Zhao et al. *Journal of Inequalities and Applications* (2018) 2018:302
https://doi.org/10.1186/s13660-018-1896-3

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

# New Jensen and Hermite–Hadamard type inequalities for *h*-convex interval-valued functions

Dafang Zhao[1,2*], Tianqing An[1], Guoju Ye[1] and Wei Liu[1]

*Correspondence:
dafangzhao@163.com
[1]College of Science, Hohai University, Nanjing, China
[2]School of Mathematics and Statistics, Hubei Normal University, Huangshi, China

**Abstract**

In this paper, we introduce the *h*-convex concept for interval-valued functions. By using the *h*-convex concept, we present new Jensen and Hermite–Hadamard type inequalities for interval-valued functions. Our inequalities generalize some known results.

**MSC:** 26D15; 26E25; 28B20

**Keywords:** Interval-valued functions; Jensen inequality; Hermite–Hadamard inequality; *h*-convex

## 1 Introduction

The theory of interval analysis has a long history which can be traced back to Archimedes' computation of the circumference of a circle. It fell into oblivion for a long time because of lack of applications to other sciences. To the best of our knowledge, significant work did not appear in this area until the 1950s. The first monograph on interval analysis is the celebrated book of R.E. Moore [28]. One of the initial uses of interval analysis was to compute the error bounds of the numerical solutions of a finite state machine. However, interval analysis has emerged as very useful over the last fifty years due to its many applications in various fields. We now see applications in automatic error analysis [42], computer graphics [45], neural network output optimization [47], and many others. For more fundamental results and applications of interval analysis theory, we refer the reader to the papers [7, 8, 10, 11, 37] and monograph [29].

Recently, several classical integral inequalities have been extended not only to the context of interval-valued functions by Chalco-Cano et al. [5, 6], Román-Flores et al. [40, 41], Flores-Franulič et al. [19], Costa and Román-Flores [12], but also to more general set-valued maps by Matkowski and Nikodem [24], Mitroi et al. [27], and Nikodem et al. [30]. In particular, Costa [9] presented a new fuzzy version of Jensen inequalities type integral for fuzzy-interval-valued functions. Motivated by Costa [9] and Dragomir [15], we introduce the *h*-convex concept for interval-valued functions. Under the *h*-convex concept, we present new Jensen type inequalities for interval-valued functions. The second objective of the article is to promote the following inequality which is known as the Hermite–

Hadamard inequality [21, 22]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2},$$

where $f:[a,b]\to\mathbb{R}$ is a convex function. For various interesting extensions and generalizations of Hermite–Hadamard inequalities, see [16–18, 20, 26, 31, 34–36, 39, 48–50]. In [43], Sarikaya et al. proved a variant of the Hermite–Hadamard inequality for $h$-convex function as follows.

**Theorem 1.1** *Let $f:[a,b]\to\mathbb{R}$ be an $h$-convex function and $h(\frac{1}{2})\neq 0$. Then*

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \big[f(a)+f(b)\big]\int_0^1 h(t)\,dt.$$

Since then, some further refinements and extensions of the Hermite–Hadamard inequalities for $h$-convex functions have been extensively studied in [2, 14, 25, 32, 33, 44]. Some Hermite–Hadamard and Jensen type inequalities for strongly $h$-convex functions were obtained also in [1, 23]. In this paper, we establish some Hermite–Hadamard type inequalities for $h$-convex interval-valued functions. Our results generalize the previous inequalities presented in [9, 32, 43].

The paper is organized as follows. After a section of preliminaries, in Sect. 3 the $h$-convex ($h$-concave, $h$-affine) concepts for interval-valued functions are given. Moreover, some Jensen type inequalities and equalities are proved, respectively. In Sect. 4, we obtain some Hermite–Hadamard type inequalities for $h$-convex interval-valued functions. In Sect. 5, we discuss the main results and limitation of the present studies. We end with Sect. 6 of conclusions and future work.

## 2 Preliminaries

In this section, we recall some basic definitions, notations, properties, and results on interval analysis, which are used throughout the paper. A real interval $[u]$ is the bounded, closed subset of $\mathbb{R}$ defined by

$$[u] = [\underline{u},\overline{u}] = \{x\in\mathbb{R} \mid \underline{u}\leq x\leq\overline{u}\},$$

where $\underline{u},\overline{u}\in\mathbb{R}$ and $\underline{u}\leq\overline{u}$. The numbers $\underline{u}$ and $\overline{u}$ are called the left and the right endpoints of $[\underline{u},\overline{u}]$, respectively. When $\underline{u}$ and $\overline{u}$ are equal, the interval $[u]$ is said to be degenerate. In this paper, the term interval will mean a nonempty interval. We call $[u]$ positive if $\underline{u}>0$ or negative if $\overline{u}<0$. The inclusion "$\subseteq$" is defined by

$$[\underline{u},\overline{u}]\subseteq[\underline{v},\overline{v}] \quad\Longleftrightarrow\quad \underline{v}\leq\underline{u}, \qquad \overline{u}\leq\overline{v}.$$

For an arbitrary real number $\lambda$ and $[u]$, the interval $\lambda[u]$ is given by

$$\lambda[\underline{u},\overline{u}] = \begin{cases} [\lambda\underline{u},\lambda\overline{u}] & \text{if } \lambda>0,\\ \{0\} & \text{if } \lambda=0,\\ [\lambda\overline{u},\lambda\underline{u}] & \text{if } \lambda<0.\end{cases}$$

For $[u] = [\underline{u}, \overline{u}]$ and $[v] = [\underline{v}, \overline{v}]$, the four arithmetic operators $(+, -, \cdot, /)$ are defined by

$$\begin{aligned}
&[u] + [v] = [\underline{u} + \underline{v}, \overline{u} + \overline{v}],\\
&[u] - [v] = [\underline{u} - \overline{v}, \overline{u} - \underline{v}],\\
&[u] \cdot [v] = \bigl[\min\{\underline{u}\underline{v}, \underline{u}\overline{v}, \overline{u}\underline{v}, \overline{u}\overline{v}\}, \max\{\underline{u}\underline{v}, \underline{u}\overline{v}, \overline{u}\underline{v}, \overline{u}\overline{v}\}\bigr],\\
&[u]/[v] = \bigl[\min\{\underline{u}/\underline{v}, \underline{u}/\overline{v}, \overline{u}/\underline{v}, \overline{u}/\overline{v}\},\\
&\qquad\qquad \max\{\underline{u}/\underline{v}, \underline{u}/\overline{v}, \overline{u}/\underline{v}, \overline{u}/\overline{v}\}\bigr], \quad \text{where } 0 \notin [\underline{v}, \overline{v}].
\end{aligned}$$

We denote by $\mathbb{R}_{\mathcal{I}}$ the set of all intervals of $\mathbb{R}$, and by $\mathbb{R}_{\mathcal{I}}^{+}$ and $\mathbb{R}_{\mathcal{I}}^{-}$ the sets of all positive intervals and negative intervals of $\mathbb{R}$, respectively. The Hausdorff–Pompeiu distance between intervals $[\underline{u}, \overline{u}]$ and $[\underline{v}, \overline{v}]$ is defined by

$$d\bigl([\underline{u}, \overline{u}], [\underline{v}, \overline{v}]\bigr) = \max\bigl\{|\underline{u} - \underline{v}|, |\overline{u} - \overline{v}|\bigr\}.$$

It is well known that $(\mathbb{R}_{\mathcal{I}}, d)$ is a complete metric space.

A division of $[a, b]$ is any finite ordered subset $D$ having the form

$$D = \{a = t_0 < t_1 < \cdots < t_n = b\}.$$

The mesh of a division $D$ is the maximum length of the subintervals comprising $D$, i.e.,

$$\text{mesh}(D) = \max\{t_i - t_{i-1} : k = 1, 2, \ldots, n\}.$$

Let $\mathcal{D}(\delta, [a, b])$ be the set of all $D \in \mathcal{D}([a, b])$ such that $\text{mesh}(D) < \delta$. In each interval $[t_{i-1}, t_i]$, where $1 \leq i \leq n$, choose an arbitrary point $\xi_i$ and form the sum

$$S(f, D, \delta) = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}),$$

where $f : [a, b] \to \mathbb{R}$ (or $\mathbb{R}_{\mathcal{I}}$). We call $S(f, D, \delta)$ a Riemann sum of $f$ corresponding to $D \in \mathcal{D}(\delta, [a, b])$.

**Definition 2.1** A function $f : [a, b] \to \mathbb{R}$ is called Riemann integrable ($R$-integrable) on $[a, b]$ if there exists $A \in \mathbb{R}$ such that, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$\bigl|S(f, \mathcal{D}, \delta) - A\bigr| < \epsilon$$

for every Riemann sum $S$ of $f$ corresponding to each $D \in \mathcal{D}(\delta, [a, b])$ and independent of the choice of $\xi_i \in [t_{i-1}, t_i]$ for $1 \leq i \leq n$. In this case, $A$ is called the $R$-integral of $f$ on $[a, b]$ and is denoted by $A = (R)\int_a^b f(t)\,dt$. The collection of all functions that are $R$-integrable on $[a, b]$ will be denoted by $\mathcal{R}_{([a,b])}$.

The following definition is a special case of the Riemann integral for set-valued maps which was earlier given by Dinghas in 1956 [13].

**Definition 2.2** A function $f : [a,b] \to \mathbb{R}_{\mathcal{I}}$ is called interval Riemann integrable (*IR*-integrable) on $[a,b]$ if there exists $A \in \mathbb{R}_{\mathcal{I}}$ such that, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$d\bigl(S(f,\mathcal{D},\delta),A\bigr) < \epsilon$$

for every Riemann sum $S$ of $f$ corresponding to each $D \in \mathcal{D}(\delta,[a,b])$ and independent of the choice of $\xi_i \in [t_{i-1},t_i]$ for $1 \le i \le n$. In this case, $A$ is called the *IR*-integral of $f$ on $[a,b]$ and is denoted by $A = (IR)\int_a^b f(t)\,dt$. The collection of all functions that are *IR*-integrable on $[a,b]$ will be denoted by $\mathcal{IR}_{([a,b])}$.

*Remark* 2.3 The concept of *IR*-integral given in Definition 2.2 is equivalent to the *IR*-integral given in [28, Definition 9.1].

The following theorem was obtained in [28].

**Theorem 2.4** *An interval-valued function* $f(t) \in \mathcal{IR}_{([a,b])}$ *if and only if* $\overline{f}(t), \underline{f}(t) \in \mathcal{R}_{([a,b])}$ *and*

$$(IR)\int_a^b f(t)\,dt = \biggl[(R)\int_a^b \underline{f}(t)\,dt, (R)\int_a^b \overline{f}(t)\,dt\biggr].$$

## 3 Generalized Jensen's inequality for interval-valued functions

The following concepts are well known.

**Definition 3.1** We say that $f : [a,b] \to \mathbb{R}$ is a convex function if for all $x,y \in [a,b]$ and $t \in [0,1]$ we have

$$f\bigl(tx + (1-t)y\bigr) \le tf(x) + (1-t)f(y). \tag{1}$$

If inequality (1) is reversed, then $f$ is said to be concave.

**Definition 3.2** (Breckner, [3]) Let $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is called an $s$-convex function (in the second sense) if

$$f\bigl(tx + (1-t)y\bigr) \le t^s f(x) + (1-t)^s f(y)$$

for each $x,y \in [0,\infty)$ and $t \in [0,1]$.

**Definition 3.3** (Dragomir et al., [17]) We say that $f : [a,b] \to \mathbb{R}$ is a $P$-function if $f$ is non-negative and for all $x,y \in [a,b]$ and $t \in [0,1]$ we have

$$f\bigl(tx + (1-t)y\bigr) \le f(x) + f(y).$$

**Definition 3.4** (Varošanec, [46]) Let $h : [c,d] \to \mathbb{R}$ be a non-negative function, $(0,1) \subseteq [c,d]$ and $h \not\equiv 0$. We say that $f : [a,b] \to \mathbb{R}$ is an $h$-convex function, or that $f$ belongs to the class $SX(h,[a,b],\mathbb{R})$, if $f$ is non-negative and for all $x,y \in [a,b]$ and $t \in (0,1)$ we have

$$f\bigl(tx + (1-t)y\bigr) \le h(t)f(x) + h(1-t)f(y). \tag{2}$$

If inequality (2) is reversed, then $f$ is said to be $h$-concave, i.e., $f \in SV(h,[a,b],\mathbb{R})$. $h$ is said to be a supermultiplicative function if

$$h(xy) \ge h(x)h(y) \tag{3}$$

for all $x, y \in [c,d]$. If inequality (3) is reversed, then $h$ is said to be a submultiplicative function. If the equality holds in (3), then $h$ is said to be a multiplicative function.

Obviously, if $h(t) = t$, then all non-negative convex functions belong to $SX(h,[a,b])$ and all non-negative concave functions belong to $SV(h,[a,b])$.

We can introduced now the following concept of function.

**Definition 3.5** Let $h : [c,d] \to \mathbb{R}$ be a non-negative function, $(0,1) \subseteq [c,d]$ and $h \not\equiv 0$. We say that $f : [a,b] \to \mathbb{R}_{\mathcal{I}}^{+}$ is an $h$-convex interval-valued function if for all $x, y \in [a,b]$ and $t \in (0,1)$ we have

$$h(t)f(x) + h(1-t)f(y) \subseteq f\big(tx + (1-t)y\big). \tag{4}$$

If set inclusion (4) is reversed, then $f$ is said to be $h$-concave. $f$ is $h$-affine if it is both $h$-concave and $h$-convex. The set of all $h$-convex ($h$-concave, $h$-affine) interval-valued functions is denoted by

$$SX\big(h,[a,b],\mathbb{R}_{\mathcal{I}}^{+}\big) \quad \big(SV\big(h,[a,b],\mathbb{R}_{\mathcal{I}}^{+}\big), \mathrm{SA}\big(h,[a,b],\mathbb{R}_{\mathcal{I}}^{+}\big), \text{respectively}\big).$$

*Remark* 3.6 It is clear that if $h(t) = t^s$, then Definition 3.5 implies a special case of convexity introduced by Breckner [4].

**Theorem 3.7** *Let $f : [a,b] \to \mathbb{R}_{\mathcal{I}}^{+}$ be an interval-valued function such that $f(t) = [\underline{f}(t), \overline{f}(t)]$. Then $f \in SX(h,[a,b],\mathbb{R}_{\mathcal{I}}^{+})$ if and only if $\underline{f} \in SX(h,[a,b],\mathbb{R}^{+})$ and $\overline{f} \in SV(h,[a,b],\mathbb{R}^{+})$.*

*Proof* Suppose that $f \in SX(h,[a,b],\mathbb{R}_{\mathcal{I}}^{+})$, and consider $x, y \in [a,b]$, $t \in (0,1)$. Then we have

$$h(t)f(x) + h(1-t)f(y) \subseteq f\big(tx + (1-t)y\big),$$

that is,

$$\begin{aligned}&\big[h(t)\underline{f}(x) + h(1-t)\underline{f}(y), h(t)\overline{f}(x) + h(1-t)\overline{f}(y)\big] \\ &\quad \subseteq \big[\underline{f}\big(tx + (1-t)y\big), \overline{f}\big(tx + (1-t)y\big)\big].\end{aligned} \tag{5}$$

It follows that

$$h(t)\underline{f}(x) + h(1-t)\underline{f}(y) \ge [\underline{f}\big(tx + (1-t)y\big)$$

and

$$h(t)\overline{f}(x) + h(1-t)\overline{f}(y) \le \overline{f}\big(tx + (1-t)y\big).$$

This shows that $\underline{f} \in SX(h,[a,b],\mathbb{R}^+)$ and $\overline{f} \in SV(h,[a,b],\mathbb{R}^+)$. Conversely, if $\underline{f} \in SX(h,[a,b],\mathbb{R}^+)$ and $\overline{f} \in SV(h,[a,b],\mathbb{R}^+)$, then from Definition 3.3 and set inclusion (5) it follows that $f \in SX(h,[a,b],\mathbb{R}_{\mathcal{I}}^+)$ and the proof is complete. □

**Theorem 3.8** *Let $f:[a,b]\to\mathbb{R}_{\mathcal{I}}^+$ be an interval-valued function such that $f(t)=[\underline{f}(t),\overline{f}(t)]$. Then $f\in SV(h,[a,b],\mathbb{R}_{\mathcal{I}}^+)$ if and only if $\underline{f}\in SV(h,[a,b],\mathbb{R}^+)$ and $\overline{f}\in SX(h,[a,b],\mathbb{R}^+)$.*

*Proof* The proof is similar to that of Theorem 3.7, so we omit it. □

**Theorem 3.9** (Varošanec, [46]) *Let $w_1,w_2,\ldots,w_n$ be positive real numbers ($n\geq 2$). If h is a non-negative supermultiplicative function and if $f\in SX(h,[a,b],\mathbb{R}^+)$, $x_1,x_2,\ldots,x_n\in[a,b]$, then*

$$f\left(\frac{1}{W_n}\cdot\sum_{i=1}^{n}w_ix_i\right)\leq\sum_{i=1}^{n}h\left(\frac{w_i}{W_i}\right)f(x_i), \tag{6}$$

*where $W_n=\sum_{i=1}^{n}w_i$. If h is submultiplicative and $f\in SV(h,[a,b],\mathbb{R}^+)$, then inequality (6) is reversed.*

**Theorem 3.10** *Let $g\in\mathcal{R}_{([a,b])}$ such that $g:[a,b]\to[m,M]$, $h:I\to[0,\infty)$ be a supermultiplicative function and $f:[m,M]\to[0,\infty)$ be h-convex and continuous. If the following limit exists, is finite, and*

$$\lim_{t\to 0^+}\frac{h(t)}{t}=k>0,$$

*then*

$$f\left(\frac{\int_a^b g(s)\,ds}{b-a}\right)\leq\frac{k}{b-a}\int_a^b f\bigl(g(s)\bigr)\,ds.$$

*Proof* Consider the division $D\in\mathcal{D}(\delta,[a,b]_{\mathbb{T}})$ be given by

$$D=\{a=t_0<t_1<\cdots<t_{n-1}<t_n=b\},$$

where $t_i=a+\frac{i}{n}(b-a)$ for $0\leq i\leq n$. In each interval $[t_{i-1},t_i]$, where $1\leq i\leq n$, choose $\xi_i=t_{i-1}=a+\frac{i-1}{n}(b-a)$ and form the Riemann sum

$$S(f,D,\delta)=\sum_{i=1}^{n}f(\xi_i)(t_i-t_{i-1}).$$

Thanks to $g\in\mathcal{R}_{([a,b])}$, then

$$\int_a^b g(s)\,ds=\lim_{n\to\infty}\sum_{i=1}^{n}g(\xi_i)(t_i-t_{i-1})=\lim_{n\to\infty}\frac{b-a}{n}\cdot\sum_{i=1}^{n}g\left(a+\frac{i}{n}(b-a)\right).$$

Since $f:[m,M]\to[0,\infty)$ is continuous, the composite function $f(g)\in\mathcal{R}_{([a,b])}$ and

$$\int_a^b f\bigl(g(s)\bigr)\,ds=\lim_{n\to\infty}\frac{b-a}{n}\cdot\sum_{i=1}^{n}f\left(g\left(a+\frac{i}{n}(b-a)\right)\right).$$

In addition, $f$ is $h$-convex, then we have

$$f\left(\frac{1}{b-a}\cdot\frac{b-a}{n}\cdot\sum_{i=1}^{n}g\left(a+\frac{i}{n}(b-a)\right)\right)$$
$$\leq\sum_{i=1}^{n}h\left(\frac{1}{n}\right)f\left(g\left(a+\frac{i}{n}(b-a)\right)\right)$$
$$=\frac{1}{b-a}\frac{h(\frac{1}{n})}{\frac{1}{n}}\sum_{i=1}^{n}\frac{b-a}{n}f\left(g\left(a+\frac{i}{n}(b-a)\right)\right).$$

Consequently, we obtain

$$\lim_{n\to\infty}f\left(\frac{1}{b-a}\cdot\frac{b-a}{n}\cdot\sum_{i=1}^{n}g\left(a+\frac{i}{n}(b-a)\right)\right)=f\left(\frac{\int_a^b g(s)\,ds}{b-a}\right).$$

Also

$$\lim_{t\to 0^+}\frac{h(t)}{t}=k>0,$$

then

$$f\left(\frac{\int_a^b g(s)\,ds}{b-a}\right)\leq\frac{k}{b-a}\int_a^b f\bigl(g(s)\bigr)\,ds.$$

The proof is complete. □

*Remark* 3.11 It is clear, if $h(t)=t$ then $k=1$, and we get the following Jensen's inequality:

$$f\left(\frac{\int_a^b g(s)\,ds}{b-a}\right)\leq\frac{1}{b-a}\int_a^b f\bigl(g(s)\bigr)\,ds.$$

**Theorem 3.12** *Let $g\in\mathcal{R}_{([a,b])}$ such that $g:[a,b]\to[m,M]$, $h:I\to[0,\infty)$ be a submultiplicative function and $f:[m,M]\to[0,\infty)$ be $h$-concave and continuous. If the following limit exists, is finite, and*

$$\lim_{t\to 0^+}\frac{h(t)}{t}=k>0,$$

*then*

$$f\left(\frac{\int_a^b g(s)\,ds}{b-a}\right)\geq\frac{k}{b-a}\int_a^b f\bigl(g(s)\bigr)\,ds.$$

*Proof* The proof is similar to that of Theorem 3.10 and hence is omitted. □

As a consequence of Theorems 3.10 and 3.12, we have the following result.

**Theorem 3.13** *Let $g \in \mathcal{R}_{([a,b])}$ such that $g : [a,b] \to [m,M]$, $h : I \to [0,\infty)$ be a multiplicative function, and $f : [m,M] \to [0,\infty)$ be h-affine and continuous. If the following limit exists, is finite, and*

$$\lim_{t\to 0^+} \frac{h(t)}{t} = k > 0,$$

*then*

$$f\left(\frac{\int_a^b g(s)\,ds}{b-a}\right) = \frac{k}{b-a}\int_a^b f\big(g(s)\big)\,ds.$$

**Theorem 3.14** *Let $g \in \mathcal{R}_{([a,b])}$ such that $g : [a,b] \to [m,M]$, $h : I \to [0,\infty)$ be a multiplicative function and $f : [m,M] \to \mathbb{R}^+_{\mathcal{I}}$ be h-convex and continuous such that $f(t) = [\underline{f}(t), \overline{f}(t)]$. If the following limit exists, is finite, and*

$$\lim_{t\to 0^+} \frac{h(t)}{t} = k > 0,$$

*then*

$$\frac{k}{b-a}(IR)\int_a^b f\big(g(s)\big)\,ds \subseteq f\left(\frac{(R)\int_a^b g(s)\,ds}{b-a}\right).$$

*Proof* The proof is a combination of Theorems 1.1, 2.4, 3.10, and 3.12. □

*Remark* 3.15 It is clear that if $h(t) = t$, then $k = 1$, and we have

$$\frac{1}{b-a}(IR)\int_a^b f\big(g(s)\big)\,ds \subseteq f\left(\frac{(R)\int_a^b g(s)\,ds}{b-a}\right).$$

If $[a,b] = [0,1]$, then we get the following Jensen's inequality [9, Theorem 3.5]:

$$(IR)\int_0^1 f\big(g(s)\big)\,ds \subseteq f\left((R)\int_0^1 g(s)\,ds\right).$$

It is important to note that the above Jensen's inequality for convex set-valued maps is due to Matkowski and Nikodem [24].

Similarly, we can get the following theorem which gives a generalization of [9, Theorem 3.4].

**Theorem 3.16** *Let $g \in \mathcal{R}_{([a,b])}$ such that $g : [a,b] \to [m,M]$, $h : I \to [0,\infty)$ be a multiplicative function, and $f : [m,M] \to \mathbb{R}^+_{\mathcal{I}}$ be h-concave and continuous such that $f(t) = [\underline{f}(t), \overline{f}(t)]$. If the following limit exists, is finite, and*

$$\lim_{t\to 0^+} \frac{h(t)}{t} = k > 0,$$

*then*

$$\frac{k}{b-a}(IR)\int_a^b f\big(g(s)\big)\,ds \supseteq f\left(\frac{(R)\int_a^b g(s)\,ds}{b-a}\right).$$

Next theorem follows from Theorems 3.14 and 3.16.

**Theorem 3.17** *Let $g \in \mathcal{R}_{([a,b])}$ such that $g : [a,b] \to [m,M]$, $h : I \to [0,\infty)$ be a multiplicative function and $f : [m,M] \to \mathbb{R}_{\mathcal{I}}^{+}$ be h-affine and continuous such that $f(t) = [\underline{f}(t), \overline{f}(t)]$. If the following limit exists, is finite, and*

$$\lim_{t\to 0^{+}} \frac{h(t)}{t} = k > 0,$$

*then*

$$f\left(\frac{(R)\int_a^b g(s)\,ds}{b-a}\right) = \frac{k}{b-a}(IR)\int_a^b f\bigl(g(s)\bigr)\,ds.$$

## 4 Hermite–Hadamard type inequality for interval-valued functions

Now, the application of Theorems 1.1, 2.4, 3.7, and 3.8 gives the following result.

**Theorem 4.1** *Let $f : [a,b] \to \mathbb{R}_{\mathcal{I}}^{+}$ be an interval-valued function such that $f(t) = [\underline{f}(t), \overline{f}(t)]$ and $f \in \mathcal{IR}_{([a,b])}$, $h : [0,1] \to \mathbb{R}$ be a non-negative function and $h(\frac{1}{2}) \neq 0$. If $f \in SX(h,[a,b],\mathbb{R}_{\mathcal{I}}^{+})$, then*

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}\int_a^b f(x)\,dx \supseteq \bigl[f(a)+f(b)\bigr]\int_0^1 h(t)\,dt.$$

*If $f \in SV(h,[a,b],\mathbb{R}_{\mathcal{I}}^{+})$, then*

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \subseteq \frac{1}{b-a}\int_a^b f(x)\,dx \subseteq \bigl[f(a)+f(b)\bigr]\int_0^1 h(t)\,dt.$$

*Remark* 4.2 It is clear that if $h(t) = t^{s}$, then Theorem 4.1 reduces to the result of Osuna-Gómez et al. [38, Theorem 4]:

$$2^{s-1} f\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}\int_a^b f(x)\,dx \supseteq \frac{1}{s+1}\bigl[f(a)+f(b)\bigr].$$

If $h(t) = t$, then Theorem 4.1 reduces to the result for convex function:

$$f\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}\int_a^b f(x)\,dx \supseteq \frac{f(a)+f(b)}{2}.$$

If $h(t) = 1$, then Theorem 4.1 reduces to the result for $P$-function:

$$\frac{1}{2} f\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}\int_a^b f(x)\,dx \supseteq \bigl(f(a)+f(b)\bigr).$$

If $\underline{f}(t) = \overline{f}(t)$, then Theorem 4.1 reduces to the result of Sarikaya et al. [43, Theorem 6]:

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \bigl[f(a)+f(b)\bigr]\int_0^1 h(t)\,dt.$$

The next result generalizes Theorem 3.1 of [32].

**Theorem 4.3** *Let $f : [a,b] \to \mathbb{R}^+_{\mathcal{I}}$ be an interval-valued function such that $f(t) = [\underline{f}(t), \overline{f}(t)]$ and $f \in \mathcal{IR}_{([a,b])}$, $h : [0,1] \to \mathbb{R}$ be a non-negative function and $h(\frac{1}{2}) \neq 0$. If $f \in SX(h,[a,b],\mathbb{R}^+_{\mathcal{I}})$, then*

$$\frac{1}{4[h(\frac{1}{2})]^2} f\left(\frac{a+b}{2}\right) \supseteq \Delta_1 \supseteq \frac{1}{b-a}\int_a^b f(x)\,dx \supseteq \Delta_2 \supseteq \left[f(a)+f(b)\right]\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\int_0^1 h(t)\,dt,$$

*where*

$$\Delta_1 = \frac{1}{4h(\frac{1}{2})}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]$$

*and*

$$\Delta_2 = \left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\int_0^1 h(t)\,dt.$$

*If $f \in SV(h,[a,b],\mathbb{R}^+_{\mathcal{I}})$, then*

$$\frac{1}{4[h(\frac{1}{2})]^2} f\left(\frac{a+b}{2}\right) \subseteq \Delta_1 \subseteq \frac{1}{b-a}\int_a^b f(x)\,dx \subseteq \Delta_2 \subseteq \left[f(a)+f(b)\right]\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\int_0^1 h(t)\,dt.$$

*Proof* The proof is completed by combining Theorems 2.4, 3.7 and the result by Noor et al. [32, Theorem 3.1]. □

*Example* 4.4 Suppose that $[a,b] = [0,2]$. Let $h(t) = t$ for all $t \in [0,1]$ and $f : [a,b] \to \mathbb{R}^+_{\mathcal{I}}$ be defined by

$$f(x) = \left[x^2, 10 - e^x\right]$$

for all $x \in [0,2]$. We have

$$\frac{1}{4[h(\frac{1}{2})]^2} f\left(\frac{a+b}{2}\right) = [1, 10-e],$$

$$\Delta_1 = \frac{1}{2}\left[f\left(\frac{1}{2}\right), f\left(\frac{3}{2}\right)\right] = \left[\frac{5}{4}, 10 - \frac{\sqrt{e}+e\sqrt{e}}{2}\right],$$

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \left[\frac{4}{3}, \frac{21-e^2}{2}\right],$$

$$\Delta_2 = \frac{1}{2}\left(\left[2, \frac{19-e^2}{2}\right] + [1, 10-e]\right) = \left[\frac{3}{2}, \frac{39-2e-e^2}{4}\right],$$

and

$$[f(a)+f(b)]\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\int_0^1 h(t)\,dt=\left[2,\frac{19-e^2}{2}\right].$$

Then we obtain that

$$[1,10-e]\supseteq\left[\frac{5}{4},10-\frac{\sqrt{e}+e\sqrt{e}}{2}\right]\supseteq\left[\frac{4}{3},\frac{21-e^2}{2}\right]\supseteq\left[\frac{3}{2},\frac{39-2e-e^2}{4}\right]\supseteq\left[2,\frac{19-e^2}{2}\right].$$

Consequently, Theorem 4.3 is verified.

The next result generalizes Theorem 7 of [43].

**Theorem 4.5** *Let* $f,g:[a,b]\to\mathbb{R}^+_{\mathcal{I}}$ *be two interval-valued functions such that* $f(t)=[\underline{f}(t),\overline{f}(t)]$, $g(t)=[\underline{g}(t),\overline{g}(t)]$ *and* $fg\in\mathcal{IR}_{([a,b])}$, $h_1,h_2:[0,1]\to\mathbb{R}$ *be non-negative continuous functions. If* $f\in SX(h_1,[a,b],\mathbb{R}^+_{\mathcal{I}})$, $g\in SX(h_2,[a,b],\mathbb{R}^+_{\mathcal{I}})$, *then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx\supseteq M(a,b)\int_0^1 h_1(t)h_2(t)\,dt+N(a,b)\int_0^1 h_1(t)h_2(1-t)\,dt,$$

*where*

$$M(a,b)=f(a)g(a)+f(b)g(b)$$

*and*

$$N(a,b)=f(a)g(b)+f(b)g(a).$$

*Proof* The proof is completed by combining Theorems 2.4, 3.7, 3.10 and the result by Sarikaya, Saglam, and Yildirim [43, Theorem 7]. □

**Theorem 4.6** *Let* $f,g:[a,b]\to\mathbb{R}^+_{\mathcal{I}}$ *be two interval-valued functions such that* $f(t)=[\underline{f}(t),\overline{f}(t)]$, $g(t)=[\underline{g}(t),\overline{g}(t)]$ *and* $fg\in\mathcal{IR}_{([a,b])}$, $h_1,h_2:[0,1]\to\mathbb{R}$ *be non-negative continuous functions and* $h_1(\frac{1}{2})h_2(\frac{1}{2})\neq 0$. *If* $f\in SX(h_1,[a,b],\mathbb{R}^+_{\mathcal{I}})$, $g\in SX(h_2,[a,b],\mathbb{R}^+_{\mathcal{I}})$, *then*

$$\begin{aligned}&\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\&\quad\supseteq\frac{1}{b-a}\int_a^b f(x)g(x)\,dx+M(a,b)\int_0^1 h_1(t)h_2(1-t)\,dt\\&\qquad+N(a,b)\int_0^1 h_1(t)h_2(t)\,dt.\end{aligned}$$

*Proof* By hypothesis, one has

$$\begin{aligned}&h_1\left(\frac{1}{2}\right)f\big(ta+(1-t)b\big)+h_1\left(\frac{1}{2}\right)f\big((1-t)a+tb\big)\subseteq f\left(\frac{a+b}{2}\right),\\&h_2\left(\frac{1}{2}\right)g\big(ta+(1-t)b\big)+h_2\left(\frac{1}{2}\right)g\big((1-t)a+tb\big)\subseteq g\left(\frac{a+b}{2}\right).\end{aligned}$$

Then

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\
&\quad\supseteq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\Big[\underline{f}\big(ta+(1-t)b\big)\underline{g}\big(ta+(1-t)b\big)+\underline{f}\big(ta+(1-t)b\big)\underline{g}\big((1-t)a+tb\big)\\
&\qquad+\underline{f}\big((1-t)a+tb\big)\underline{g}\big(ta+(1-t)b\big)+\underline{f}\big((1-t)a+tb\big)\underline{g}\big((1-t)a+tb\big),\\
&\qquad\overline{f}\big(ta+(1-t)b\big)\overline{g}\big(ta+(1-t)b\big)+\overline{f}\big(ta+(1-t)b\big)\overline{g}\big((1-t)a+tb\big)\\
&\qquad+\overline{f}\big((1-t)a+tb\big)\overline{g}\big(ta+(1-t)b\big)+\overline{f}\big((1-t)a+tb\big)\overline{g}\big((1-t)a+tb\big)\Big]\\
&\quad=h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\Big[\underline{f}\big(ta+(1-t)b\big)\underline{g}\big(ta+(1-t)b\big),\overline{f}\big(ta+(1-t)b\big)\overline{g}\big(ta+(1-t)b\big)\Big]\\
&\qquad+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\Big[\underline{f}\big(ta+(1-t)b\big)\underline{g}\big((1-t)a+tb\big),\overline{f}\big(ta+(1-t)b\big)\overline{g}\big((1-t)a+tb\big)\Big]\\
&\qquad+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\Big[\underline{f}\big((1-t)a+tb\big)\underline{g}\big(ta+(1-t)b\big),\overline{f}\big((1-t)a+tb\big)\overline{g}\big(ta+(1-t)b\big)\Big]\\
&\qquad+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\Big[\underline{f}\big((1-t)a+tb\big)\underline{g}\big((1-t)a+tb\big),\overline{f}\big((1-t)a+tb\big)\overline{g}\big((1-t)a+tb\big)\Big]\\
&\quad=h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\Big[f\big(ta+(1-t)b\big)g\big(ta+(1-t)b\big)+f\big((1-t)a+tb\big)g\big((1-t)a+tb\big)\Big]\\
&\qquad+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\Big[f\big(ta+(1-t)b\big)g\big((1-t)a+tb\big)\\
&\qquad+f\big((1-t)a+tb\big)g\big(ta+(1-t)b\big)\Big]\\
&\quad\supseteq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\Big[f\big(ta+(1-t)b\big)g\big(ta+(1-t)b\big)+f\big((1-t)a+tb\big)g\big((1-t)a+tb\big)\Big]\\
&\qquad+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\Big[\big(h_1(t)f(a)+h_1(1-t)f(b)\big)\big(h_2(1-t)g(a)+h_2(t)g(b)\big)\\
&\qquad+\big(h_1(1-t)f(a)+h_1(t)f(b)\big)\big(h_2(t)g(a)+h_2(1-t)g(b)\big)\Big]\\
&\quad=h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\Big[f\big(ta+(1-t)b\big)g\big(ta+(1-t)b\big)+f\big((1-t)a+tb\big)g\big((1-t)a+tb\big)\Big]\\
&\qquad+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\Big[\big(h_1(t)h_2(1-t)+h_1(1-t)h_2(t)\big)M(a,b)\\
&\qquad+\big(h_1(t)h_2(t)+h_1(1-t)h_2(1-t)\big)N(a,b)\Big].
\end{aligned}$$

Integrating over $[0,1]$, we have

$$\begin{aligned}
&\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\
&\quad\supseteq\frac{1}{b-a}\int_a^b f(x)g(x)\,dx+M(a,b)\int_0^1 h_1(t)h_2(1-t)\,dt\\
&\qquad+N(a,b)\int_0^1 h_1(t)h_2(t)\,dt.
\end{aligned}$$

This concludes the proof. □

## 5 Results and discussion

We obtain some Jensen and Hermite–Hadamard type inequalities for $h$-convex interval-valued functions. Our results not only improve upon work by Costa but also generalize the results of Sarikaya et al. Because of the lack of "interval derivatives" with good properties, we have not investigated inequalities involving interval derivatives.

## 6 Conclusions

This paper introduced the $h$-convex (concave, affine) concept for interval-valued functions. Under the above concept, we presented some Jensen and Hermite–Hadamard type inequalities for interval-valued functions. Our results generalize the previous inequalities presented by Costa et al. The next step in the research direction proposed here is to investigate Jensen and Hermite–Hadamard type inequalities for interval-valued functions and fuzzy-valued functions on time scales.

**Acknowledgements**
The authors are very grateful to two anonymous referees, for several valuable and helpful comments, suggestions and questions, which helped them to improve the paper into present form.

**Funding**
This research is supported by the Fundamental Research Funds for the Central Universities (2017B19714 and 2017B07414) and Natural Science Foundation of Jiangsu Province (BK20180500).

**Availability of data and materials**
Not applicable.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

## Publisher's Note



Received: 19 June 2018 Accepted: 1 November 2018 Published online: 07 November 2018

**References**

1. Angulo, H., Gimenez, J., Moros, A.M., Nikodem, K.: On strongly $h$-convex functions. Ann. Funct. Anal. **2**, 87–93 (2011)
2. Bombardelli, M., Varošanec, S.: Properties of $h$-convex functions related to the Hermite–Hadamard–Fejér inequalities. Comput. Math. Appl. **58**, 1869–1877 (2009)
3. Breckner, W.W.: Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Räumen. Publ. Inst. Math. **23**, 13–20 (1978)
4. Breckner, W.W.: Continuity of generalized convex and generalized concave set-valued functions. Rev. Anal. Numér. Théor. Approx. **22**, 39–51 (1993)
5. Chalco-Cano, Y., Flores-Franulič, A., Román-Flores, H.: Ostrowski type inequalities for interval-valued functions using generalized Hukuhara derivative. Comput. Appl. Math. **31**, 457–472 (2012)
6. Chalco-Cano, Y., Lodwick, W.A., Condori-Equice, W.: Ostrowski type inequalities and applications in numerical integration for interval-valued functions. Soft Comput. **19**, 3293–3300 (2015)
7. Chalco-Cano, Y., Rufián-Lizana, A., Román-Flores, H., Jiménez-Gamero, M.D.: Calculus for interval-valued functions using generalized Hukuhara derivative and applications. Fuzzy Sets Syst. **219**, 49–67 (2013)
8. Chalco-Cano, Y., Silva, G.N., Rufián-Lizana, A.: On the Newton method for solving fuzzy optimization problems. Fuzzy Sets Syst. **272**, 60–69 (2015)
9. Costa, T.M.: Jensen's inequality type integral for fuzzy-interval-valued functions. Fuzzy Sets Syst. **327**, 31–47 (2017)
10. Costa, T.M., Bouwmeester, H., Lodwick, W.A., Lavor, C.: Calculating the possible conformations arising from uncertainty in the molecular distance geometry problem using constraint interval analysis. Inf. Sci. **415–416**, 41–52 (2017)
11. Costa, T.M., Chalco-Cano, Y., Lodwick, W.A., Silva, G.N.: Generalized interval vector spaces and interval optimization. Inf. Sci. **311**, 74–85 (2015)
12. Costa, T.M., Román-Flores, H.: Some integral inequalities for fuzzy-interval-valued functions. Inf. Sci. **420**, 110–125 (2017)
13. Dinghas, A.: Zum Minkowskischen Integralbegriff abgeschlossener Mengen. Math. Z. **66**, 173–188 (1956)
14. Dragomir, S.S.: Inequalities of Hermite–Hadamard type for $h$-convex functions on linear spaces. Proyecciones **34**, 323–341 (2015)

15. Dragomir, S.S.: Integral inequalities of Jensen type for λ-convex functions. Mat. Vesn. **68**, 45–57 (2016)
16. Dragomir, S.S.: Inequalities of Hermite–Hadamard type for functions of selfadjoint operators and matrices. J. Math. Inequal. **11**, 241–259 (2017)
17. Dragomir, S.S., Pečarié, J., Persson, L.E.: Some inequalities of Hadamard type. Soochow J. Math. **21**, 335–341 (1995)
18. Farissi, A.E.: Simple proof and refinement of Hermite–Hadamard inequality. J. Math. Inequal. **4**, 365–369 (2010)
19. Flores-Franulič, A., Chalco-Cano, Y., Román-Flores, H.: An Ostrowski type inequality for interval-valued functions. In: IFSA World Congress and NAFIPS Annual Meeting IEEE, vol. 35, pp. 1459–1462 (2013)
20. Gao, X.: A note on the Hermite–Hadamard inequality. J. Math. Inequal. **4**, 587–591 (2010)
21. Hadamard, J.: Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. J. Math. Pures Appl. **58**, 171–215 (1893)
22. Hermite, C.: Sur deux limites d'une intégrale définie. Mathesis **3**, 82 (1883)
23. Lara, T., Merentes, N., Nikodem, K.: Strong *h*-convexity and separation theorems. Int. J. Anal. **2016**, Article ID 7160348 (2016)
24. Matkowski, J., Nikodem, K.: An integral Jensen inequality for convex multifunctions. Results Math. **26**, 348–353 (1994)
25. Matłoka, M.: On Hadamard's inequality for *h*-convex function on a disk. Appl. Math. Comput. **235**, 118–123 (2014)
26. Mihai, M.V., Noor, M.A., Noor, K.I., Awan, M.U.: Some integral inequalities for harmonic *h*-convex functions involving hypergeometric functions. Appl. Math. Comput. **252**, 257–262 (2015)
27. Mitroi, F.-C., Nikodem, K., Wąsowicz, S.: Hermite–Hadamard inequalities for convex set-valued functions. Demonstr. Math. **46**, 655–662 (2013)
28. Moore, R.E.: Interval Analysis. Prentice Hall, Englewood Cliffs (1966)
29. Moore, R.E., Kearfott, R.B., Cloud, M.J.: Introduction to Interval Analysis. SIAM, Philadelphia (2009)
30. Nikodem, K., Snchez, J.L., Snchez, L.: Jensen and Hermite–Hadamard inequalities for strongly convex set-valued maps. Math. Æterna **4**, 979–987 (2014)
31. Noor, M.A., Cristescu, G., Awan, M.U.: Generalized fractional Hermite–Hadamard inequalities for twice differentiable *s*-convex functions. Filomat **24**, 191–197 (2014)
32. Noor, M.A., Noor, K.I., Awan, M.U.: A new Hermite–Hadamard type inequality for *h*-convex functions. Creative Math. Inform. **24**, 191–197 (2015)
33. Noor, M.A., Noor, K.I., Awan, M.U., Costache, S.: Some integral inequalities for harmonically *h*-convex functions. Sci. Bull. "Politeh." Univ. Buchar., Ser. A, Appl. Math. Phys. **77**, 5–16 (2015)
34. Noor, M.A., Noor, K.I., Awan, M.U., Li, J.: On Hermite–Hadamard inequalities for *h*-preinvex functions. Filomat **24**, 1463–1474 (2014)
35. Noor, M.A., Noor, K.I., Iftikhar, S., Ionescu, C.: Hermite–Hadamard inequalities for co-ordinated harmonic convex functions. Sci. Bull. "Politeh." Univ. Buchar., Ser. A, Appl. Math. Phys. **79**, 25–34 (2017)
36. Noor, M.A., Noor, K.I., Mihai, M.V., Awan, M.U.: Fractional Hermite–Hadamard inequalities for some classes of differentiable preinvex functions. Sci. Bull. "Politeh." Univ. Buchar., Ser. A, Appl. Math. Phys. **78**, 163–174 (2016)
37. Osuna-Gómez, R., Chalco-Cano, Y., Hernández-Jiménez, B., Ruiz-Garzón, G.: Optimality conditions for generalized differentiable interval-valued functions. Inf. Sci. **321**, 136–146 (2015)
38. Osuna-Gómez, R., Jimenez-Gaméro, M.D., Chalco-Cano, Y., Rojas-Medar, M.A.: Hadamard and Jensen inequalities for *s*-convex fuzzy processes. In: Soft Methodology and Random Information Systems, pp. 645–652. Springer, Berlin (2004)
39. Pečarié, J., Proschan, F., Tong, Y.L.: Convex Functions, Partial Orderings, and Statistical Applications. Academic Press, San Diego (1992)
40. Román-Flores, H., Chalco-Cano, Y., Lodwick, W.A.: Some integral inequalities for interval-valued functions. Comput. Appl. Math. **35**, 1–13 (2016)
41. Román-Flores, H., Chalco-Cano, Y., Silva, G.N.: A note on Gronwall type inequality for interval-valued functions. In: IFSA World Congress and NAFIPS Annual Meeting IEEE, vol. 35, pp. 1455–1458 (2013)
42. Rothwell, E.J., Cloud, M.J.: Automatic error analysis using intervals. IEEE Trans. Ed. **55**, 9–15 (2012)
43. Sarikaya, M.Z., Saglam, A., Yildirim, H.: On some Hadamard-type inequalities for *h*-convex functions. J. Math. Inequal. **2**, 335–341 (2008)
44. Sarikaya, M.Z., Set, E., Özdemir, M.E.: On some new inequalities of Hadamard-type involving *h*-convex functions. Acta Math. Univ. Comen. **LXXIX**(2), 265–272 (2010)
45. Snyder, J.M.: Interval analysis for computer graphics. SIGGRAPH Comput. Graph. **26**, 121–130 (1992)
46. Varošanec, S.: On *h*-convexity. J. Math. Anal. Appl. **326**, 303–311 (2007)
47. Weerdt, E.de., Chu, Q.P., Mulder, J.A.: Neural network output optimization using interval analysis. IEEE Trans. Neural Netw. **20**, 638–653 (2009)
48. Xi, B.Y., He, C.Y., Qi, F.: Some new inequalities of the Hermite–Hadamard type for extended $((s_1,m_1)-(s_2,m_2))$-convex functions on co-ordinates. Cogent Math. **3**, Article ID 1267300 (2016)
49. Xi, B.Y., Qi, F.: Integral inequalities of Hermite–Hadamard type for $((\alpha,m),\log)$-convex functions on co-ordinates. Probl. Anal. Issues Anal. **4**, 73–92 (2015)
50. Xi, B.Y., Qi, F.: Inequalities of Hermite–Hadamard type for extended *s*-convex functions and applications to means. J. Nonlinear Convex Anal. **6**, 873–890 (2015)