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New inequalities of Opial type for conformable fractional integrals

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Abstract: In this paper, some Opial-type inequalities for conformable fractional integrals are obtained using the remainder function of Taylor's theorem for conformable integrals.

Key words: Opial inequality, Hölder's inequality, conformable fractional integrals

1. Introduction

In 1960, Opial established the following interesting integral inequality [10]:

Theorem 1 *Let $x(t) \in C^{(1)}[0,h]$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then the following inequality holds:*

$$\int_0^h |x(t)x'(t)|\,dt \leq \frac{h}{4}\int_0^h (x'(t))^2\,dt \tag{1.1}$$

The constant $h/4$ is the best possible.

Opial's inequality and its generalizations, extensions, and discretizations play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations. Over the last 20 years a large number of papers have appeared in the literature that deals with the simple proofs, various generalizations, and discrete analogues of Opial's inequality and its generalizations; see [2,4,5,11–14,16,17].

The purpose of this paper is to establish some Opial-type inequalities for conformable integrals. The structure of this paper is as follows. In Section 2, we give the definitions of conformable derivatives and conformable integrals and introduce several useful notations for conformable integrals used in our main results. In Section 3, the main result is presented. Using the remainder function of Taylor's theorem for conformable integrals, we establish several Opial-type inequalities.

2. Definitions and properties of conformable fractional derivatives and integrals

The following definitions and theorems with respect to conformable fractional derivatives and integrals were referred to (see [1], [3], [6]–[9]).

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Definition 1 (Conformable fractional derivative) *Given a function $f:[0,\infty)\rightarrow\mathbb{R}$. Then the "conformable fractional derivative" of f of order α is defined by*

$$D_{\alpha}(f)(t)=\lim_{\epsilon\rightarrow 0}\frac{f\left(t+\epsilon t^{1-\alpha}\right)-f(t)}{\epsilon} \tag{2.1}$$

for all $t>0$, $\alpha\in(0,1)$. If f is $\alpha-$ differentiable in some $(0,a)$, $\alpha>0$, $\lim_{t\rightarrow 0^{+}}f^{(\alpha)}(t)$ exist, then define

$$f^{(\alpha)}(0)=\lim_{t\rightarrow 0^{+}}f^{(\alpha)}(t). \tag{2.2}$$

We can write $f^{(\alpha)}(t)$ for $D_{\alpha}(f)(t)$ to denote the conformable fractional derivatives of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is $\alpha-$ differentiable.

Theorem 2 *Let $\alpha\in(0,1]$ and f,g be $\alpha-$ differentiable at a point $t>0$. Then*

i. $D_{\alpha}(af+bg)=aD_{\alpha}(f)+bD_{\alpha}(g)$, for all $a,b\in\mathbb{R}$,

ii. $D_{\alpha}(\lambda)=0$, for all constant functions $f(t)=\lambda$,

iii. $D_{\alpha}(fg)=fD_{\alpha}(g)+gD_{\alpha}(f)$,

iv. $D_{\alpha}\left(\frac{f}{g}\right)=\frac{fD_{\alpha}(g)-gD_{\alpha}(f)}{g^{2}}$.

If f is differentiable, then

$$D_{\alpha}(f)(t)=t^{1-\alpha}\frac{df}{dt}(t). \tag{2.3}$$

Definition 2 (Conformable fractional integral) *Let $\alpha\in(0,1]$ and $0\leq a<b$. A function $f:[a,b]\rightarrow\mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral*

$$\int_{a}^{b}f(x)\,d_{\alpha}x:=\int_{a}^{b}f(x)\,x^{\alpha-1}dx \tag{2.4}$$

exists and is finite. All α-fractional integrable on $[a,b]$ is indicated by $L_{\alpha}^{1}([a,b])$.

Remark 1

$$I_{\alpha}^{a}(f)(t)=I_{1}^{a}\left(t^{\alpha-1}f\right)=\int_{a}^{t}\frac{f(x)}{x^{1-\alpha}}dx,$$

where the integral is the usual Riemann improper integral, and $\alpha\in(0,1]$.

Theorem 3 *Let $f:(a,b)\rightarrow\mathbb{R}$ be differentiable and $0<\alpha\leq 1$. Then, for all $t>a$ we have*

$$I_{\alpha}^{a}D_{\alpha}^{a}f(t)=f(t)-f(a). \tag{2.5}$$

Theorem 4 (Integration by parts) *Let $f, g : [a, b] \to \mathbb{R}$ be two functions such that fg is differentiable. Then*

$$\int_a^b f(x) D_\alpha^a (g)(x) d_\alpha x = fg|_a^b - \int_a^b g(x) D_\alpha^a (f)(x) d_\alpha x. \tag{2.6}$$

Theorem 5 *Assume that $f : [a, \infty) \to \mathbb{R}$ such that $f^{(n)}(t)$ is continuous and $\alpha \in (n, n+1]$. Then, for all $t > a$ we have*

$$D_\alpha^a I_\alpha^a f(t) = f(t).$$

We can give Hölder's inequality in conformable integral as follows:

Lemma 1 *Let $f, g \in C[a, b]$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\int_a^b |f(x)g(x)| d_\alpha x \leq \left(\int_a^b |f(x)|^p d_\alpha x \right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q d_\alpha x \right)^{\frac{1}{q}}.$$

Remark 2 *If we take $p = q = 2$ in Lemma 1, then we have the Cauchy–Schwarz inequality for conformable integrals.*

Theorem 6 (Taylor's Formula) *[3] Let $\alpha \in (0, 1]$ and $n \in \mathbb{N}$. Suppose f is $n+1$ times $\alpha-$fractional differentiable on $[0, \infty)$, and $s, t \in [0, \infty)$. Then we have*

$$f(t) = \sum_{k=0}^{n} \frac{1}{k!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^k D_\alpha^k f(s) + \frac{1}{n!} \int_s^t \left(\frac{t^\alpha - \tau^\alpha}{\alpha} \right)^n D_\alpha^{n+1} f(\tau) d_\alpha \tau.$$

Using Taylor's theorem, we define the remainder function by

$$R_{-1,f}(., s) := f(s),$$

and for $n > -1$,

$$\begin{aligned} R_{n,f}(t, s) \quad : \quad &= f(s) - \sum_{k=0}^{n} \frac{1}{k!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^k D_\alpha^k f(s) \\ &= \frac{1}{n!} \int_s^t \left(\frac{t^\alpha - \tau^\alpha}{\alpha} \right)^n D_\alpha^{n+1} f(\tau) d_\alpha \tau. \end{aligned} \tag{2.7}$$

Opial's inequality can be represented for conformable fractional integral forms as follows [15]:

Theorem 7 *Let $\alpha \in (0, 1]$ and u be an α-fractional differentiable function on $(0, h)$ with $u(0) = u(h) = 0$. Then the following inequality for conformable fractional integrals holds:*

$$\int_0^h |u(t) D_\alpha (u)(t)| d_\alpha t \leq \frac{h^\alpha}{4\alpha} \int_0^h |D_\alpha (u)(t)|^2 d_\alpha t. \tag{2.8}$$

Now we present the main results,

3. Opial-type inequalities for conformable fractional integrals

Let $\alpha \in (0,1]$. In the following we adapt to the α-fractional setting some results from [2] by applying the fractional Opial inequality.

Theorem 8 *Let $\alpha \in (0,1]$, $f : [a,b] \to \mathbb{R}$ be an $n+1$ times $\alpha-$fractional differentiable function, $p,q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $t \geq x_0$, $t, x_0 \in [a,b]$. Then we have the following inequality:*

$$\begin{aligned} & \int_{x_0}^{t} |R_{n,f}(x_0,\tau)| \left|D_\alpha^{n+1} f(\tau)\right| d_\alpha \tau \\ \leq \quad & \frac{(t^\alpha - x_0^\alpha)^{n+2/p}}{\alpha^{n+2/p} 2^{\frac{1}{q}} n! \left[(np+1)(np+2)\right]^{1/p}} \left(\int_{x_0}^{t} \left|D_\alpha^{n+1} f(\tau)\right|^q d_\alpha \tau \right)^{\frac{2}{q}}. \end{aligned} \tag{3.1}$$

Proof From (2.7), we have

$$R_{n,f}(x_0,t) = \frac{1}{n!} \int_{x_0}^{t} \left(\frac{t^\alpha - \tau^\alpha}{\alpha} \right)^n D_\alpha^{n+1} f(\tau) d_\alpha \tau, \ x_0, t \in [a,b].$$

By using Hölder's inequality for conformable integrals, it follows that

$$\begin{aligned} |R_{n,f}(x_0,t)| & \leq \frac{1}{\alpha^n n!} \int_{x_0}^{t} (t^\alpha - \tau^\alpha)^n \left|D_\alpha^{n+1} f(\tau)\right| d_\alpha \tau \\ & \leq \frac{1}{\alpha^n n!} \left(\int_{x_0}^{t} (t^\alpha - \tau^\alpha)^{np} d_\alpha \tau \right)^{\frac{1}{p}} \left(\int_{x_0}^{t} \left|D_\alpha^{n+1} f(\tau)\right|^q d_\alpha \tau \right)^{\frac{1}{q}} \\ & = \frac{1}{\alpha^{n+1/p} n!} \frac{(t^\alpha - x_0^\alpha)^{n+1/p}}{(np+1)^{1/p}} (z(t))^{\frac{1}{q}} \end{aligned} \tag{3.2}$$

where

$$z(t) = \int_{x_0}^{t} \left|D_\alpha^{n+1} f(\tau)\right|^q d_\alpha \tau, \ x_0 \leq t \leq b, \ z(x_0) = 0.$$

Thus,

$$D_\alpha z(t) = \left|D_\alpha^{n+1} f(t)\right|^q$$

and

$$\left|D_\alpha^{n+1} f(t)\right| = (D_\alpha z(t))^{1/q}. \tag{3.3}$$

By (3.2) and (3.3), we get

$$|R_{n,f}(x_0,t)| \left|D_\alpha^{n+1} f(t)\right| \leq \frac{1}{\alpha^{n+1/p} n!} \frac{(t^\alpha - x_0^\alpha)^{n+1/p}}{(np+1)^{1/p}} (z(t) D_\alpha z(t))^{\frac{1}{q}}. \tag{3.4}$$

Integrating the inequality (3.4) and using Hölder's inequality for conformable integrals, we have

$$\begin{aligned}
&\int_{x_0}^{t} |R_{n,f}(x_0,\tau)| \left|D_\alpha^{n+1} f(\tau)\right| d_\alpha \tau \\
&\le \frac{1}{\alpha^{n+1/p} n! (np+1)^{1/p}} \int_{x_0}^{t} (\tau^\alpha - x_0^\alpha)^{n+1/p} \left(z(\tau) D_\alpha z(\tau)\right)^{\frac{1}{q}} d_\alpha \tau \\
&\le \frac{1}{\alpha^{n+1/p} n! (np+1)^{1/p}} \left(\int_{x_0}^{t} (\tau^\alpha - x_0^\alpha)^{np+1} d_\alpha \tau \right)^{\frac{1}{p}} \left(\int_{x_0}^{t} z(\tau) D_\alpha z(\tau) d_\alpha \tau \right)^{\frac{1}{q}} \\
&= \frac{(t^\alpha - x_0^\alpha)^{n+2/p}}{\alpha^{n+2/p} n! \left[(np+1)(np+2)\right]^{1/p}} \frac{(z(t))^{\frac{2}{q}}}{2^{\frac{1}{q}}}
\end{aligned}$$

which completes the proof. □

Corollary 1 *Under the assumption of Theorem 8 with $p = q = 2$, we get*

$$\int_{x_0}^{t} |R_{n,f}(x_0,\tau)| \left|D_\alpha^{n+1} f(\tau)\right| d_\alpha \tau \le \frac{(t^\alpha - x_0^\alpha)^{n+1}}{2\alpha^{n+2} n! \sqrt{(2n+1)(n+1)}} \int_{x_0}^{t} \left|D_\alpha^{n+1} f(\tau)\right|^2 d_\alpha \tau.$$

Theorem 9 *Let $\alpha \in (0,1]$, $f : [a,b] \to \mathbb{R}$ be an $n+1$ times $\alpha-$fractional differentiable function, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $t \le x_0$, $t, x_0 \in [a,b]$. Then we have the following inequality:*

$$\begin{aligned}
&\int_{t}^{x_0} |R_{n,f}(x_0,\tau)| \left|D_\alpha^{n+1} f(\tau)\right| d_\alpha \tau \qquad (3.5)\\
&\le \frac{(x_0^\alpha - t^\alpha)^{n+2/p}}{\alpha^{n+1+2/p} 2^{\frac{1}{q}} n! \left[(np+1)(np+2)\right]^{1/p}} \left(\int_{t}^{x_0} \left|D_\alpha^{n+1} f(\tau)\right|^q d_\alpha \tau \right)^{\frac{2}{q}}.
\end{aligned}$$

Proof From (2.7), we have

$$\begin{aligned}
|R_{n,f}(x_0,t)| &= \frac{1}{\alpha^n n!} \left| \int_{x_0}^{t} (t^\alpha - \tau^\alpha)^n D_\alpha^{n+1} f(\tau) d_\alpha \tau \right| \qquad (3.6)\\
&\le \frac{1}{\alpha^n n!} \int_{t}^{x_0} (\tau^\alpha - t^\alpha)^n \left|D_\alpha^{n+1} f(\tau)\right| d_\alpha \tau \\
&\le \frac{1}{\alpha^n n!} \left(\int_{t}^{x_0} (\tau^\alpha - t^\alpha)^{np} d_\alpha \tau \right)^{\frac{1}{p}} \left(\int_{t}^{x_0} \left|D_\alpha^{n+1} f(\tau)\right|^q d_\alpha \tau \right)^{\frac{1}{q}} \\
&= \frac{1}{\alpha^{n+1/p} n!} \frac{(x_0^\alpha - t^\alpha)^{n+1/p}}{(np+1)^{1/p}} (z(t))^{\frac{1}{q}}
\end{aligned}$$

where

$$z(t)=\int_{t}^{x_0}\left|D_{\alpha}^{n+1}f(\tau)\right|^{q}d_{\alpha}\tau,\ a\le t\le x_0,\ z(x_0)=0.$$

Therefore,

$$D_{\alpha}z(t)=-\left|D_{\alpha}^{n+1}f(t)\right|^{q}$$

and

$$\left|D_{\alpha}^{n+1}f(t)\right|=\left(-D_{\alpha}z(t)\right)^{1/q}. \tag{3.7}$$

From (3.6) and (3.7), it follows that

$$\left|R_{n,f}(x_0,t)\right|\left|D_{\alpha}^{n+1}f(t)\right|\le\frac{1}{\alpha^{n+1/p}n!}\frac{\left(x_0^{\alpha}-t^{\alpha}\right)^{n+1/p}}{(np+1)^{1/p}}\left(-z(t)D_{\alpha}z(t)\right)^{\frac{1}{q}}. \tag{3.8}$$

Integrating the inequality (3.8) and using Hölder's inequality for conformable integrals, we have

$$\begin{aligned}
&\int_{t}^{x_0}\left|R_{n,f}(x_0,\tau)\right|\left|D_{\alpha}^{n+1}f(\tau)\right|d_{\alpha}\tau\\
\le\ &\frac{1}{\alpha^{n+1/p}n!(np+1)^{1/p}}\int_{t}^{x_0}\left(x_0^{\alpha}-\tau^{\alpha}\right)^{n+1/p}\left(z(\tau)D_{\alpha}z(\tau)\right)^{\frac{1}{q}}d_{\alpha}\tau\\
\le\ &\frac{1}{\alpha^{n+1/p}n!(np+1)^{1/p}}\left(\int_{t}^{x_0}\left(x_0^{\alpha}-\tau^{\alpha}\right)^{np+1}d_{\alpha}\tau\right)^{\frac{1}{p}}\left(\int_{t}^{x_0}\left(-z(\tau)D_{\alpha}z(\tau)\right)d_{\alpha}\tau\right)^{\frac{1}{q}}\\
=\ &\frac{\left(x_0^{\alpha}-t^{\alpha}\right)^{n+2/p}}{\alpha^{n+2/p}n!\left[(np+1)(np+2)\right]^{1/p}}\frac{(z(t))^{\frac{2}{q}}}{2^{\frac{1}{q}}}.
\end{aligned}$$

This completes the proof. □

Corollary 2 *Under the assumption of Theorem 9 with $p=q=2$, we get*

$$\int_{t}^{x_0}\left|R_{n,f}(x_0,\tau)\right|\left|D_{\alpha}^{n+1}f(\tau)\right|d_{\alpha}\tau\le\frac{\left(x_0^{\alpha}-t^{\alpha}\right)^{n+1}}{2\alpha^{n+2}n!\sqrt{(2n+1)(n+1)}}\int_{t}^{x_0}\left|D_{\alpha}^{n+1}f(\tau)\right|^{2}d_{\alpha}\tau.$$

Theorem 10 *Let $\alpha\in(0,1]$, $f:[a,b]\to\mathbb{R}$ be an $n+1$ times $\alpha-$fractional differentiable function, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, and $t,x_0\in[a,b]$. Then we have the following inequality:*

$$\begin{aligned}
&\left|\int_{x_0}^{t}\left|R_{n,f}(x_0,\tau)\right|\left|D_{\alpha}^{n+1}f(\tau)\right|d_{\alpha}\tau\right| \qquad (3.9)\\
\le\ &\frac{\left|t^{\alpha}-x_0^{\alpha}\right|^{n+2/p}}{\alpha^{n+2/p}2^{\frac{1}{q}}n!\left[(np+1)(np+2)\right]^{1/p}}\left|\int_{x_0}^{t}\left|D_{\alpha}^{n+1}f(\tau)\right|^{q}d_{\alpha}\tau\right|^{\frac{2}{q}}.
\end{aligned}$$

Proof Combining Theorem 8 and Theorem 9, we can easily get the required result. □

Corollary 3 *Under the assumption of Theorem 10 with $p=q=2$, we get*

$$\left|\int_{x_0}^{t} |R_{n,f}(x_0,\tau)| \left|D_\alpha^{n+1} f(\tau)\right| d_\alpha \tau\right| \leq \frac{|t^\alpha - x_0^\alpha|^{n+1}}{2\alpha^{n+1} n! \sqrt{(n+1)(2n+1)}} \left|\int_{x_0}^{t} \left|D_\alpha^{n+1} f(\tau)\right|^2 d_\alpha \tau\right|.$$

Using Theorem 10 and Corollary 3, we obtain the following important inequality.

Corollary 4 *Let $\alpha \in (0,1]$, $f:[a,b] \to \mathbb{R}$ be an $n+1$ times $\alpha-$fractional differentiable function, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, and $t,x_0 \in [a,b]$. If $D_\alpha^k f(x_0)=0$, $k=0,1,...,n$, then we have the following Opial-type inequality:*

$$\begin{aligned}
&\left|\int_{x_0}^{t} |f(\tau)| \left|D_\alpha^{n+1} f(\tau)\right| d_\alpha \tau\right| \\
\leq\ & \min\left\{ \frac{|t^\alpha - x_0^\alpha|^{n+2/p}}{\alpha^{n+2/p} 2^{\frac{1}{q}} n! \left[(np+1)(np+2)\right]^{1/p}} \left|\int_{x_0}^{t} \left|D_\alpha^{n+1} f(\tau)\right|^q d_\alpha \tau\right|^{\frac{2}{q}}, \right. \\
& \left. \frac{|t^\alpha - x_0^\alpha|^{n+1}}{2\alpha^{n+1} n! \sqrt{(n+1)(2n+1)}} \left|\int_{x_0}^{t} \left|D_\alpha^{n+1} f(\tau)\right|^2 d_\alpha \tau\right| \right\}.
\end{aligned}$$

Corollary 5 *If we choose $n=0$ Corollary 4, then we have the following inequality:*

$$\begin{aligned}
&\left|\int_{x_0}^{t} |f(\tau)| \left|D_\alpha f(\tau)\right| d_\alpha \tau\right| \\
\leq\ & \frac{1}{2}\min\left\{ \frac{|t^\alpha - x_0^\alpha|^{2/p}}{2\alpha^{2/p}} \left|\int_{x_0}^{t} \left|D_\alpha f(\tau)\right|^q d_\alpha \tau\right|^{\frac{2}{q}}, \frac{|t^\alpha - x_0^\alpha|}{2\alpha} \left|\int_{x_0}^{t} \left|D_\alpha f(\tau)\right|^2 d_\alpha \tau\right| \right\}.
\end{aligned}$$

Theorem 11 *Let $\alpha \in (0,1]$, $f:[a,b] \to \mathbb{R}$ be an $n+1$ times $\alpha-$fractional differentiable function, $p=1$, $q=\infty$ and $t \in [x_0,b]$. Then we have the inequality*

$$\int_{x_0}^{t} |R_{n,f}(x_0,\tau)| \left|D_\alpha^{n+1} f(\tau)\right| d_\alpha \tau \leq \frac{(t^\alpha - x_0^\alpha)^{n+2}}{\alpha^{n+2}(n+2)!} \left\|D_\alpha^{n+1} f\right\|_{\infty,[x_0,b]}^2 \tag{3.10}$$

where

$$\left\|D_\alpha^{n+1} f\right\|_\infty := \sup_{x\in[a,b]} \left|D_\alpha^{n+1} f(x)\right|.$$

Proof From (2.7), we have

$$\begin{aligned} |R_{n,f}(x_0,t)| &\leq \frac{1}{\alpha^n n!}\int_{x_0}^{t}(t^\alpha-\tau^\alpha)^n\left|D_\alpha^{n+1}f(\tau)\right|d_\alpha\tau \\ &\leq \frac{1}{\alpha^n n!}\left\|D_\alpha^{n+1}f\right\|_{\infty,[x_0,b]}\int_{x_0}^{t}(t^\alpha-\tau^\alpha)^n d_\alpha\tau \\ &= \frac{\left\|D_\alpha^{n+1}f\right\|_{\infty,[x_0,b]}}{\alpha^{n+1}(n+1)!}\left(t^\alpha-x_0^\alpha\right)^{n+1}. \end{aligned} \tag{3.11}$$

Moreover, we get

$$\left|D_\alpha^{n+1}f(t)\right| \leq \left\|D_\alpha^{n+1}f\right\|_{\infty,[x_0,b]}$$

for all $t\in[x_0,b]$.

Therefore it follows that

$$|R_{n,f}(x_0,t)|\left|D_\alpha^{n+1}f(t)\right| \leq \frac{\left\|D_\alpha^{n+1}f\right\|_{\infty,[x_0,b]}^2}{\alpha^{n+1}(n+1)!}\left(t^\alpha-x_0^\alpha\right)^{n+1}. \tag{3.12}$$

Integrating the inequality (3.12), we have

$$\begin{aligned} \int_{x_0}^{t}|R_{n,f}(x_0,\tau)|\left|D_\alpha^{n+1}f(\tau)\right|d_\alpha\tau &\leq \frac{\left\|D_\alpha^{n+1}f\right\|_{\infty,[x_0,b]}^2}{\alpha^{n+1}(n+1)!}\int_{x_0}^{t}\left(\tau^\alpha-x_0^\alpha\right)^{n+1}d_\alpha\tau \\ &= \frac{\left(t^\alpha-x_0^\alpha\right)^{n+2}}{\alpha^{n+2}(n+2)!}\left\|D_\alpha^{n+1}f\right\|_{\infty,[x_0,b]}^2 \end{aligned}$$

This completes the proof of the inequality (3.10). □

Theorem 12 *Let* $p=1,\ q=\infty$ *and* $t\in[a,x_0]$. *Then we have the inequality*

$$\int_{x_0}^{t}|R_{n,f}(x_0,\tau)|\left|D_\alpha^{n+1}f(\tau)\right|d_\alpha\tau \leq \frac{\left(x_0^\alpha-t^\alpha\right)^{n+2}}{\alpha^{n+2}(n+2)!}\left\|D_\alpha^{n+1}f\right\|_{\infty,[a,x_0]}^2. \tag{3.13}$$

Proof From (2.7), we get

$$\begin{aligned} |R_{n,f}(x_0,t)| &= \left| \frac{1}{\alpha^n n!} \int_{x_0}^{t} (t^\alpha - \tau^\alpha)^n D_\alpha^{n+1} f(\tau) d_\alpha \tau \right| \\ &\leq \frac{1}{\alpha^n n!} \int_{t}^{x_0} (\tau^\alpha - t^\alpha)^n \left| D_\alpha^{n+1} f(\tau) \right| d_\alpha \tau \\ &\leq \frac{1}{\alpha^n n!} \left\| D_\alpha^{n+1} f \right\|_{\infty,[x_0,b]} \int_{t}^{x_0} (\tau^\alpha - t^\alpha)^n d_\alpha \tau \\ &= \frac{\left\| D_\alpha^{n+1} f \right\|_{\infty,[x_0,b]}}{\alpha^{n+1}(n+1)!} (x_0^\alpha - t^\alpha)^{n+1}. \end{aligned} \tag{3.14}$$

Furthermore, we have

$$\left| D_\alpha^{n+1} f(t) \right| \leq \left\| D_\alpha^{n+1} f \right\|_{\infty,[a,x_0]} \tag{3.15}$$

for all $t \in [a, x_0]$.

Thus, we obtain

$$\begin{aligned} \int_{x_0}^{t} |R_{n,f}(x_0,\tau)| \left| D_\alpha^{n+1} f(\tau) \right| d_\alpha \tau &\leq \frac{\left\| D_\alpha^{n+1} f \right\|_{\infty,[a,x_0]}^2}{\alpha^{n+1}(n+1)!} \int_{x_0}^{t} (x_0^\alpha - \tau^\alpha)^{n+1} d_\alpha \tau \\ &= \frac{(x_0^\alpha - t^\alpha)^{n+2}}{\alpha^{n+2}(n+2)!} \left\| D_\alpha^{n+1} f \right\|_{\infty,[a,x_0]}^2 \end{aligned}$$

which completes the proof of the inequality (3.13). □

Combining Theorem 11 and Theorem 12, we have the following result.

Corollary 6 *Let $\alpha \in (0,1]$, $f : [a,b] \to \mathbb{R}$ be an $n+1$ times $\alpha-$fractional differentiable function, $p = 1$, $q = \infty$ and $t \in [a,b]$. Then the following inequality holds:*

$$\left| \int_{x_0}^{t} |R_{n,f}(x_0,\tau)| \left| D_\alpha^{n+1} f(\tau) \right| d_\alpha \tau \right| \leq \frac{|t^\alpha - x_0^\alpha|^{n+2}}{\alpha^{n+2}(n+2)!} \left\| D_\alpha^{n+1} f \right\|_\infty^2.$$

Using the Corollary 6, we obtain the following important inequality.

Corollary 7 *Let $\alpha \in (0,1]$, $f : [a,b] \to \mathbb{R}$ be an $n+1$ times $\alpha-$fractional differentiable function, $p = 1$, $q = \infty$ and $t \in [a,b]$. If $D_\alpha^k f(x_0) = 0$, $k = 0,1,...,n$, then we have the following Opial-type inequality:*

$$\left| \int_{x_0}^{t} |f(\tau)| \left| D_\alpha^{n+1} f(\tau) \right| d_\alpha \tau \right| \leq \frac{|t^\alpha - x_0^\alpha|^{n+2}}{\alpha^{n+2}(n+2)!} \left\| D_\alpha^{n+1} f \right\|_\infty^2.$$

Corollary 8 *If we choose* $n = 0$ *Corollary 7, then we have the following inequality:*

$$\left| \int_{x_0}^{t} |f(\tau)| \, |D_\alpha f(\tau)| \, d_\alpha \tau \right| \leq \frac{|t^\alpha - x_0^\alpha|^2}{2\alpha^2} \|D_\alpha f\|_\infty^2 .$$

4. Conclusions

In this study, we presented some Opial-type inequalities for conformable fractional integrals via using the remainder function of Taylor's theorem for conformable integrals. A further study could assess weighted versions of these inequalities.

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