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NEW INTEGRABILITY CONDITIONS OF DERIVATIONAL EQUATIONS OF A SUBMANIFOLD IN A GENERALIZED RIEMANNIAN SPACE

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Abstract

The present work is a continuation of [5] and [6]. In [5] we have obtained derivational equations of a submanifold X_M of a generalized Riemannian space GR_N. Since the basic tensor in GR_N is asymmetric and in this way the connection is also asymmetric, in a submanifold the connection is generally asymmetric too. By reason of this, we define 4 kinds of covariant derivative and obtain 4 kinds of derivational equations. In [6] we have obtained integrability conditions and Gauss-Codazzi equations using the 1^{st} and the 2^{st} kind of covariant derivative.

The present work deals in the cited matter, using the 3^{rd} and the 4^{th} kind of covariant derivative. One obtains three new integrability conditions for derivational equations of tangents and three such conditions for normals of the submanifold, as the corresponding Gauss-Codazzi equations too.

1 Introduction

1.1. A generalized Riemannian space GR_N is a differentiable manifold equipped with an asymmetric basic tensor $G_{ij}(x^1, ..., x^N)$ (the components) where x^i are the local coordinates. The symmetric, respectively antisymmetric part of G_{ij} are H_{ij} and K_{ij}.

For the lowering and rasing of indices in GR_N one uses H_{ij}, respectively H^{ij}, where

$$(H^{ij}) = (H_{ij})^{-1}, \quad (det(H_{ij}) \neq 0). \tag{1.1}$$

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Cristoffel symbols at GR_N are

$$\Gamma_{i.jk} = \frac{1}{2}(G_{ji,k} - G_{jk,i} + G_{ik,j}), \quad \Gamma^i_{jk} = H^{ip}\Gamma_{p.jk}, \tag{1.2}$$

where, for example, $G_{ji,k} = \partial G_{ji}/\partial x^k$. Based on the asymmetry of G_{ij}, it follows that the Cristoffel symbols are also asymmetric with respect to j, k in (1.2).

By equations

$$x^i = x^i(u^1, ..., u^M) \equiv x^i(u^\alpha), \quad i = 1, .., N, \tag{1.3}$$

a submanifold X_M is defined in local coordinates. If $rank(B^i_\alpha) = M$ ($B^i_\alpha = \partial x^i/\partial u^\alpha$) and

$$g_{\alpha\beta} = B^i_\alpha B^j_\beta G_{ij}, \tag{1.4}$$

X_M becomes $GR_M \subset GR_N$, with **induced basic tensor** (1.4), which is generally also asymmetric. Note that in the present work Latin indices $i, j, ...$ take values $1, \dots, N$ and refer to the GR_N, while the Greek ones take values $1, \dots, M$ and refer to the GR_M.

In the GR_M are valid the relations similar to (1.1) and (1.2). The symmetric part of $g_{\alpha\beta}$ is denoted with $h_{\alpha\beta}$, and antisymmetric one with $k_{\alpha\beta}$, where e.g.

$$h_{\alpha\beta} = B^i_\alpha B^j_\beta H_{ij}, \quad (h^{\alpha\beta}) = (h_{\alpha\beta})^{-1}. \tag{1.5}$$

Cristoffel symbols $\widetilde{\Gamma}_{\alpha.\beta\gamma}$, $\widetilde{\Gamma}^\alpha_{\beta\gamma} = h^{\alpha\pi}\widetilde{\Gamma}_{\pi.\beta\gamma}$ are expressed by $g_{\alpha\beta}$ analogously to (1.2).

For the unit, mutually orthogonal vectors N^i_A, which are orthogonal to the GR_M too, we have [1]

$$H_{ij}N^i_A N^j_B = e_A\delta^A_B = h_{AB},\ e_A \in \{-1, 1\},\ H_{ij}N^i_A B^j_\alpha = 0, \tag{1.6}$$

where $A, B, \dots \in \{M + 1, \dots, N\}$.

As it is known, the following relations between Cristoffel symbols of a generalized Riemannian space and its subspace are valid:

$$\widetilde{\Gamma}_{\alpha.\beta\gamma} = \Gamma_{i.jk}B^i_\alpha B^j_\beta B^k_\gamma + H_{ij}B^i_\alpha B^j_{\beta,\gamma}, \tag{1.7}$$

$$\widetilde{\Gamma}^\alpha_{\beta\gamma} = h^{\pi\alpha}\widetilde{\Gamma}_{\pi.\beta\gamma} = h^{\pi\alpha}(\Gamma_{i.jk}B^i_\pi B^j_\beta B^k_\gamma + H_{ij}B^i_\pi B^j_{\beta,\gamma}), \tag{1.8}$$

i.e.

$$\widetilde{\Gamma}^\alpha_{\beta\gamma} = h^{\pi\alpha}H_{pi}B^p_\pi(\Gamma^i_{jk}B^j_\beta B^k_\gamma + B^i_{\beta,\gamma}). \tag{1.8'}$$

1.2. The set of normals of the submanifold $X_M \subset GR_N$ make a **normal bundle** for X_M, and we note it X^N_{N-M}. One can introduce a metric tensor on X^N_{N-M}

$$g_{AB} = G_{ij}N^i_A N^j_B, \tag{1.9}$$

which is asymmetric in a general case.

The symmetric part is

$$(1.10)\qquad h_{AB}=H_{ij}N^i_AN^j_B \underset{(1.5)}{=} e_A\delta^A_B=h_{BA}=\begin{cases} e_A, & \text{A=B},\\ 0, & \text{otherwise.}\end{cases},\ e_A\in\{-1,1\}.$$

If

$$(h^{AB})=(h_{AB})^{-1},$$

we have

$$h^{AB}=e_A\delta^A_B=h_{AB}=h^{BA}.$$

On X^N_{N-M} one can define in two manners connection coefficients

$$(1.11)\qquad \overline{\Gamma}^A_{\substack{1\\2}B\mu}=H_{ij}h^{AQ}N^j_Q(N^i_{B,\mu}+\Gamma^i_{\substack{pq\\qp}}N^p_BB^q_\mu).$$

Being the coefficients Γ, $\widetilde{\Gamma}$, $\overline{\Gamma}$ non-symmetric in general, for a tensor, defined at points of GR_M, is possible define four kinds of covariant derivative. For example

$$(1.12)\qquad \begin{aligned}\underset{\substack{1\\2\\3\\4}}{\nabla}_\mu t^{i\alpha A}_{j\beta B}\equiv t^{i\alpha A}_{j\beta B\underset{\substack{1\\2\\3\\4}}{|}\mu}=t^{i\alpha A}_{j\beta B,\mu}+\Gamma^i_{\substack{pm\\mp\\pm\\mp}}t^{p\alpha A}_{j\beta B}B^m_\mu-\Gamma^p_{\substack{jm\\mj\\mj\\jm}}t^{i\alpha A}_{p\beta B}B^m_\mu\\ +\widetilde{\Gamma}^\alpha_{\substack{\pi\mu\\\mu\pi\\\pi\mu\\\mu\pi}}t^{i\pi A}_{j\beta B}-\widetilde{\Gamma}^\pi_{\substack{\beta\mu\\\mu\beta\\\mu\beta\\\beta\mu}}t^{i\alpha A}_{j\pi B}+\overline{\Gamma}^A_{\substack{1\\2\\1\\2}P\mu}t^{i\alpha P}_{j\beta B}-\overline{\Gamma}^P_{\substack{1\\2\\2\\1}B\mu}t^{i\alpha A}_{j\beta P}\end{aligned}$$

In this way four connection $\underset{\theta}{\nabla}$, $\theta\in\{1,\dots,4\}$, on $X_M\subset GR_N$ are defined. We shall note the obtained structures $(X_M\subset GR_N,\underset{\theta}{\nabla},\ \theta\in\{1,\dots,4\})$.

2 New first and second kind integrability conditions of derivational equations

2.0. In [5] are obtained derivational equations of a submanifold in a GR_N, and in [6] integrability conditions of these equations in the structure $(X_M\subset GR_N,\ \underset{\theta}{\nabla},\ \theta\in\{1,2\})$. In the present work we engage in this problem for the structure $(X_M\subset GR_N,\ \underset{\theta}{\nabla},\ \theta\in\{3,4\})$.

As it is proved in [5] (Th. 1.2.), *derivational equations* in the considered case for a tangent are

$$(2.1)\qquad B^i_{\alpha\underset{\theta}{|}\mu}=\sum_P\underset{\theta}{\Omega}_{P\alpha\mu}N^i_P,\quad \theta\in\{3,4\},$$

and then for induced torsion in X_M is valid

$$\widetilde{T}^{\alpha}_{\beta\gamma}=0 \quad (\widetilde{\Gamma}^{\alpha}_{\beta\gamma}=\widetilde{\Gamma}^{\alpha}_{\gamma\beta}). \tag{2.2}$$

By virtue of the Th. 2.3. in [5], for unit normal is

$$N^i_{A\underset{\theta}{|}\mu}=-e_A\underset{\theta}{\Omega}_{A\rho\mu}h^{\pi\rho}B^i_\pi, \quad \theta\in\{3,4\}, \tag{2.3}$$

and

$$\underset{1}{\overline{\Gamma}}{}^A_{B\mu}=\underset{2}{\overline{\Gamma}}{}^A_{B\mu}=\overline{\Gamma}^A_{B\mu}, \tag{2.4}$$

in (1.12), and based on (1.8) in [5]

$$\underset{\substack{1\\2}}{\Omega}_{P\alpha\mu}=e_PH_{ij}N^i_P(B^j_{\alpha,\mu}+\underset{mp}{\Gamma^j_{pm}}B^p_\alpha B^m_\mu)=\underset{\substack{3\\4}}{\Omega}_{P\alpha\mu}. \tag{2.5}$$

In relation with (2.2,4), the addends in (1.12), related to X_M and to X^N_{N-M} are not different for separate kinds of derivatives, and (1.12) now becomes

$$\begin{aligned} t^{i\alpha A}_{j\beta B\underset{\substack{1\\2\\3\\4}}{|}\mu}&=t^{i\alpha A}_{j\beta B,\mu}+\underset{\substack{mp\\pm\\mp}}{\Gamma^i_{pm}}t^{p\alpha A}_{j\beta B}B^m_\mu-\underset{\substack{mj\\mj\\jm}}{\Gamma^p_{jm}}t^{i\alpha A}_{p\beta B}B^m_\mu\\ &+\widetilde{\Gamma}^\alpha_{\pi\mu}t^{i\pi A}_{j\beta B}-\widetilde{\Gamma}^\pi_{\beta\mu}t^{i\alpha A}_{j\pi B}+\overline{\Gamma}^A_{P\mu}t^{i\alpha P}_{j\beta B}-\overline{\Gamma}^P_{B\mu}t^{i\alpha A}_{j\beta P}, \end{aligned} \tag{2.6}$$

where the coefficients $\widetilde{\Gamma}$ are symmetric, and $\overline{\Gamma}$ are unique ($\underset{1}{\overline{\Gamma}}=\underset{2}{\overline{\Gamma}}=\overline{\Gamma}$). If in a differentiated tensor no exists indices as $i,j,...$, we write $|\mu$ instead of $\underset{\theta}{|}\mu$.

Using (2.1,3), we get (see (2.4) in [6])

$$\begin{aligned} B^i_{\alpha\underset{\theta}{|}\mu\underset{\omega}{|}\nu}-B^i_{\alpha\underset{\omega}{|}\nu\underset{\theta}{|}\mu}&=\sum_P[e_Ph^{\pi\rho}(-\underset{\theta}{\Omega}_{P\alpha\mu}\underset{\omega}{\Omega}_{P\rho\nu}+\underset{\omega}{\Omega}_{P\alpha\nu}\underset{\theta}{\Omega}_{P\rho\mu})B^i_\pi\\ &+(\underset{\theta}{\Omega}_{P\alpha\mu\underset{\omega}{|}\nu}-\underset{\omega}{\Omega}_{P\alpha\nu\underset{\theta}{|}\mu})N^i_P], \quad \theta,\omega\in\{3,4\}. \end{aligned} \tag{2.7}$$

2.1. With respect of Ricci-type identities (12) and (13) from [2], and taking into consideration (2.2), we have

$$B^i_{\alpha\underset{\theta}{|}\mu\underset{\theta}{|}\nu}-B^i_{\alpha\underset{\theta}{|}\nu\underset{\theta}{|}\mu}=\underset{\theta-2}{R}{}^i_{pmn}B^p_\alpha B^m_\mu B^n_\nu-\widetilde{R}^\pi_{\alpha\mu\nu}B^i_\pi,\ \theta\in\{3,4\}, \tag{2.8}$$

where

$$\underset{1}{R}{}^i_{jmn}=\Gamma^i_{jm,n}-\Gamma^i_{jn,m}+\Gamma^p_{jm}\Gamma^i_{pn}-\Gamma^p_{jn}\Gamma^i_{pm}, \tag{2.9a}$$

(2.9*b*) $$\underset{2}{R}^i_{jmn} = \Gamma^i_{mj,n} - \Gamma^i_{nj,m} + \Gamma^p_{mj}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{mp}$$

are **curvature tensors of the 1st, respectively 2nd kind** of GR_N and $\widetilde{R}^\alpha_{\beta\mu\nu}$ is, with respect of (2.2), curvature tensor of $R_M \subset GR_N$.

We obtained in [6] three kinds integrability conditions for derivational equation of a tangent B^i_α, i.e. for $B^i_{\alpha\underset{\theta}{|}\mu}$, $\theta \in \{1,2\}$. We shall consider here such conditions for $\theta \in \{3,4\}$.

If one substitutes $\theta = \omega \in \{3,4\}$ into (2.7) and compares with (2.8), taking into consideration (2.5) and (2.6), we get

(2.10) $$\begin{aligned}\underset{\theta-2}{R}^{\ i}_{pmn}B^p_\alpha B^m_\mu B^n_\nu = [\widetilde{R}^\pi_{\alpha\mu\nu} - \sum_P e_P h^{\pi\rho}(\underset{\theta}{\Omega}_{P\alpha\mu}\underset{\theta}{\Omega}_{P\rho\nu} - \underset{\theta}{\Omega}_{P\alpha\nu}\underset{\theta}{\Omega}_{P\rho\mu})]B^i_\pi \\ + \sum_P [\underset{\theta}{\Omega}_{P\alpha\mu|\nu} - \underset{\theta}{\Omega}_{P\alpha\nu|\mu}]N^i_P, \quad \theta \in \{3,4\},\end{aligned}$$

which are **the 1st and the 2nd integrability conditions** of derivational equation (2.1) in the structure $(X_M \subset GR_N,\ \underset{\theta}{\nabla},\ \theta \in \{3,4\})$.

a) Composing the previous equation with $H^{ij}B^j_\beta$, one gets

(2.11) $$\underset{\theta-2}{R}_{jpmn}B^j\beta B^p_\alpha B^m_\mu B^n_\nu = \widetilde{R}_{\beta\alpha\mu\nu} - \sum_P e_P(\underset{\theta}{\Omega}_{P\alpha\mu}\underset{\theta}{\Omega}_{P\beta\nu} - \underset{\theta}{\Omega}_{P\alpha\nu}\underset{\theta}{\Omega}_{P\beta\mu}),\ \theta \in \{3,4\},$$

where

(2.12 *a, b*) $$\underset{\theta-2}{R}_{jpmn} = H_{ij}\underset{\theta-2}{R}^{\ i}_{pmn}, \quad \widetilde{R}_{\beta\alpha\mu\nu} = h_{\pi\beta}\widetilde{R}^\pi_{\alpha\mu\nu},\ \theta \in \{3,4\}.$$

Taking into count the antisymmetry of the tensors (2.12) with respect of the first two indices and substituting i in place of p, the equation (2.11) becomes

(2.13) $$\widetilde{R}_{\alpha\beta\mu\nu} = \underset{\theta-2}{R}_{ijmn}B^i_\alpha B^j_\beta B^m_\mu B^n_\nu - \sum_P e_P(\underset{\theta}{\Omega}_{P\alpha\mu}\underset{\theta}{\Omega}_{P\beta\nu} - \underset{\theta}{\Omega}_{P\alpha\nu}\underset{\theta}{\Omega}_{P\beta\mu}),\ \theta \in \{3,4\},$$

which are **Gauss equations of the 1st and the 2nd kind** in the structure $(X_M \subset GR_N,\ \underset{\theta}{\nabla},\ \theta \in \{3,4\})$.

b) Composing the equation (2.10) with $H_{ij}N^j_Q$ we obtain finally

(2.14) $$\underset{\theta-2}{R}_{ijmn}B^i_\alpha N^j_Q B^m_\mu B^n_\nu = e_Q(\underset{\theta}{\Omega}_{Q\alpha\nu|\mu} - \underset{\theta}{\Omega}_{Q\alpha\mu|\nu}),\ \theta \in \{3,4\},$$

and that are **the 1st Codazzi equations of the 1st and the 2nd kind** at the cited structure.

2.2. Consider the same matter for the unit normal N^i_A. Using (2.3,1), we obtain (see (2.13) in [6]):

$$\begin{aligned}N^i_{A\underset{\theta}{|}\mu\underset{\omega}{|}\nu} - N^i_{A\underset{\omega}{|}\nu\underset{\theta}{|}\mu} &= -e_A h^{\pi\rho}[(\underset{\theta}{\Omega}_{A\rho\mu\underset{\omega}{|}\nu} - \underset{\omega}{\Omega}_{A\rho\nu\underset{\theta}{|}\mu})B^i_\pi \\ &\quad + \sum_P (\underset{\theta}{\Omega}_{A\rho\mu}\underset{\omega}{\Omega}_{P\pi\nu} - \underset{\omega}{\Omega}_{A\rho\nu}\underset{\theta}{\Omega}_{P\pi\mu})N^i_P].\end{aligned} \tag{2.15}$$

In order to find corresponding Ricci-type identity for the left side of this equation for $\theta = \omega \in \{3,4\}$, we use (2.6). Firstly, we have

$$N^i_{A\underset{3}{|}\mu} = N^i_{A,\mu} + \Gamma^i_{pm} N^p_A B^m_\mu - \overline{\Gamma}^P_{A\mu} N^i_P, \tag{2.16}$$

and further

$$\begin{aligned}N^i_{A\underset{3}{|}\mu\underset{3}{|}\nu} &= (N^i_{A\underset{3}{|}\mu})_{,\nu} + \Gamma^i_{sn} N^s_{A\underset{3}{|}\mu} B^n_\nu - \widetilde{\Gamma}^\sigma_{\mu\nu} N^s_{A\underset{3}{|}\sigma} - \overline{\Gamma}^S_{A\nu} N^i_{S\underset{3}{|}\mu} \\ &= N^i_{A,\mu\nu} + \Gamma^i_{pm,n} N^p_A B^m_\mu B^n_\nu + \Gamma^i_{pm} N^p_{A,\nu} B^m_\mu + \Gamma^i_{pm} N^p_A B^m_{\mu,\nu} \\ &- \overline{\Gamma}^P_{A\mu,\nu} N^i_P - \overline{\Gamma}^P_{A\mu} N^i_{P,\nu} + \Gamma^i_{sn} N^s_{A,\mu} B^n_\nu + \Gamma^i_{sn}\Gamma^s_{pm} B^n_\nu N^p_A B^m_\mu \\ &- \Gamma^i_{sn} N^s_P \overline{\Gamma}^P_{A\mu} B^n_\nu - \widetilde{\Gamma}^\sigma_{\mu\nu} N^i_{A,\sigma} - \widetilde{\Gamma}^\sigma_{\mu\nu}\Gamma^i_{pm} N^p_A B^m_\sigma + \widetilde{\Gamma}^\sigma_{\mu\nu}\overline{\Gamma}^P_{A\sigma} N^i_P \\ &- \overline{\Gamma}^S_{A\nu} N^i_{S,\mu} - \overline{\Gamma}^S_{A\nu}\Gamma^i_{pm} N^p_S B^m_\mu + \overline{\Gamma}^S_{A\nu}\overline{\Gamma}^P_{S\mu} N^i_P,\end{aligned}$$

wherefrom

$$N^i_{A\underset{3}{|}\mu\underset{3}{|}\nu} - N^i_{A\underset{3}{|}\nu\underset{3}{|}\mu} = \underset{1}{\overline{R}}{}^i_{pmn} N^p_A B^m_\mu B^n_\nu - \overline{R}^P_{A\mu\nu} N^i_P, \tag{2.17}$$

where

$$\overline{R}^A_{B\mu\nu} = \overline{\Gamma}^A_{B\mu,\nu} - \overline{\Gamma}^A_{B\nu,\mu} + \overline{\Gamma}^P_{B\mu}\overline{\Gamma}^A_{P\nu} - \overline{\Gamma}^P_{B\nu}\overline{\Gamma}^A_{P\mu}, \tag{2.18}$$

is **curvature tensor of the space$\mathbf{GR_N}$ with respect to the normal submanifold** in the structure $(X_M \subset GR_N,\ \underset{\theta}{\nabla},\ \theta \in \{3,4\})$.

By means of the 4^{th} kind of covariant derivative we obtain an equation corresponding to (2.17), and we conclude

$$N^i_{A\underset{\theta}{|}\mu\underset{\theta}{|}\nu} - N^i_{A\underset{\theta}{|}\nu\underset{\theta}{|}\mu} = \underset{\theta-2}{R}{}^i_{pmn} N^p_A B^m_\mu B^n_\nu - \overline{R}^P_{A\mu\nu} N^i_P,\ \ \theta \in \{3,4\}. \tag{2.19}$$

If one substitutes into (2.15) $\theta = \omega \in \{3,4\}$ and equilizes the right sides of obtained equation and (2.19), we get **the $\mathbf{1^{st}}$ and the $\mathbf{2^{nd}}$ kind integrability conditions of derivational equation (2.3)** in the structure $(X_M \subset GR_N,\ \underset{\theta}{\nabla},\ \theta \in \{3,4\})$:

$$\begin{aligned}\underset{\theta-2}{R}{}^i_{pmn} N^p_A B^m_\mu B^n_\nu &= e_A h^{\pi\rho}(\underset{\theta}{\Omega}_{A\rho\mu|\nu} - \underset{\theta}{\Omega}_{A\rho\nu|\mu})B^i_\pi \\ &+ [\overline{R}^P_{A\mu\nu} - e_A h^{\pi\rho}\sum_P(\underset{\theta}{\Omega}_{A\rho\mu}\underset{\theta}{\Omega}_{P\pi\nu} - \underset{\theta}{\Omega}_{A\rho\nu}\underset{\theta}{\Omega}_{P\pi\mu})]N^i_P,\ \ \theta \in \{3,4\}.\end{aligned} \tag{2.20}$$

a) If we compose this equation with $H_{ij}B^j_\beta$ one obtains an equation equivalent with (2.14),that is the 1^{st} Codazzi equation of the 1^{st} and the 2^{nd} kind for the structure $(X_M \subset GR_N,\ \underset{\theta}{\nabla},\ \theta \in \{3,4\})$.

b) By composing the equation (2.20) with $H_{ij}N^j_B$, one obtains endly

$$\underset{\theta-2}{R}{}_{ijmn}N^i_A N^j_B B^m_\mu B^n_\nu = \overline{R}_{AB\mu\nu} + e_A e_B h^{\pi\rho}(\underset{\theta}{\Omega}{}_{A\pi\mu}\underset{\theta}{\Omega}{}_{B\rho\nu} - \underset{\theta}{\Omega}{}_{A\pi\nu}\underset{\theta}{\Omega}{}_{B\rho\mu}), \tag{2.21}$$

where

$$\overline{R}_{AB\mu\nu} = h_{AP}\overline{R}^P_{B\mu\nu}. \tag{2.22}$$

The equation (2.21) is **the $2^{\mathbf{nd}}$ Codazzi equation of the $1^{\mathbf{st}}$ and the $2^{\mathbf{nd}}$ kind** for the structure $(X_M \subset GR_N,\ \underset{\theta}{\nabla},\ \theta \in \{3,4\})$.

Based on expressed above, the next theorems are valid:

Theorem 2.1. *The 1^{st} and the 2^{nd} kind integrability conditions for derivational equations* (2.1), (2.3) *in the in the structure* $(X_M \subset GR_N,\ \underset{\theta}{\nabla},\ \theta \in \{3,4\})$ *are given by equations* (2.10), (2.20) *respectively, where* $\underset{\theta}{\Omega}$ *is given in* (2.5), $\underset{1}{R}$, $\underset{2}{R}$ *in* (2.9), $\widetilde{R}$ *is curvature tensor of the symmetric connection* $\widetilde{\Gamma}$, *while* $\overline{R}$ *is given in* (2.18), (2.22).

Theorem 2.2. *The Gauss equations of the 1^{st} and the 2^{nd} kind in the structure* $(X_M \subset GR_N,\ \underset{\theta}{\nabla},\ \theta \in \{3,4\})$ *are given in* (2.13), *the 1^{st} Codazzi equations of the 1^{st} and the 2^{nd} kind in* (2.14), *and the 2^{nd} Codazzi equations of the 1^{st} and the 2^{nd} kind in* (2.21) *in the same structure.*

3 Third kind integrability condition of derivational equations

3.1. Using simultaneously the 3^{rd} and the 4^{th} kind of covariant derivative by virtue of (2.6), we obtain Ricci-type identity (eq. (46) in [2]):

$$B^i_{\alpha\underset{3}{|}\mu\underset{4}{|}\nu} - B^i_{\alpha\underset{4}{|}\nu\underset{3}{|}\mu} = \underset{4}{R}{}^i_{p\mu\nu}B^p_\alpha - \widetilde{R}^\pi_{\alpha\mu\nu}B^i_\pi, \tag{3.1}$$

where

$$\underset{4}{R}{}^i_{j\mu\nu} = (\Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm})B^m_\mu B^n_\nu + T^i_{jm}(B^m_{\mu,\nu} - \widetilde{\Gamma}^\pi_{\nu\mu}B^m_\pi) \tag{3.2}$$

is **curvature tensor of the $4^{\mathbf{th}}$ kind of $\mathbf{GR_N}$ with respect to $\mathbf{X_M \subset GR_N}$**.

On the other hand, if we put into (2.7) $\theta = 3$, $\omega = 4$ and compare the obtained equation with (3.1), we obtain **the 3[rd] kind integrability condition** of derivational equation (2.1) in the structure $(X_M \subset GR_N,\ \underset{\theta}{\nabla},\ \theta \in \{3,4\})$:

$$\begin{aligned}R^i_{\underset{4}{}p\mu\nu} B^p_\alpha = [\widetilde{R}^\pi_{\alpha\mu\nu} - \sum_P e_P h^{\pi\rho}(\underset{1}{\Omega}_{P\alpha\mu}\underset{2}{\Omega}_{P\rho\nu} - \underset{2}{\Omega}_{P\alpha\nu}\underset{1}{\Omega}_{P\rho\mu})]B^i_\pi \\ + \sum_P(\underset{1}{\Omega}_{P\alpha\mu|\nu} - \underset{2}{\Omega}_{P\alpha\nu|\mu})N^i_P.\end{aligned} \tag{3.3}$$

a) Composing previous equation with $H_{ij}B^j_\beta$, we get

$$\underset{4}{R}_{jp\mu\nu} B^j_\beta B^p_\alpha = \widetilde{R}_{\beta\alpha\mu\nu} - \sum_P e_P(\underset{1}{\Omega}_{P\alpha\mu}\underset{2}{\Omega}_{P\beta\nu} - \underset{2}{\Omega}_{P\alpha\nu}\underset{1}{\Omega}_{P\beta\mu}),$$

i.e., exchanging $j \to i$, $p \to j$, $\alpha \leftrightarrow \beta$, it follows that

$$\widetilde{R}_{\alpha\beta\mu\nu} = \underset{4}{R}_{ij\mu\nu} B^i_\alpha B^j_\beta - \sum_P e_P(\underset{1}{\Omega}_{P\alpha\mu}\underset{2}{\Omega}_{P\beta\nu} - \underset{2}{\Omega}_{P\alpha\nu}\underset{1}{\Omega}_{P\beta\mu}), \tag{3.4}$$

where

$$\underset{4}{R}_{ij\mu\nu} = H_{ip}\underset{4}{R}^p_{j\mu\nu}. \tag{3.5}$$

The equation (3.4) is **Gauss equation of the 3[rd] in the structure** $(X_M \subset GR_N,\ \underset{\theta}{\nabla},\ \theta \in \{3,4\})$.

b) Composing (3.4) with $H_{ij}N^j_Q$, we obtain

$$\underset{4}{R}_{ij\mu\nu} N^i_Q B^j_\alpha = e_Q(\underset{1}{\Omega}_{Q\alpha\mu|\nu} - \underset{2}{\Omega}_{Q\alpha\nu|\mu}).$$

This is **the 1[st] Codazzi equation of the 3[rd] kind** in the cited structure.

3.2. On the base of (2.6) and (2.16) we have

$$\begin{aligned}N^i_{A\underset{3}{|}\mu\underset{4}{|}\nu} &= (N^i_{A\underset{3}{|}\mu})_{,\nu} + \Gamma^i_{ns}N^s_{A\underset{3}{|}\mu}B^n_\nu - \widetilde{\Gamma}^\sigma_{\mu\nu}N^i_{A\underset{3}{|}\sigma} - \overline{\Gamma}^S_{A\nu}N^i_{S\underset{3}{|}\mu} \\
&= N^i_{A,\mu\nu} + \Gamma^i_{pm,n}N^p_A B^m_\mu B^n_\nu + \Gamma^i_{pm}N^p_{A,\nu}B^m_\mu + \Gamma^i_{pm}N^p_A B^m_{\mu,nu} \\
&- \overline{\Gamma}^P_{A\mu,\nu}N^i_P - \overline{\Gamma}^P_{A\mu}N^i_{P,\nu} + \Gamma^i_{ns}N^s_{A,\mu}B^n_\nu + \Gamma^i_{ns}\Gamma^s_{pm}B^n_\nu N^p_A B^m_\mu \\
&- \Gamma^i_{ns}N^s_P\overline{\Gamma}^P_{A\mu}B^n_\nu - \widetilde{\Gamma}^\sigma_{\mu\nu}N^i_{A,\sigma} - \widetilde{\Gamma}^\sigma_{\mu\nu}\Gamma^i_{pm}N^p_A B^m_\sigma + \widetilde{\Gamma}^\sigma_{\mu\nu}\overline{\Gamma}^P_{A\sigma}N^i_P \\
&- \overline{\Gamma}^S_{A\nu}N^i_{S,\mu} - \overline{\Gamma}^S_{A\nu}\Gamma^i_{pm}N^p_S B^m_\mu + \overline{\Gamma}^S_{A\nu}\overline{\Gamma}^P_{S\mu}N^i_P,\end{aligned}$$

and

$$N^i_{A\underset{3}{|}\mu\underset{4}{|}\nu} - N^i_{A\underset{4}{|}\nu\underset{3}{|}\mu} = \underset{4}{R}^i_{p\mu\nu}N^p_A - \overline{R}^P_{A\mu\nu}N^i_P, \tag{3.6}$$

where $\underset{4}{R}$ is given in (3.2), and $\overline{R}$ in (2.18).

By substituting into (2.15) $\theta = 3$, $\omega = 4$ and comparing the obtained equation with (3.6), we obtain **the 3$^{\mathbf{rd}}$ kind integrability condition of derivational equation (2.3)** in the structure $(X_M \subset GR_N,\ \underset{\theta}{\nabla},\ \theta \in \{3,4\})$:

$$\begin{aligned}\underset{4}{R}{}^i_{p\mu\nu}N^p_A &= -e_A h^{\pi\rho}(\underset{1}{\Omega}_{A\rho\mu|\nu} - \underset{2}{\Omega}_{A\rho\nu|\mu})B^i_\pi \\ &\quad + [\overline{R}^P_{A\mu\nu} - e_A h^{\pi\rho}\sum_P(\underset{1}{\Omega}_{A\rho\mu}\underset{2}{\Omega}_{P\pi\nu} - \underset{2}{\Omega}_{A\rho\nu}\underset{1}{\Omega}_{P\pi\mu})]N^i_P.\end{aligned} \tag{3.11}$$

a) Composing this equation with $H_{ij}B^j_\beta$ one obtains the equation of the form (3.5), that is the 1^{st} Codazzi of the 3^{rd} kind.

b) Composing (3.7) with $H_{ij}N^j_B$, we obtain **the 2$^{\mathbf{nd}}$ Codazzi equation of the 3^{rd} kind** in the above cited structure:

$$\underset{4}{R}_{ij\mu\nu}N^i_A N^j_B = \overline{R}_{AB\mu\nu} + e_A e_B h^{\pi\rho}(\underset{1}{\Omega}_{A\rho\mu}\underset{2}{\Omega}_{B\pi\nu} - \underset{2}{\Omega}_{A\rho\nu}\underset{1}{\Omega}_{B\pi\mu}). \tag{3.8}$$

From exposed, the following theorems are valid.

Theorem 3.1. *The 3^{rd} kind integrability conditions of derivational equations* (2.1, 3) *for $(X_M \subset GR_N$, with the structure $(X_M \subset GR_N,\ \underset{\theta}{\nabla},\ \theta \in \{3,4\})$, where the connection $\underset{\theta}{\nabla}$ is defined in* (2.6)*, are given:*

- for tangents B^i_α by equation (3.3)*,*

- for normals N^i_A by equation (3.7)*.*

Theorem 3.2. *In the same structure (from the previous theorem) the Gauss equation of the 3^{rd} kind for $X_M \subset GR_N$ is given in* (3.4)*, the 1^{st} Codazzi equation of the 3^{rd} kind by* (3.5)*, and the 2^{nd} Codazzi equation of the 3^{rd} kind by* (3.8)*.*

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