}{\partial x^2 \partial y^2} = 4D_4(x,y) := 4D\begin{pmatrix} 1, & \begin{matrix} \frac{1}{R}, \ldots, \frac{1}{R^{n-4}} \\ \\ \frac{1}{R}, \ldots, \frac{1}{R^{n-4}} \end{matrix} \end{pmatrix}.$$

Now we proceed to evaluate $Z(0,0)$, $(\partial Z/\partial x)(0,0)$, $(\partial Z/\partial y)(0,0)$, etc. First we observe that

$$Z(0,0) = |M_2(0,n)| = (1 - R^{-2})^n.$$

Then *we recall that* $Z(y, x) = Z(x, y)$. The value of the determinant $D_{1,1}(0,0)$ is zero since its first column is $1/R$ times its second column. Also the value of $D_{1,2}(0,0)$ is zero since its nth row is $1/R$ times its $(n-1)$th row. Hence,

$$\frac{\partial Z}{\partial x}(0,0) = \frac{\partial Z}{\partial y}(0,0) = 0.$$

Also,

$$\frac{\partial^2 Z}{\partial x^2}(0,0) = \frac{\partial^2 Z}{\partial y^2}(0,0) = 0$$

since the $(n-1)$th row of $D_2(0,0)$ is $1/R$ times its $(n-2)$th row. In order to evaluate $D_{2,1}(0,0)$, we subtract $1/R$ times the $(k+1)$th row from the kth row for $k = 1, 2, \ldots, n-2$ and readily obtain

$$D_{2,1}(0,0) = (1 + R^{-2})(1 - R^{-2})^{n-2}.$$

Precisely the same operation gives us

$$D_{2,2}(0,0) = R^{-2}(1 - R^{-2})^{n-2}.$$

Also,

$$D_{2,3}(0,0) = R^{-2}(1 - R^{-2})^{n-2},$$

as is seen by subtracting $1/R$ times the $(k+1)$th column from the kth column for $k = 1, 2, \ldots, n-2$. Subtracting $1/R$ times the kth row from the $(k+1)$th row for $k = n-2, n-3, \ldots, 1$, we see at once that

$$D_{2,4}(0,0) = (1 + R^{-2})(1 - R^{-2})^{n-2}.$$

Thus,

$$\frac{\partial^2 Z}{\partial x \partial y}(0,0) = -2(1 + 2R^{-2})(1 - R^{-2})^{n-2}.$$

The value of the determinant $D_{3,1}(0,0)$ is zero, since its first row is $1/R$ times its second row. Also, the value of $D_{3,2}(0,0)$ is zero since its $(n-2)$th row is $1/R$ times its $(n-3)$th row. Therefore,

$$\frac{\partial^3 Z}{\partial x^2 \partial y}(0,0) = \frac{\partial^3 Z}{\partial x \partial y^2}(0,0) = 0.$$

Finally, we observe that $D_4(0,0) = m_{2,n-3}$ (see (3.4)), so

$$\frac{\partial^4 Z}{\partial x^2 \partial y^2}(0,0) = 4(1 - R^{-2})^{n-4}.$$

Since the partial derivatives of order higher than four are all zero, we obtain

$$Z(x, y) = (1 - R^{-2})^n - 2(1 + 2R^{-2})(1 - R^{-2})^{n-2}xy + (1 - R^{-2})^{n-4}x^2y^2,$$

and, in particular,

$$Z(\alpha, \bar{\alpha}) = (1 - R^{-2})^{n-4}\left\{(1 - R^{-2})^4 - 2(1 + 2R^{-2})(1 - R^{-2})^2|\alpha|^2 + |\alpha|^4\right\}.$$

Its leading principal minors of order k $(1 \leqslant k \leqslant n-1)$, being equal to $m_{2,1}$, $m_{2,2},\ldots,m_{2,n-1}$, respectively, are all positive for $R > 1$. Hence, $\zeta(\alpha, \bar{\alpha})$ is positive definite for those values of α for which $Z(\alpha, \bar{\alpha})$ and its leading principal minor of order n are *both* positive, and it is *not even semidefinite* for those values of α for which *at least one* of them is negative. Referring to (3.2), we recognize that the leading principal minor of order n of $Z(\alpha, \bar{\alpha})$ is nothing but $\det(M_2(\alpha, n-1))$ and, as such, is equal to (see (3.3))

$$(1 - R^{-2})^{n-3}\left\{(1 - R^{-2})^2 - |\alpha|^2\right\}.$$

This and the above expression for $Z(\alpha, \bar{\alpha})$ readily lead us to the desired result for $n \geqslant 4$.

The cases $n = 2$ and $n = 3$ can be handled in the same way. However, in the case $n = 3$ the fourth degree equation

$$(1 - R^{-2})^3 - 2(1 - R^{-2})(1 + 2R^{-2})x^2 - 4R^{-2}x^3 + x^4 = 0$$

must be solved. For this we may make the substitution $x = (1 - R^{-2})y$, which gives $(1 - y^2)^2 = R^{-2}y^2(y + 2)^2$.

The case $n = 1$ is trivial.

The method can also be used to study the dependence of $\|P'\|$ and $M_P(R)$ on an arbitrary a_ν $(2 \leqslant \nu \leqslant n-1)$, but we will not do this because the calculations become very long. We will, however, prove the following result, which has various interesting consequences.

THEOREM 5. *If $P(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree at most n, then*

$$\|zP'(z) - \tfrac{1}{2}a_n z^n + \tfrac{1}{4}a_0\| \leqslant \left(n - \tfrac{1}{2}\right)\|P\| - \gamma_n|a_0|, \tag{4.5}$$

where

$$\gamma_n := \begin{cases} 1/4, & n \equiv 1 \pmod 2,\ n \geqslant 1, \\ 5/12, & n = 2, \\ 11/20, & n = 4, \\ (n+3)/4(n-1), & n \equiv 0 \pmod 2,\ n \geqslant 6. \end{cases} \tag{4.6}$$

The constant γ_n is the best possible for each n.

PROOF. Clearly,

$$\left\|zP'(z) - \frac{1}{2}a_n z^n + \frac{1}{4}a_0\right\| + \gamma_n|a_0|$$
$$= \left\|\left\{\left(\gamma_n e^{i\beta} + \frac{1}{4}\right) + \sum_{k=1}^{n-1} kz^k + \left(n - \frac{1}{2}\right)z^n\right\} * P(z)\right\|$$

for some $\beta \in \mathbf{R}$. In view of (2.2) it is therefore enough to prove that

$$f(z;\alpha) := 1 + 2\sum_{k=1}^{n-1} \frac{n-k}{2n-1} z^k + \frac{4\alpha + 1}{2(2n-1)} z^n$$

belongs to $\mathscr{B}_n^0$ for $|\alpha| < \gamma_n$, whereas to every $t > \gamma_n$ there corresponds a complex number α_t with $|\alpha_t| = t$ such that $f(z; \alpha_t) \notin \mathscr{B}_n^0$. As such, we consider the matrix

$$\eta(\alpha) := M\begin{pmatrix} & c_1, c_2, \ldots, c_n \\ c_0, & \\ & \bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n \end{pmatrix},$$

where $c_0 = n - 1/2$, $c_k = n - k$ for $1 \leqslant k \leqslant n - 1$, and $c_n = \alpha + 1/4$. The determinant of $\eta(\alpha)$ can be developed in the form

$$\det(\eta(\alpha)) = C_n^* + 2(-1)^n B_n^* \operatorname{Re}(\alpha + 1/4) - A_n^* |\alpha + 1/4|^2,$$

where

$$A_n^* := D\begin{pmatrix} & n-1, n-2, \ldots, 2 \\ n - \frac{1}{2}, & \\ & n-1, n-2, \ldots, 2 \end{pmatrix},$$

$$B_n^* := D\begin{pmatrix} & n-\frac{1}{2}, n-1, n-2, \ldots, 3, 2 \\ n-1, & \\ & n-2, n-3, n-4, \ldots, 1, 0 \end{pmatrix},$$

and

$$C_n^* := D\begin{pmatrix} & n-1, n-2, \ldots, 2, 1, 0 \\ n - \frac{1}{2}, & \\ & n-1, n-2, \ldots, 2, 1, 0 \end{pmatrix}.$$

In order to evaluate A_n^* we perform the following operations one after the other:

(a) Subtract the $(k+1)$th row from the kth row for $k = 1, 2, \ldots, n-2$; then subtract the (new) kth row from the (new) $(k+1)$th row for $k = 1, 2, \ldots, n-2$.

(b) Subtract from the last row $2k$ times the $(2k)$th row for $k = 1, 2, \ldots, [(n-2)/2]$.

(c) Subtract the first row from the second.

(d) Subtract from the third row $1/3$ times the second row and then the (new) third row from the fourth; subtract from the fifth row $3/5$ times the (new) fourth row and the (new) fifth row from the sixth. Continuing in this way, we subtract from the $(2k+1)$th row $(2k-1)/(2k+1)$ times the (new) $(2k)$th row and the (new) $(2k+1)$th row from the $(2k+2)$th row. This is done for $k = 1, 2, \ldots, [(n-2)/2]$. Finally, in the case of odd n we subtract from the $(n-2)$th row $(n-4)/(n-2)$ times the (new) $(n-3)$th row, and then subtract from the $(n-1)$th row n times the (new) $(n-2)$th row. We thus end up with a determinant whose elements below the main diagonal are zero. The elements along the main diagonal turn out to be

$$\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{5}{6}, \frac{1}{2}, \frac{7}{10}, \ldots, \frac{1}{2}, \frac{n-1}{2(n-3)}, \frac{n+1}{2} \quad \text{if } n \text{ is even,}$$

and

$$\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{5}{6}, \frac{1}{2}, \frac{7}{10}, \ldots, \frac{n-3}{2(n-5)}, \frac{1}{2}, \frac{n^2}{2(n-2)} \quad \text{if } n \text{ is odd.}$$

Consequently,

$$A_n^* = \begin{cases} (n^2 - 1)/2^{n-1} & \text{if } n \text{ is even}, \\ n^2/2^{n-1} & \text{if } n \text{ is odd}. \end{cases}$$

In order to evaluate B_n^* we perform the following succession of operations:

(i) Subtract the $(k+1)$th row from the kth row for $k = 1, 2, \ldots, n-1$.

(ii) Subtract the kth row from the $(k+1)$th row for $k = 1, 2, \ldots, n-2$.

(iii) Subtract from the last row $2k$ times the $(2k)$th row for $k = 1, 2, \ldots, [(n-1)/2]$.

We end up with a determinant whose elements below the main diagonal are all zero. The elements along the (main) diagonal itself turn out to be

$$1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{n}{2} \quad \text{if } n \text{ is even},$$

and

$$1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, 0 \quad \text{if } n \text{ is odd}.$$

As such,

$$B_n^* = \begin{cases} n(1/2)^{n-1} & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}. \end{cases}$$

In order to evaluate C_n^* we perform the following operations successively:

(A) Subtract the $(k+1)$th row from the kth row for $k = 1, 2, \ldots, n$; then subtract the (new) kth row from the $(k+1)$th row for $k = 1, 2, \ldots, n-1$.

(B) Subtract from the last row $2k$ times the $(2k+1)$th row for $k = 1, 2, \ldots, [(n-1)/2]$.

(C) Subtract the first row from the second.

(D) Subtract from the third row $1/3$ times the second row and then the (new) third row from the fourth; subtract from the fifth row $3/5$ times the (new) fourth row and the (new) fifth row from the sixth. Continuing in this way, we subtract from the $(2k+1)$th row $(2k-1)/(2k+1)$ times the (new) $(2k)$th row and the (new) $(2k+1)$th row from the $(2k+2)$th row. This is done for $k = 1, 2, \ldots, [(n-2)/2]$. Then in the case of even n we subtract from the $(n+1)$th row $n(n-1)/(n+1)$ times the (new) nth row, whereas, in the case of odd n we subtract from the nth row $(n-2)/n$ times the (new) $(n-1)$th row. In the resulting determinant all the elements below the main diagonal are zero. The elements along the main diagonal are

$$\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{5}{6}, \ldots, \frac{n+1}{2(n-1)}, \frac{n-1}{2} \quad \text{if } n \text{ is even},$$

and

$$\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{5}{6}, \ldots, \frac{n}{2(n-2)}, \frac{n}{2} \quad \text{if } n \text{ is odd}.$$

Hence,

$$C_n^* = \begin{cases} (n^2-1)(1/2)^{n+1} & \text{if } n \text{ is even,} \\ n^2(1/2)^{n+1} & \text{if } n \text{ is odd.} \end{cases}$$

Before studying the determinant of $\eta(\alpha)$ any further, we wish to point out that all its leading principal minors of order $\leqslant n$ are positive. In fact,

$$D\begin{pmatrix} n-\frac{1}{2}, & n-1,\ldots,n-k \\ & n-1,\ldots,n-k \end{pmatrix} = \begin{cases} \dfrac{2(k+1)n-(k+1)^2}{2^{k+1}}, & k \equiv 0 \pmod 2, \\ \dfrac{2(k+1)n-(k^2+2k)}{2^{k+1}}, & k \equiv 1 \pmod 2, \end{cases}$$

$0 \leqslant k \leqslant n-1$, where for $k=0$, the left side is to be interpreted as $n-1/2$. As such, $f(z;\alpha) \in \mathscr{B}_n^0$ if $\det(\eta(\alpha)) > 0$, and $f(z;\alpha) \notin \mathscr{B}_n^0$ if $\det(\eta(\alpha)) < 0$.

Now let n be odd. Using the values of A_n^*, B_n^*, and C_n^* calculated above, we see that

$$\det(\eta(\alpha)) = n^2(1/2)^{n-1}(1/4 - |\alpha+1/4|^2) > 0$$

if $|\alpha+1/4| < 1/2$, and so certainly if $|\alpha| < 1/4$. On the other hand, if $\alpha > 1/4$ then $\alpha + 1/4 > 1/2$, so $\det(\eta(\alpha)) < 0$. Thus $f(z;\alpha)$ belongs to $\mathscr{B}_n^0$ if $|\alpha| < 1/4$ and does not belong to $\mathscr{B}_n^0$ if $\alpha > 1/4$. This proves Theorem 5 for odd values of n.

Finally, let n be even. In this case $\det(\eta(\alpha)) > 0$ if

$$f_2(\alpha) := 4(n^2-1) + 8n\,\mathrm{Re}(4\alpha+1) - (n^2-1)|4\alpha+1|^2 > 0,$$

and so certainly if

$$(3n^2+8n-3) - 8|n^2-4n-1|\,|\alpha| - 16(n^2-1)|\alpha|^2 > 0,$$

i.e., if

$$|\alpha| \leqslant \gamma_n := \begin{cases} 5/12 & \text{in case } n=2, \\ 11/20 & \text{in case } n=4, \\ (n+3)/4(n-1) & \text{in case } n \equiv 0 \pmod 2,\ n \geqslant 6. \end{cases}$$

On the other hand, for every $t > \gamma_n$ there corresponds a complex number α_t, with $|\alpha_t| = t$, such that $f_2(\alpha_t) < 0$ and, in turn, $\det(\eta(\alpha_t)) < 0$, i.e., $f(z;\alpha_t) \notin \mathscr{B}_n^0$. This settles the case of even n and the proof of Theorem 5 is complete.

5. Polynomials in $\mathscr{R}$ and the class $\mathscr{B}_N^0$. We know how polynomials in $\mathscr{B}_n^0$ can give rise to interesting inequalities. It is therefore pertinent to find out ways of manufacturing such polynomials. The next result shows how polynomials in $\mathscr{R}$ can be used for this purpose. We recall that f is said to belong to $\mathscr{R}$ if it is analytic, with $\mathrm{Re}\,f(z) > 1/2$, in $|z| < 1$ and $f(0) = 1$.

THEOREM 6. *Let $Q \in \mathscr{P}_n$. If $Q \in \mathscr{R}$, then for all positive integers m and all $\alpha \in \mathbf{C}$, such that $|\alpha| \leqslant 1$, we have*

$$Q + \alpha z^m \tilde{Q} \in \mathscr{B}^0_{n+m}. \tag{5.1}$$

The proof depends on the following lemma, which is, as we shall see, a result of independent interest with a variety of applications.

LEMMA 4. *Let $Q \in \mathscr{P}_n$, $Q(0) = 1$, $N > n$. Then for $\gamma \in \mathbf{R}$ and $P \in \mathscr{P}_N$,*

$$\begin{aligned} P(z) * \left(Q(z) + e^{i\gamma} z^N \overline{Q(1/\bar{z})}\right) \\ = \frac{1}{N} \sum_{k=1}^{N} \left(2 \operatorname{Re} Q\left(e^{-(2k\pi+\gamma)i/N}\right) - 1\right) P\left(z e^{(2k\pi+\gamma)i/N}\right). \end{aligned} \tag{5.2}$$

PROOF. Consider the integral

$$I_\rho(z) := \frac{1}{2\pi i} \int_{|\zeta|=\rho} \frac{Q(\zeta) + e^{i\gamma} \zeta^N \left(\overline{Q(1/\bar{\zeta})} - 1\right)}{(\zeta - z)(\zeta^N - e^{-i\gamma})} d\zeta.$$

Since $Q \in \mathscr{P}_n$, $Q(0) = 1$, and $N > n$, the polynomial $Q(\zeta) + e^{i\gamma}\zeta^N(\overline{Q(1/\bar{\zeta})} - 1)$ is of degree $< N$. Consequently, as $\rho \to \infty$ the integral $I_\rho(z)$ tends to 0 uniformly for all z belonging to a compact subset of the complex plane. On the other hand, using the residue theorem, we have, for all $\rho > 1$ and $|z| < \rho$,

$$I_\rho(z) = \frac{Q(z) + e^{i\gamma} z^N \left(\overline{Q(1/\bar{z})} - 1\right)}{z^N - e^{-i\gamma}} + \frac{1}{N} \sum_{k=1}^{N} \frac{2 \operatorname{Re} Q\left(e^{-(2k\pi+\gamma)i/N}\right) - 1}{1 - z e^{(2k\pi+\gamma)i/N}} e^{i\gamma},$$

whence

$$\begin{aligned} Q(z) + e^{i\gamma} z^N \left(\overline{Q(1/\bar{z})} - 1\right) \\ = \frac{1}{N} \sum_{k=1}^{N} \left(2 \operatorname{Re} Q\left(e^{-(2k\pi+\gamma)i/N}\right) - 1\right) \frac{1 - z^N e^{i\gamma}}{1 - z e^{(2k\pi+\gamma)i/N}}. \end{aligned}$$

From this identity it follows that if $P(z) := \sum_{j=0}^{N} a_j z^j$, then

$$\begin{aligned} P(z) * \left(Q(z) + e^{i\gamma} z^N \left(Q(1/\bar{z}) - 1\right)\right) \\ &= \frac{1}{N} \sum_{k=1}^{N} \left\{2 \operatorname{Re} Q\left(e^{-(2k\pi+\gamma)i/N}\right) - 1\right\} \left\{P(z) * \left(\frac{1 - z^N e^{i\gamma}}{1 - z e^{(2k\pi+\gamma)i/N}}\right)\right\} \\ &= \frac{1}{N} \sum_{k=1}^{N} \left\{2 \operatorname{Re} Q\left(e^{-(2k\pi+\gamma)i/N}\right) - 1\right\} \left\{P(z) * \sum_{j=1}^{N-1} e^{(2k\pi+\gamma)ji/N} z^j\right\} \\ &= \frac{1}{N} \sum_{k=1}^{N} \left\{2 \operatorname{Re} Q\left(e^{-(2k\pi+\gamma)i/N}\right) - 1\right\} \left\{P\left(z e^{(2k\pi+\gamma)i/N}\right) - a_N e^{i\gamma} z^N\right\}. \end{aligned}$$

Now, in order to obtain the desired formula (5.2), we need only observe that

$$P(z) * \left(e^{i\gamma} z^N\right) = a_N e^{i\gamma} z^N,$$

and

$$\frac{1}{N}\sum_{k=1}^{N}\left\{2\operatorname{Re}Q\left(e^{-(2k\pi+\gamma)i/N}\right)-1\right\}=2\operatorname{Re}Q(0)-1=1. \tag{5.3}$$

PROOF OF THEOREM 6. We apply (5.2) with $N = n + m$. Since, by hypothesis, $Q \in \mathscr{R}$, the coefficients $2\operatorname{Re}Q(e^{-(2k\pi+\gamma)i/N}) - 1$ in the right member of (5.2) are nonnegative for each k $(1 \leqslant k \leqslant N)$. Consequently, for all $P \in \mathscr{P}_N$ and $|z| \leqslant 1$,

$$\begin{aligned}&\left|P(z)*\left(Q(z)+e^{i\gamma}z^N\overline{Q(1/\bar z)}\right)\right|\\&\qquad\leqslant\frac{1}{N}\sum_{k=1}^{N}\left\{2\operatorname{Re}Q\left(e^{-(2k\pi+\gamma)i/N}\right)-1\right\}\left|P\left(ze^{(2k\pi+\gamma)i/N}\right)\right|\leqslant\|P\|\end{aligned}$$

by (5.3). This is equivalent to the desired result for $|\alpha| = 1$ since $z^N\overline{Q(1/\bar z)}$ is nothing but $z^m\tilde{Q}(z)$. That $|\alpha|$ can be allowed to be less than 1 is a simple consequence of the maximum modulus principle.

6. Some related results. Lemma 4 yields some generalizations of (1.1) which we present next.

THEOREM 7. *Let $Q \in \mathscr{P}_n$, $Q(0) = 0$ such that $\tilde{Q} \in \mathscr{R}$. Then for $P \in \mathscr{P}_n$, we have*

$$\|Q * P\| \leqslant \max_{|z|=1}|\operatorname{Re}P(z)|. \tag{6.1}$$

PROOF. We may suppose that $\max_{|z|=1}|\operatorname{Re}P(z)| = 1$. Since $Q(0) = 0$, the polynomial Q belongs to $\mathscr{P}_{n-1}$. Taking $\tilde{Q}$ instead of Q and $N = n$ in (5.2), we obtain that for all $P \in \mathscr{P}_n$ and $\gamma \in \mathbf{R}$,

(6.2)

$$P(z)*\left(\tilde{Q}(z)+e^{i\gamma}Q(z)\right)=\frac{1}{n}\sum_{k=1}^{n}\left(2\operatorname{Re}\tilde{Q}\left(e^{-(2k\pi+\gamma)i/n}\right)-1\right)P\left(ze^{(2k\pi+\gamma)i/n}\right).$$

Since $\tilde{Q} \in \mathscr{R}$, the coefficients $2\operatorname{Re}\tilde{Q}(e^{-(2k\pi+\gamma)i/n}) - 1$ in the right member are nonnegative for each k $(1 \leqslant k \leqslant n)$. Besides, their arithmetic mean is equal to 1 since $\tilde{Q}$ clearly satisfies (5.3) with $N = n$. Thus, the right member of (6.2) is a convex linear combination of the numbers $P(ze^{(2k\pi+\gamma)i/n})$, $1 \leqslant k \leqslant n$. Since, by hypothesis, $-1 \leqslant \operatorname{Re}P(z) \leqslant 1$ for $|z| \leqslant 1$, we conclude that

$$-1 \leqslant \operatorname{Re}\left\{P(z)*\tilde{Q}(z)+\alpha\left(P(z)*Q(z)\right)\right\} \leqslant 1$$

for $|z| \leqslant 1$ and $|\alpha| \leqslant 1$. This means that the disk with center at the point $P(z)*\tilde{Q}(z)$ and radius $|P(z)*Q(z)|$ is contained in the strip $-1 \leqslant \operatorname{Re}w \leqslant 1$. Since the maximal radius of such a disk is 1, the desired result follows.

THEOREM 8. *If $P \in \mathscr{P}_n$ then*

$$\|P'\| \leqslant n \max_{1\leqslant k\leqslant 2n}\left|P\left(e^{k\pi i/n}\right)\right|, \tag{6.3}$$

i.e. in (1.1), $\|P\|$ may be replaced by the maximum of $|P(z)|$ in the $(2n)$th roots of unity. On the other hand, the maximum in the $(n+m)$th roots of unity, with $m < n$, does not suffice.

What we need for the proof of Theorem 8 is the following special case of Lemma 4.

LEMMA 4′. *If $P \in \mathscr{P}_n$, then for all real γ and $R \geqslant 1$,*

$$(6.4)\qquad e^{i\gamma}P(Re^{i\theta}) = e^{i\gamma}P(e^{i\theta}) + \frac{1}{2n}\sum_{k=1}^{2n}(-1)^k A_k(R,\gamma)P(e^{i(\theta+(k\pi+\gamma)/n)}),$$

where

$$A_k(R,\gamma) := R^n - 1 + 2\sum_{j=1}^{n-1}(R^{n-j}-1)\cos j\left(\frac{k\pi+\gamma}{n}\right).$$

The coefficients $A_k(R,\gamma)$ are all nonnegative and

$$(6.5)\qquad \frac{1}{2n}\sum_{k=1}^{2n}A_k(R,\gamma) = R^n - 1.$$

Since it needs to be explained why Lemma 4′ is a special case of Lemma 4, we prefer to give a direct

PROOF OF LEMMA 4′. Let $P(e^{i\theta}) := \sum_{\nu=0}^{n} a_\nu e^{i\nu\theta}$. Substituting for $A_k(R,\gamma)$, $P(e^{i(\theta+(k\pi+\gamma)/n)})$, then using the fact that

$$\frac{1}{2n}\sum_{k=1}^{2n}(-1)^k e^{\nu k\pi i/n} = \begin{cases} 1 & \text{if } \nu = n, \\ 0 & \text{if } 0 \leqslant \nu < n, \end{cases}$$

and writing

$$\frac{1}{2}e^{ij(k\pi+\gamma/n)} + \frac{1}{2}e^{-ij(k\pi+\gamma/n)} \quad \text{for } \cos j\left(\frac{k\pi+\gamma}{n}\right),$$

we obtain

$$\begin{aligned}
&\frac{1}{2n}\sum_{k=1}^{2n}(-1)^k A_k(R,\gamma)P(e^{i(\theta+(k\pi+\gamma)/n)}) \\
&\quad = (R^n-1)\frac{1}{2n}\sum_{k=1}^{2n}\sum_{\nu=0}^{n}(-1)^k a_\nu e^{i\nu(\theta+(k\pi+\gamma)/n)} \\
&\qquad + \frac{1}{n}\sum_{k=1}^{2n}\sum_{j=1}^{n-1}\sum_{\nu=0}^{n}(-1)^k a_\nu(R^{n-j}-1)\cos j\left(\frac{k\pi+\gamma}{n}\right)e^{i\nu(\theta+(k\pi+\gamma)/n)} \\
&\quad = (R^n-1)a_n e^{in\theta+i\gamma} \\
&\qquad + \frac{1}{2n}\sum_{j=1}^{n-1}\sum_{\nu=0}^{n}\sum_{k=1}^{2n} a_\nu(R^{n-j}-1)e^{i\nu\theta+i(\nu+j)\gamma/n}e^{(n+\nu+j)k\pi i/n} \\
&\qquad + \frac{1}{2n}\sum_{j=1}^{n-1}\sum_{\nu=0}^{n}\sum_{k=1}^{2n} a_\nu(R^{n-j}-1)e^{i\nu\theta+i(\nu-j)\gamma/n}e^{(n+\nu-j)k\pi i/n}.
\end{aligned}$$

Since $n+1 \leqslant n+\nu+j \leqslant 3n-1$, whereas $1 \leqslant n+\nu-j \leqslant 2n-1$, we have

$$\frac{1}{2n}\sum_{k=1}^{2n}e^{(n+\nu+j)k\pi i/n} = \begin{cases} 1 & \text{if } n+\nu+j = 2n, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\frac{1}{2n}\sum_{k=1}^{2n} e^{(n+\nu-j)k\pi i/n} = 0,$$

whence

$$\frac{1}{2n}\sum_{k=1}^{2n}(-1)^k A_k(R,\gamma)P\left(e^{i(\theta+(k\pi+\gamma)/n)}\right)$$

$$= (R^n-1)a_n e^{in\theta+i\gamma} + \sum_{\nu=1}^{n-1}(R^\nu-1)a_\nu e^{i\nu\theta+i\gamma}$$

$$= e^{i\gamma}P(Re^{i\theta}) - e^{i\gamma}P(e^{i\theta}).$$

This proves (6.4).

The property $A_k(R,\gamma) \geqslant 0$ follows from a result of Rogosinski and Szegö [**15**, p. 75], according to which

$$\lambda_0 + 2\sum_{j=1}^{n}\lambda_j \cos j\theta \geqslant 0 \qquad (\theta \in \mathbf{R}) \tag{6.6}$$

if $\lambda_n \geqslant 0$, $\lambda_{n-1} - 2\lambda_n \geqslant 0$, and $\lambda_{j-1} - 2\lambda_j + \lambda_{j+1} \geqslant 0$ for $1 \leqslant j \leqslant n-1$.

Finally, in order to verify the identity (6.5), we may simply set $P(e^{i\theta}) = e^{in\theta}$ in (6.4).

Proof of Theorem 8. Let θ be an arbitrary real number. Then choosing $\gamma = -n\theta$ in Lemma 4′, we obtain

$$|P(Re^{i\theta}) - P(e^{i\theta})| \leqslant \frac{1}{2n}\left\{\sum_{k=1}^{2n} A_k(R,-n\theta)\right\}\max_{1\leqslant k\leqslant 2n}|P(e^{k\pi i/n})|,$$

which, in conjunction with (6.5), gives us

$$|P(Re^{i\theta}) - P(e^{i\theta})| \leqslant (R^n - 1)\max_{1\leqslant k\leqslant 2n}|P(e^{k\pi i/n})|. \tag{6.7}$$

Dividing both sides of (6.7) by $R-1$ and letting R tend to 1, we obtain the first part of Theorem 8. In order to prove the second part let $\varepsilon > 0$ and consider the polynomial $P_\varepsilon(z) := z^n - \varepsilon z^{n-m} + \varepsilon$, $1 \leqslant m < n$. It is easily checked that

$$\max_{1\leqslant k\leqslant n+m}|P_\varepsilon(e^{2k\pi i/(n+m)})| \leqslant (1+4\varepsilon^2)^{1/2},$$

whereas $\|P_\varepsilon'\| = n + (n-m)\varepsilon$. Hence

$$\|P_\varepsilon'\| > n\max_{1\leqslant k\leqslant n+m}|P_\varepsilon(e^{2k\pi i/(n+m)})|$$

for all sufficiently small ε (> 0).

7. Remarks and applications.

7.1. *Some consequences of Theorem* 2. (i) Theorem 2 constitutes a generalization of a useful inequality due to Visser [**22**], namely: *if* $P(z) := \sum_{\nu=0}^{n} a_\nu z^\nu \in \mathscr{P}_n$, *then*

$$|a_0| + |a_n| \leqslant \|P\|. \tag{7.1}$$

Inequality (7.1) is in fact a limiting case of (1.9): dividing both sides of (1.9) by R^n and letting R tend to ∞, we obtain (7.1).

(ii) Applying Theorem 2 to the polynomial $z^nP(1/z)$, we obtain

THEOREM 2′. *Let* $n \geqslant 2$. *If* $P(z) := \sum_{\nu=0}^{n} a_\nu z^\nu \in \mathcal{P}_n$, *then*

$$M_P(R) + (1 - R^2)|a_n| \leqslant \|P\| \qquad (R \leqslant 1). \tag{7.2}$$

The coefficient of $|a_n|$ *is the best possible for each* R.

(iii) Theorem 2 also leads us to

COROLLARY 2. *Let* $n \geqslant 2$. *If* $P(z) := \sum_{\nu=0}^{n} a_\nu z^\nu \in \mathcal{P}_n$, *then*

$$M_P(R) \geqslant (1 - R^2)|a_0| + R^n\|P\| \quad \text{if } R \leqslant 1, \tag{7.3}$$

whereas

$$M_P(R) \geqslant (R^n - R^{n-2})|a_n| + \|P\| \quad \text{if } R \geqslant 1. \tag{7.4}$$

Inequality (1.5), when applied to $P(z/R)$, gives us (7.3), which in turn, when applied to the polynomial $z^nP(1/z)$, yields (7.4).

7.2. *Some consequences of Theorem* 6. (i) Bernstein's inequality (1.1) is known (see [**12**, p. 8]) to admit the following refinement: *for* $P \in \mathcal{P}_n$,

$$|P'(z)| + |\tilde{P}'(z)| \leqslant n\|P\| \qquad (|z| \leqslant 1), \tag{7.5}$$

where $\tilde{P}(z) := z^n\overline{P(1/\bar{z})}$ *is the "inverse" of* P.

A polynomial $P \in \mathcal{P}_n$ is said to be "self-inverse" if there exists $u \in \mathbf{C}$, $|u| = 1$ such that $P = u\tilde{P}$; let S_n denote the class of such polynomials.

The following result, which is a significant generalization of (7.5), is a consequence of Theorem 6.

COROLLARY 3. *Let* $Q \in \mathcal{P}_n$, $Q(0) = 0$, *such that* $\tilde{Q} \in \mathcal{R}$. *Then for* $P \in \mathcal{P}_n$ *and* $|z| \leqslant 1$, *we have*

$$|(Q * P)(z)| + |(Q * \tilde{P})(z)| \leqslant \|P\|. \tag{7.6}$$

In particular for $P \in S_n$

$$\|Q * P\| \leqslant \tfrac{1}{2}\|P\|. \tag{7.7}$$

The choice

$$Q(z) := \sum_{k=1}^{n} \frac{R^k - 1}{R^n - 1} z^k \qquad (R > 1),$$

which is admissible according to the above-mentioned result of Rogosinski and Szegö (see (6.6)), gives

$$|P(Rz) - P(z)| + |\tilde{P}(Rz) - \tilde{P}(z)| \leqslant (R^n - 1)\|P\| \qquad (|z| \leqslant 1, R > 1). \tag{7.8}$$

Dividing both sides of (7.8) by $R - 1$ and letting R tend to 1, we obtain (7.5). Inequality (7.8) also implies that

$$|P(Rz)| + |\tilde{P}(Rz)| \leqslant (R^n + 1)\|P\| \qquad (|z| \leqslant 1, R \geqslant 1). \tag{7.9}$$

PROOF OF COROLLARY 3. The assumptions and Theorem 6, with $m = 1$ and $n - 1$ instead of n, show that $\tilde{Q} + \alpha Q \in \mathscr{B}_n^0$ $(|\alpha| \leqslant 1)$. (Here it is to be noted that $\tilde{Q} \in \mathscr{P}_{n-1}$.) Thus for $P \in \mathscr{P}_n$,

$$\|(P * \tilde{Q})(z) + \alpha(P * Q)(z)\| \leqslant \|P\| \qquad (|\alpha| \leqslant 1),$$

so that

$$|(P * \tilde{Q})(z)| + |(P * Q)(z)| \leqslant \|P\| \qquad (|z| = 1).$$

Now, (7.6) follows from the observation that for $|z| = 1$, $|(P * \tilde{Q})(z)| = |(Q * \tilde{P})(z)|$.

(ii) Here is another consequence of Theorem 6.

COROLLARY 4. *For all* $P \in \mathscr{P}_n$,

$$\|zP'(z) - (P(z) - P(0))\| \leqslant (n-1)\|P\| - |P(0)|. \tag{7.10}$$

It can be shown that the coefficient of $|P(0)|$ on the right side of (7.10) cannot be replaced by a smaller number.

As an application of (7.10) we wish to mention:

If $P \in \mathscr{P}_n$ *and*

$$g_\varepsilon(z) := P(z) - P(0) + \varepsilon n\|P\|z \qquad (|\varepsilon| = 1) \tag{7.11}$$

then

$$|zg_\varepsilon'(z)/g_\varepsilon(z) - 1| < 1 \quad \text{for } |z| < 1, \tag{7.12}$$

which implies, *in particular*, that *the function*

$$h_\varepsilon(z) := g_\varepsilon(z)/g_\varepsilon'(0) = z + \cdots \tag{7.13}$$

is normalized starlike univalent in $|z| < 1$.

PROOF OF COROLLARY 4. Let

$$Q(z) = \sum_{k=0}^{n-2} \frac{n-1-k}{n-1} z^k;$$

considering it to be an element of $\mathscr{P}_{n-1}$, let $\tilde{Q}$ be its inverse. Noting that $Q \in \mathscr{R}$, we may apply Theorem 6 with $m = n - 1$ to conclude that $Q + \bar{\alpha} z^{n-1}\tilde{Q} \in \mathscr{B}_{2n-2}^0$ for all α such that $|\alpha| \leqslant 1$. By truncation, $Q + \bar{\alpha} z^n/(n-1) \in \mathscr{B}_n^0$. This, in view of (2.2), implies that

$$\frac{\alpha}{n-1} + \sum_{k=2}^{n} \frac{k-1}{n-1} z^k \in \mathscr{B}_n,$$

so (7.10) holds.

As we have claimed, the coefficient of $|a_0|$ on the right side of (7.10) cannot be replaced by any smaller number. In fact, (7.10) can also be proved by showing that

$$\sum_{k=0}^{n-1} \frac{n-k-1}{n-1} z^k + \frac{\alpha}{n-1} z^n$$

belongs to $\mathscr{B}_n^0$ if $|\alpha| < 1$ and does not belong to $\mathscr{B}_n^0$ if $|\alpha| > 1$. The matrix to be considered is

$$M\begin{pmatrix} n-1, & n-2, \ldots, 0, \alpha \\ & n-2, \ldots, 0, \bar{\alpha} \end{pmatrix}.$$

(iii) In order to place the next corollary of Theorem 6 in perspective, we wish to recall that if $f(z) := \sum_{k=0}^{\infty} a_k z^k$ is analytic in $|z| < 1$, where it satisfies $|f(z)| \leqslant 1$, then [**9**, Exercise 9, p. 172]

$$|a_0|^2 + |a_k| \leqslant 1 \qquad (1 \leqslant k < \infty), \tag{7.14}$$

and, consequently,

$$|a_0| + \tfrac{1}{2}|a_k| \leqslant 1 \qquad (1 \leqslant k < \infty). \tag{7.15}$$

The example

$$f(z) := \frac{z^k + a_0}{a_0 z^k + 1} \qquad \left(a_0 = \frac{3 - 2c}{1 + 2c}\right)$$

shows that in (7.15) *the coefficient of* $|a_k|$ *cannot, in general, be replaced by any number c greater than* $1/2$. In view of a well-known property of the Cesàro means

$$\sigma_n(z) := \sum_{\nu=0}^{n} \frac{n + 1 - \nu}{n + 1} a_\nu z^\nu \qquad (n = 0, 1, 2, \ldots),$$

namely,

$$\|\sigma_n\| \leqslant \sup_{|z|<1} \left| \sum_{k=0}^{\infty} a_k z^k \right|,$$

it cannot be done even for a polynomial if the degree is allowed to be arbitrary. Nevertheless, we have:

COROLLARY 5. *If* $P(z) := \sum_{\nu=0}^{n} a_\nu z^\nu \in \mathscr{P}_n$ *and* $\|P\| \leqslant 1$, *then*

$$|a_0| + \tfrac{1}{2}\big(|a_k| + |a_l|\big) \leqslant 1 \quad \textit{for } 1 \leqslant k \leqslant l, \textit{ with } l \geqslant n + 1 - k; \tag{7.16}$$

by symmetry,

$$|a_n| + \tfrac{1}{2}\big(|a_j| + |a_k|\big) \leqslant 1 \quad \textit{for } 0 \leqslant j < n, \textit{ with } j \leqslant n + 1 - k. \tag{7.16$'$}$$

PROOF. Let $Q(z) := 1 + (\varepsilon/2)z^k$, $|\varepsilon| \leqslant 1$, so that, by Theorem 6,

$$1 + (\varepsilon/2)z^k + (\bar{\varepsilon}\alpha/2)z^{n+m-k} \in \mathscr{B}_n^0 \quad \text{for } |\alpha| \leqslant 1 \text{ and } m \in \mathbf{N};$$

hence, for $|z| \leqslant 1$,

$$|a_0 + (\varepsilon/2)a_k z^k + (\bar{\varepsilon}\alpha/2)a_{n+m-k} z^{n+m-k}| \leqslant 1.$$

We may assume $a_0 > 0$, and the choice $z = 1$, $\varepsilon = \bar{a}_k/|a_k|$, $\alpha = \varepsilon\bar{a}_{n+m-k}/|a_{n+m-k}|$ establishes the assertion.

REMARK. In (7.16) the restriction on l, namely $l \geqslant n + 1 - k$, cannot, in general, be relaxed. The quantity $|a_0| + \frac{1}{2}(|a_k| + |a_l|)$ may indeed be greater than 1 if l is allowed to be $n - k$.

(I) *The case* $k < l$. We prove the existence of a polynomial $P(z) := \sum_{\nu=0}^{3} a_\nu z^\nu$ such that $|a_0| + \frac{1}{2}(|a_1| + |a_2|) > \|P\|$. It is clearly enough to show that

$$1 + \tfrac{1}{2}z + \tfrac{1}{2}e^{i\gamma}z^2 \notin \mathscr{B}_3^0 \tag{7.17}$$

for some $\gamma \in \mathbf{R}$. Since the determinant

$$D\begin{pmatrix} & \frac{1}{2}, \frac{1}{2}e^{i\gamma}, 0 \\ 1, & \\ & \frac{1}{2}, \frac{1}{2}e^{-i\gamma}, 0 \end{pmatrix} = -\frac{1}{4} + \frac{1}{2}\cos\gamma$$

is negative if $\cos\gamma < 1/2$, it follows from Lemmas 2 and 3 that (7.17) holds for such values of γ.

(II) *The case* $k = l$. Let α_0 be the smaller of the two positive roots of the equation $3\alpha^3 - 12\alpha + 8 = 0$; then for each $\alpha \in (-\alpha_0, \alpha_0)$, the polynomial

$$P(z) := -\alpha + (1 - \alpha^2)z^k + \alpha(1 - \alpha^2)z^{2k}$$

serves as a counterexample.

7.3. *A few special cases of* (6.1). (i) With the choice

$$Q(z) := \sum_{\nu=0}^{n} \frac{R^\nu - 1}{R^n - 1} z^\nu \qquad (R > 1)$$

in (6.1), we obtain: *for all* $P \in \mathscr{P}_n$,

$$\|P(Rz) - P(z)\| \leqslant (R^n - 1)\max_{|z|=1}|\operatorname{Re} P(z)| \qquad (R > 1). \tag{7.18}$$

This latter inequality implies: *if* $P \in \mathscr{P}_n$ *then*

$$\|P'\| \leqslant n \max_{|z|=1}|\operatorname{Re} P(z)|, \tag{7.18$'$}$$

which is an interesting generalization of Bernstein's inequality (1.1). For other proofs of (7.18′) see **[20, 16, 8]**.

(ii) Another special case of (6.1), which is obtained by taking $Q(z) := z^n + \frac{1}{2}z^j$, says that *if* $P(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ *then, for* $0 < j < n$,

$$|a_n| + \tfrac{1}{2}|a_j| \leqslant \max_{|z|=1}|\operatorname{Re} P(z)|. \tag{7.19}$$

This inequality is to be compared with (7.15). An example of the form $i\alpha + z^n$, with an appropriate $\alpha \in \mathbf{R}$, shows that in (7.19) neither j can be allowed to be 0 nor can $|a_n|$ be replaced by $|a_0|$.

7.4. *Remarks on Theorem* 8. (i) According to a well-known result of A. Markoff, if $P \in \mathscr{P}_n$ then

$$\max_{-1 \leqslant x \leqslant 1}|P'(x)| \leqslant n^2 \max_{-1 \leqslant x \leqslant 1}|P(x)|. \tag{7.20}$$

Over forty years ago it was shown by Duffin and Schaeffer **[1]** that in (7.20), $\max_{-1 \leqslant x \leqslant 1}|P(x)|$ may be replaced by the maximum of $|P(x)|$ in the extrema $\{\cos(k\pi/n)\}_{k=0}^{n}$ of the nth Chebyshev polynomial of the first kind. Theorem 8 represents the corresponding refinement of Bernstein's inequality (1.1).

(ii) It may be mentioned that for an arbitrary $P \in \mathscr{P}_n$, $\|P\|$ may be considerably larger than $\max_{1 \leqslant k \leqslant 2n}|P(e^{k\pi i/n})|$, as is shown by the simple example $1 + iz^n$.

(iii) It is natural to ask whether in (1.2) as well $\|P\|$ can be replaced by the quantity $\max_{1 \leqslant k \leqslant 2n}|P(e^{k\pi i/n})|$. The answer is essentially "no". Indeed, the example

$1 + i(R_0^n - R_0^{-n})z^n/2$ shows that for given $R_0 > 1$ there exists a polynomial P_0 such that

(7.21) $$\max_{|z|=R} |P_0(z)| > R^n \max_{1 \leqslant k \leqslant 2n} |P_0(e^{k\pi i/n})| \quad \text{for } 1 \leqslant R < R_0.$$

However, we do have:

THEOREM 9. *For each given polynomial $P \in \mathscr{P}_n$ there exists a number R_* depending on P such that*

(7.22) $$\max_{|z|=R} |P(z)| \leqslant R^n \max_{1 \leqslant k \leqslant 2n} |P(e^{k\pi i/n})| \quad \textit{for } R \geqslant R_*.$$

PROOF. Let $P(z) := \sum_{k=0}^{m} a_k z^k$, where $m \leqslant n$ and $a_m \neq 0$. In view of Parseval's identity,

$$\|P'\|^2 = \sum_{k=0}^{m} k^2 |a_k|^2.$$

In particular,

(7.23) $$\|P'\| \geqslant m|a_m|,$$

where equality holds if and *only if*

(7.24) $$P(z) \equiv a_m z^m.$$

Thus, if (7.24) fails to hold, then

(7.25) $$\|P'\| = (1 + \varepsilon) m |a_m|$$

for some $\varepsilon > 0$. Now let us choose $R_* > 0$ such that

$$v(R_*) := 1 + (1/|a_m|)\left(|a_{m-1}| R_*^{-1} + |a_{m-2}| R_*^{-2} + \cdots + |a_0| R_*^{-m}\right) = 1 + \varepsilon.$$

Then for $R \geqslant R_*$, we have

$$\begin{aligned} \max_{|z|=R} |P(z)| &\leqslant |a_m| R^m v(R_*) = (1+\varepsilon)|a_m| R^m \\ &= R^m \|P'\|/m \quad \text{by (7.25)} \\ &\leqslant R^n \max_{1 \leqslant k \leqslant 2n} |P(e^{k\pi i/n})| \quad \text{by Theorem 8.} \end{aligned}$$

On the other hand, if $P(z) \equiv a_m z^m$, then (7.22) holds for all $R \geqslant 1$.

7.5. *Self-reciprocal polynomials and an application of Theorem* 5. From (7.5) it follows, in particular, that

(7.26) $$\|P'\| \leqslant (n/2)\|P\| \quad \text{if } P \in S_n.$$

We call a polynomial $P \in \mathscr{P}_n$ "self-reciprocal" if it satisfies the condition $z^n P(1/z) \equiv P(z)$; let $\mathscr{S}_n$ be the class of such functions. For the pertinence of the class $\mathscr{S}_n$ see Frappier and Rahman [**3**]. The first attempt at determining

(7.27) $$U_n := \sup_{P \in \mathscr{S}_n} \{\|P'\|/\|P\|\}$$

was made by Govil, Jain, and Labelle [**6**], who had an example to show that

(7.28) $$U_n \geqslant n/\sqrt{2},$$

where equality holds for $n = 2$. In view of (7.26) and the fact that all the zeros of the extremal polynomial in (1.1) lie at the origin, one might have suspected that U_n may not be much larger. However, the problem turns out to be quite intriguing, and the precise value of U_n for $n \geqslant 3$ remains unknown. With the help of Theorem 5 and Corollary 1 we are able to prove that

$$U_n \leqslant n - \delta_n \quad \text{where } \delta_n \to 2/5 \text{ as } n \to \infty. \tag{7.29}$$

On the other hand, we show that

$$U_n \geqslant n - 1. \tag{7.30}$$

For (7.30) let us consider the polynomial

$$P_+(z) := (1 - iz)^2 + z^{n-2}(z - i)^2,$$

which clearly belongs to $\mathscr{S}_n$. We note that $\|P_+\| = 4$. In fact

$$\begin{aligned}\|P_+\| &\leqslant \max_{|z|=1}\left(|1 - iz|^2 + |z - i|^2\right)\\ &= \max_{|z|=1}\left(|1 - iz|^2 + |1 + iz|^2\right) = 4 = |P_+(1)|.\end{aligned}$$

Since $\|P'_+\| \geqslant |P'_+(-i)| = 4(n - 1)$, estimate (7.30) holds.

Proof of (7.29). Let $\alpha := 8n/(9n + 2)$. Multiply (4.5) by α and (1.4) by $1 - \alpha$; adding them up and using the triangle inequality, we obtain

$$\left\|zP'(z) - \frac{4n}{9n + 2}a_n z^n\right\| \leqslant \left(n - \frac{4n}{9n + 2}\right)\|P\| - \frac{8n}{9n + 2}\gamma_n|a_0|. \tag{7.31}$$

Under our hypothesis $a_0 = a_n$. Let us assume that $|a_0| = |a_n| = \lambda\|P\|$. Then from (7.31) we obtain

$$\|P'\| \leqslant \left\{\frac{9n - 2}{9n + 2}n + \frac{4(1 - 2\gamma_n)n}{9n + 2}\lambda\right\}\|P\|, \tag{7.32}$$

whereas (1.4) gives us

$$\|P'\| \leqslant (n - 2n\lambda/(n + 2))\|P\|. \tag{7.33}$$

The last two inequalities imply that

$$U_n \leqslant n - 4n/(11n - 4(n + 2)\gamma_n + 6),$$

so (7.29) holds.

7.6. *An application of Theorem 5 to polynomials with a prescribed zero.* For an arbitrary $\rho \in (0, \infty)$, let $\mathscr{P}_{n,\rho}$ denote the class of all polynomials $P \in \mathscr{P}_n$ which vanish at the point ρ, and let

$$\Lambda_{n,\rho} := \left\{\sup\|P'\|/\|P\| : P \in \mathscr{P}_{n,\rho}\right\}.$$

The problem of determining $\Lambda_{n,\rho}$ was proposed by R. P. Boas, Jr. in 1957 but still remains open. It is known (see Giroux and Rahman [**5**] and Newman [**10**]) that $n - C_1/n < \Lambda_{n,1} < n - C_2/n$, where C_1 and C_2 are positive constants independent of n; in fact, C_2 may be taken to be $(2 - \sqrt{2})/4$. As remarked by Newman [**10**, pp. 265–266], it is quite difficult to pin down the value of $\Lambda_{n,1}$. It therefore seems to be of interest to point out that (4.5) leads us to a considerably better upper bound for

$\Lambda_{n,1}$, namely

$$(7.34)\qquad \Lambda_{n,1} \leqslant n - \frac{1}{2}\left(1 - \left(\cos\frac{\pi}{2(n+1)}\right)^{n+1}\right) = n - \frac{\pi^2}{16}\frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

In fact, if $P(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ belongs to $\mathscr{P}_{n,1}$ then so does $Q(z) := z^n P(1/z)$. By a result of Lachance, Saff, and Varga [**7**],

$$(7.35)\qquad |a_n| = |Q(0)| \leqslant \left(\cos\frac{\pi}{2(n+1)}\right)^{n+1}\|Q\| = \left(\cos\frac{\pi}{2(n+1)}\right)^{n+1}\|P\|.$$

This, in conjunction with (4.5), readily gives us (7.34).

Using a nontrivial upper bound for

$$\sup\left\{\frac{|a_n|}{\|P\|} : P(z) := \sum_{\nu=0}^{n} a_\nu z^\nu \in \mathscr{P}_{n,\rho}\right\},$$

obtainable from [**13**, Theorem 3], we may similarly estimate $\Lambda_{n,\rho}$ for other values of ρ.

7.7. *A generalization of Corollary* 1. Corollary 1 is a special case of the following more general result.

THEOREM 10. *If $P \in \mathscr{P}_n$ and $0 \leqslant \omega \leqslant n/2$, then*

$$(7.36)\qquad \|zP'(z) - \omega P(z)\| + \varepsilon_n(\omega)|P(0)| \leqslant (n - \omega)\|P\|,$$

where

$$\varepsilon_1(\omega) := 1 - 2\omega, \qquad \varepsilon_n(\omega) := 2(n - 2\omega)/(n - 2\omega + 2) \quad (n \geqslant 2).$$

For each value of the parameter ω the constant $\varepsilon_n(\omega)$ is best possible.

PROOF. Since the left side of (7.36) is equal to

$$\sup_{|\alpha| < \varepsilon_n(\omega)} \|zP'(z) - \omega P(z) + \bar{\alpha}P(0)\|,$$

we study the definiteness of the matrix

$$\eta(\alpha, \omega) := M\begin{pmatrix} & c_1, \ldots, c_n \\ c_0, & \\ & \bar{c}_1, \ldots, \bar{c}_n \end{pmatrix},$$

where $c_k := n - k - \omega$ for $0 \leqslant k < n$ and $c_n := \alpha - \omega$. For $1 \leqslant k \leqslant n$ the value of the kth order leading principal minor is $2^{k-2}(2n - 2\omega - k + 1)$, a positive quantity. Besides, as in the case of (3.1), the determinant of $\eta(\alpha, \omega)$ can be developed in the form

$$\det(\eta(\alpha, \omega)) = \det(\eta(\omega, \omega)) + 2(-1)^n D_n(\omega)\operatorname{Re}(\alpha - \omega) - C_n(\omega)|\alpha - \omega|^2,$$

where

$$C_n(\omega) = 2^{n-3}(n + 2 - 2\omega), \quad D_n(\omega) = (-1)^n 2^{n-3}\{2\omega^2 - (n+2)\omega + 2\},$$

and

$$\det(\eta(\omega, \omega)) = 2^{n-3}\{2\omega^3 - (n+2)\omega^2 - 4\omega + 4n\}.$$

We are thus led to the equation $(4n - 8\omega) - 4x - (n + 2 - 2\omega)x^2 = 0$, whose only positive root is the number $\varepsilon_n(\omega)$ defined in the statement of Theorem 10.

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